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ON THE SUPERIORITY OF APPROVAL VS PLURALITY: A COUNTEREXAMPLE*

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ABSTRACT. We present a simple voting environment where the Condorcet winner exists. Under plurality rule, the derived game has a stable set where such a candidate is elected with probability one. However, no stable set of the approval game elects the Condorcet winner with positive probability.

KEYWORDS: Approval voting, Plurality voting, Sophisticated voting, Mertens Stability.
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1. Introduction

Comparing alternative voting procedures has always been of primary interest to voting theorists. Economic theory provides the methodology; construct game theoretical models under different voting rules while holding voters' preferences over candidates constant; then compare the equilibrium outcomes emerging from those voting models. But this approach should be taken with care. It is typical for voting games to display unreasonable Nash equilibria that, as a consequence, do not capture plausible voting behavior. Indeed, from the pioneering work of Farquharson (1969), voting theorists have commonly accepted the *sophisticated voting principle*: reasonable voting equilibria should survive iterated deletion of dominated strategies. Not only did the subsequent literature on equilibrium refinements provide a game theoretical foundation to that principle (see Kohlberg and Mertens (1986)), but also expanded the list of requirements that reasonable equilibria should satisfy. It seems that *Mertens Stability* (Mertens, 1989) is the equilibrium concept that satisfies the most comprehensive list of such requirements.[1]

For instance, De Sinopoli (2000) shows that Mertens stable sets satisfy the sophisticated voting principle in voting games with *plurality rule*. (Under plurality rule, voters may vote for just one candidate or abstain, the candidate with the most votes wins the election and ties are broken randomly.) De Sinopoli (2000) also shows the inadequacy of Nash, *perfect* and *proper* equilibrium in these games. Even the sophisticated voting principle does not fully capture voters' strategic behavior; this principle does not distinguish among the different Nash equilibria of the reduced game obtained after the rounds of elimination. On the other hand, a Mertens stable set of the plurality game always selects a stable set of the resulting reduced game, thus, refining the set of sophisticated equilibria.

An application of Mertens stability to *approval voting* games can be found in De Sinopoli et al. (2006). (Under approval voting, every voter gives an approval vote to as many candidates as she wishes, the candidate with most approval votes wins the election and ties are broken randomly.) In particular, De Sinopoli et al. (2006) provide an example of an approval voting game where the *Condorcet winner*[2] is not elected in the unique stable set of the game.

The outcome of such an example contrasts with many arguments in favor of approval voting and its superiority over other voting procedures such as plurality rule. These arguments range from its greater flexibility (under approval voters

[1] Mertens stability is a set-valued concept (in contrast with Nash equilibrium). *Stable sets* are connected sets of normal form *perfect* equilibria and contain a stable set of the game obtained by deleting strategies that are at minimum probability in any *ε-perfect* equilibrium close to it. Stable sets satisfy existence—every game has an stable set—and have many desirable properties when compared to other equilibrium refinements (see van Damme (1991) for an excellent review).

[2] The Condorcet winner is the candidate that is elected in any pairwise contest.

can cast the same votes as they would do under plurality) to an increase in voting turnout (Brams and Fishburn (1978) argue that voters have more incentive to vote if they are more able to express their preferences) and include a better protection of minorities (voters do not "waste" a vote voting for a minority candidate because they can also vote for a stronger candidate at the same time). But the key argument focuses on the election of the Condorcet winner; if a Condorcet winner exists, the approval voting game has a Nash equilibrium in undominated strategies that selects it (Fishburn and Brams (1981); Brams and Sanver (2006)). The analogous claim does not hold for plurality. In fact, even in the example provided by De Sinopoli et al. (2006) in which the Condorcet winner is not elected in its unique stable set, we may still accept approval rule as the "lesser evil" for one can show that this candidate is not elected either in any stable set of the corresponding plurality game.

This note proposes a voting environment where we compare approval voting with plurality rule through the stable sets of equilibria of the corresponding games. In this example there is a preeminent candidate that is the Condorcet winner, the (relative) utilitarian candidate,[3] and the only candidate that satisfies the Rawlsian maximin criterion.[4] However, the approval voting game has a unique sophisticated equilibrium (hence, a unique stable set) and under this strategy combination this privileged candidate is not elected. In contrast, the plurality game has a stable set where this candidate is elected with probability one.

2. The example

We consider an election with five voters. During the analysis we may also refer to the voters as *players*. Each voter i has an ideal position x_i^* in the political spectrum $[0,1]$. Let $x_1^* = x_2^* = 0$, $x_3^* = 0.5$ and $x_4^* = x_5^* = 1$. Each voter only cares about the distance d between her ideal position and the elected candidate's political ideology. We let $u(d) = 1 - \sqrt{d}$ be the utility function representing those preferences. Therefore, the payoff to voter i if a candidate with with ideology $x \in [0,1]$ wins the election is $v(x, x_i^*) = u(|x - x_i^*|) = 1 - |x - x_i^*|^{1/2}$.

There are three candidates L, C and R with respective political ideologies 0, 0.5 and 1. A ballot is represented by the voted candidate(s).

Remark. What follows holds as long as there are two equally sized groups of voters at both sides of the political spectrum and one voter centered at 0.5. Leftist and rightist voters do not necessarily need to be all located at 0 and 1 and the utility function u can take other shapes and (or) be voter specific. It is enough that

[3] Dhillon and Mertens (1999) define the relative utilitarian candidate as the one that maximizes the sum of the voter's utilities after they are normalized so that voters' utilities form their top and bottom candidates are the same across voters.

[4] In other words, it is the candidate that maximizes the utility of the worst-off voter after the election. See Rawls (1971).

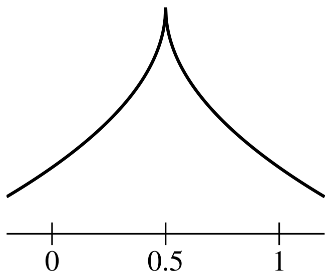

FIGURE 1. Voter 3's payoff function $v(\cdot, 0.5)$.

each voter's utility for her middle-ranked candidate is less than the average of the utilities for her top- and bottom-ranked candidates.

2.1. *Approval voting*

Under approval voting each voter may give an approval vote to as many candidates as she wishes. It is not difficult to see (cf. Brams and Fishburn (1978)) that in every undominated strategy a voter approves her most preferred candidate(s) and does not approve her least preferred one(s). In our example, voters 1 and 2 have two undominated strategies, L and LC; player 3 has only one, C; and players 4 and 5 have two, R and RC. We now show that in the reduced game where voters can only use undominated strategies voting just for the most preferred candidate is a dominant strategy.

Consider voter 1. If voter 1's opponents play undominated strategies then they give one vote to candidate L, two votes to candidate R and anything between one and four to candidate C depending on the particular undominated strategies that they use. Thus, we have four cases to consider. If candidate C receives one vote then voter 1 prefers to vote L and give rise to fair lottery between L and R rather than vote LC and induce a three-candidate lottery. If C receives two votes, voter 1 now prefers the three-candidate lottery to an outright win by candidate C. Therefore, she would vote L. Finally, If candidate C receives three or four votes this candidate wins the election outright whatever ballot is casted by voter 1.

In the same vein, one can show that voter $i = 2, 4, 5$ only votes for her most preferred candidate whenever voter i's opponents use undominated strategies. We conclude that there is only one strategy combination that survives iterated deletion of dominated strategies and that, under this strategy combination, both L and R receive two votes and candidate C only one. Consequently, the Condorcet winner C is not elected under approval voting.

2.2. *Plurality*

Under Plurality rule voters may vote for one candidate or abstain. Abstention and voting for the least preferred candidate(s) are dominated strategies.

In this section we show that the strategy combination $\tilde{c} = (C, C, C, R, R)$ is a Mertens stable set of the plurality voting game. We prove it showing that $\tilde{c}$ is an *absorbing retract* (Kalai and Samet, 1984) and therefore contains a stable set (Mertens, 1992).[5]

First of all, notice that under $\tilde{c}$, players 1, 2 and 3 play a strict best reply against $\tilde{c}$. That implies that for these voters C is also a best reply to any strategy combination sufficiently close to $\tilde{c}$. To prove that $\tilde{c}$ is an absorbing retract we have to show that for every strategy combination sufficiently close to $\tilde{c}$ players 4 and 5 prefer voting for R to voting for C.

Let us show this for player 4. Notice that as long as players 1, 2 and 3 vote for C, player 5's vote does not have any impact because C is elected anyway. Now consider a deviation from $\tilde{c}$ of just one of the first three players. We show that for any possible deviation voter 4 still prefers voting for R. If the deviating player abstains or votes for L, voting for R induces an equal probability lottery between C and R which is strictly preferred to the election of C for sure if she votes for C. If the deviating player votes for R, voter 4 strictly prefers to vote for R so that R wins the election rather than vote for C so that C wins the election. All the above proves that R is voter 4's best reply to any strategy combination sufficiently close to $\tilde{c}$. A symmetric argument applies to player 5 and, consequently, $\tilde{c}$ is an absorbing retract.

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[5] A strategy combination is an absorbing retract if it is a best reply to all sufficiently close strategy combinations.

References

Brams, S., Fishburn, P., 1978. Approval voting. American Political Science Review 72, 831–847.

Brams, S., Sanver, M., 2006. Critical strategies under approval voting: Who gets ruled in and ruled out. Electoral Studies 25 (2), 287–305.

De Sinopoli, F., 2000. Sophisticated voting and equilibrium refinements under plurality rule. Social Choice and Welfare 17 (4), 655–672.

De Sinopoli, F., Dutta, B., Laslier, J.-F., 2006. Approval voting: three examples. International Journal of Game Theory 35 (1), 27–38.

Dhillon, A., Mertens, J.-F., 1999. Relative utilitarianism. Econometrica 67 (3), 471–498.

Farquharson, R., 1969. Theory of Voting. New Haven: Yale University Press.

Fishburn, P., Brams, S., 1981. Approval voting, Condorcet's principle, and runoff elections. Public Choice 36 (1), 89–114.

Kalai, E., Samet, D., 1984. Persistent equilibria in strategic games. International Journal of Game Theory 13 (3), 129–144.

Kohlberg, E., Mertens, J.-F., 1986. On the strategic stability of equilibria. Econometrica 54, 1003–1038.

Mertens, J.-F., 1989. Stable equilibria–a reformulation, part I: Definition and basic properties. Mathematics of Operations Research 14 (4), 575–625.

Mertens, J.-F., 1992. The small worlds axiom for stable equilibria. Games and Economic Behavior 4 (4), 553–564.

Rawls, J., 1971. A Theory of Justice. Cambridge, Mass.

van Damme, E., 1991. Stability and Perfection of Nash Equilibria. Springer-Verlag, Berlin.