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Unpuzzling the Purchasing Power Parity Puzzle

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UNPUZZLING THE PURCHASING POWER PARITY PUZZLE

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The empirical validation of the purchasing power parity (PPP) theory is generally based on real exchange rates built using consumer price indexes (CPI). The empirical evidence does not generally support the theory and this fact goes under the name of *purchasing power parity puzzle*.

In this paper we show by theoretical arguments that, even if the law of one price holds for all the goods traded in two countries, real exchange rates based on CPI are not mean-reverting and therefore statistical tests based on them should reject the PPP hypothesis. We prove that such real exchange rates are neither stationary nor integrated, and so both unit-root and stationarity tests should reject the null according to their power properties.

The performance of the most common unit-root and stationarity tests in situations in which the law of one price holds is studied by means of a simulation experiment, based on real European CPI weights and price behaviours.

KEYWORDS: Purchasing power parity, Law of one price, Stationarity, Unit root.

JEL Codes: C30, C22

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1. INTRODUCTION

The purchasing power parity (PPP) and the law of one price (LOP) are among the most relevant issues in the academic debate: restricting the search to Google Scholar, the exact sequence of words "purchasing power parity" scored 57,600 hits[1]. The reason of such popularity lies in the fact that the basic relationship underlying these concepts is one of the founding elements of international economics. Nevertheless, when undergone to empirical scrutiny the PPP quickly became one of the major puzzles in international finance.

Initially the problem was related to the inability to detect mean reversion in real exchange rates during the recent float through standard univariate ADF tests. The common explanation was based on the well known lack of power of standard unit root tests in small samples (see among others Adler and Lehmann, 1983; Huizinga, 1987; Meese and Rogoff, 1988). The reaction was therefore to improve the power of such tests by increasing the length and the width of the sample under investigation. In the first case longer time series were considered (see for example Lothian and Taylor, 1996; Taylor, 2002) while in the latter case the attention was devoted to panel of countries (Abuaf and Jorion, 1990; Frankel and Rose, 1996; Taylor and Sarno, 1998).

At the same time several authors investigated the possibility of using more powerful tests such as Elliott et al.'s (1996) DF-GLS test (Cheung and Lai, 1998, 2000) or stationarity tests such as the KPSS (Kwiatkowski et al., 1992) obtaining stronger evidence of PPP even during the recent float. More recently Elliott and Pesavento (2006) found stronger rejections of the null of integration using covariate-augmented tests while Lopez et al. (2005) stressed the importance of the lag selection method. Finally a strand of the literature (Sarno et al., 2004; Taylor et al., 2001; Taylor, 2001) used non linear models to account for the possibility that the form of mean-reversion of real exchange rates might be nonlinear. Considering the different samples and techniques used, the consensus of the literature is that there is general evidence of the existence of the long-run PPP, however the puzzle remains since the estimated degree of mean reversion is far too low as compared to the type of shocks that are likely to hit prices and exchange rates. In other words the estimated persistence in real exchange rates is too high even in those cases in which mean-reversion is apparently assessed.

The PPP puzzle took a further twist when the availability of more complete microeconomic datasets allowed more direct tests of the LOP. In particular Crucini et al. (2005) study good-by-good deviations from the LOP for over 1,800 retail goods and services between all EU countries finding roughly as many overpriced goods as there are underpriced goods between any pair of countries and that good-by-good measures of cross-sectional price dispersion are negatively related to the tradeability of the good. Moreover Crucini and Shintani (2008) use an extensive micro-price panel, finding evidence of strong mean reversion across goods

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[1]On 21 December 2011, using double quotes to impose exact phrase search. The search "purchasing power parity puzzle" scores 3,540 entries in Google Scholar.

implying a level of PPP persistence much lower than previously estimated in the literature.

The tension between studies on individual prices and on aggregate prices called for a possible explanation based on the presence of an aggregation bias in the construction of the real exchange rate. Initially Taylor (2001) stressed the role of the temporal aggregation bias generated by the fact that sampling the data at low frequencies (generally annual and quarterly) does not allow the identification of the high-frequency component in prices' adjustment process. Subsequently, in a highly influential paper Imbs et al. (2005) show that the high persistence of the real exchange rate can be caused by the the aggregation bias deriving from the heterogeneous dynamics in its price components.

In this paper we provide a decisive contribution in addressing the PPP puzzle. We claim that the research so far failed to detect a fundamental problem at the heart of the poor empirical performances of PPP tests: traditional CPI-based real exchange rates are constructed such that they do not preserve the possible stationarity properties of the ratios of the individual prices of the same good across two countries. In particular, we prove that CPI-based real exchange rates are not mean reverting even when the LOP holds for all the goods traded in two countries.

Moreover, we prove that, under reasonable assumptions on the price dynamics, CPI-based real exchange rates are neither stationary nor integrated (i.e. unit-root processes), and therefore both stationarity and unit-root tests should reject their respective null hypotheses. This result casts some doubts on studies that claim to have found mean-reversion in real exchange rates using unit-root tests. Indeed, rejecting the null of a unit-root does not necessarily imply that stationarity holds and, thus, stationarity testing should substitute the common practice of testing the PPP using tests such as unit-root tests, which do not have PPP as null hypothesis[2].

We suggest a way for solving the aggregation problem by providing a sufficient condition that indicates how to compute meaningful real exchange rates that let the LOP translate into PPP.

Finally, we investigate the behaviour of common stationarity and unit-root tests applied to CPI-based real exchange rates under the LOP by carrying out simulation experiments for different weighting schemes of the CPI. The simulations confirm the theoretical results. Building a CPI-based real exchange rate from individual price series constructed such that the LOP holds yields a real exchange rate which is neither stationary nor integrated. In addition the simulation reveals a decisive role of the weights used in the CPI indexes. When the LOP holds, if the CPI weights of the two countries are equal, the CPI-based real exchange rate process is harder to distinguish from a mean-reverting process in finite samples, while, if the CPI weights are distant, the real exchange rates be-

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[2]As stressed by Caner and Kilian (2001) also stationarity tests are not immune from problems, albeit of a different nature. In fact they suffer from size distortions when the data generating process is highly persistent.

haviour is more similar to that of an integrated process. Both the stationarity and unit root tests we consider seem to be consistent (i.e. they reject their null with probability approaching one as the sample size increases and the null is false) whatever the choice of the weighting scheme.

The remainder of the paper is structured as follows: section 2 contains formal definitions of the LOP and of the PPP, illustrates the sufficient condition for building real exchange rates so that the LOP is translated into PPP and describes the theoretical results for CPI-based real exchange rates. Section 3 presents the simulation experiment; Section 4 concludes.

2. THEORY

In this section we prove that CPI-based real exchange rates are not suitable for assessing the validity of the PPP theory, and we show how to build meaningful real exchange rates that preserve the PPP when the LOP holds for all the individual price pairs.

The section is organised as follows:

1. we provide a formal statistical definitions of the LOP and of the PPP;
2. we state a sufficient condition for building real exchange rates that let the LOP translate into the PPP;
3. we show that CPI based real exchange rates do not comply with the above sufficient condition and we prove by counterexamples that real exchange rates constructed in such a way are not mean reverting even when the LOP holds.

2.1. *Statistical definition of LOP and PPP*

Let us denote the price of good $n \in \{1, \ldots, N\}$ in country $l \in \{a, b\}$ at time $t \in \{1, \ldots, T\}$ by $p_{n,l,t}$. The law of one price (LOP) states that, in the long run, the ratio of the prices of the same good in two different countries, expressed in the same currency, should be one (strong LOP) or at least constant (weak LOP):

$$(2.1) \qquad \frac{p_{n,a,t}}{p_{n,b,t}} = \exp\{\eta_{n,t}\}, \qquad \forall n, t$$

where $\exp\{\eta_{n,t}\}$ is a mean-reverting process such that $\mathbb{E} \exp\{\eta_{n,t}\} = 1$ in the strong case, and $\mathbb{E} \exp\{\eta_{n,t}\} = b_n < \infty$ in the weak case, with b_n positive constant. Notice that in (2.1) we used the exponential function as a device to let $\eta_{n,t}$ take values in $\mathbb{R}$ and, at the same time, guarantee that the price ratios are positive.

Generally formal definitions of mean-reversion are model dependent (i.e. based on specific assumptions on the dynamics of the data generating process), thus, we use a general, model-independent substitute: the probabilistic notion of *strong mixing*, which implies mean-reversion[3] (if the mean exists) and allows for the

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[3]See the discussion in Davidson (1994, p.214)

form of *short-memory* that economists expect in real exchange rates when PPP holds.[4] Moreover, since the real exchange rate is a nonlinear functions of prices, strong mixing has to be supplemented with *strict stationarity*, which guarantees that its mean (or median in case the mean does not exist) is constant over time.[5]

Let $\{X_t\}$ be a (possibly vectorial) random sequence and let $\mathcal{F}^t_{-\infty}$ and $\mathcal{F}^\infty_{t+m}$ be the σ-fields generated by, respectively, $\{\ldots, X_{t-1}, X_t\}$ and $\{X_{t+m}, X_{t+m+1}, \ldots\}$; and define

$$\alpha_m := \sup_{A \in \mathcal{F}^t_{-\infty}, B \in \mathcal{F}^\infty_{t+m}} |\Pr(A \cap B) - \Pr(A)\Pr(B)|.$$

The sequence $\{X_t\}$ is said *strong mixing* if $\lim_{m \to \infty} \alpha_m = 0$.

In addition, $\{X_t\}$ is *strictly stationary* if the shift transformation is measure-preserving, i.e. the sequences $\{X_{t_1}, \ldots, X_{t_h}\}$ and $\{X_{t_1+k}, \ldots, X_{t_h+k}\}$ have the same joint distribution for every $h, k \in \mathbb{Z}$ and for every $\{t_1, \ldots, t_h\} \in \mathbb{Z}^h$.

Following these definitions we can state the law of one price as follows.

DEFINITION 1 (Law of One Price – LOP)   For a given pair of countries a and b, the law of one price holds for a subset of goods $\Omega \subseteq \{1, \ldots, N\}$ if (2.1) holds with $\boldsymbol{\eta}_t := \{\eta_{n,t}\}_{n \in \Omega}$ strictly stationary strong mixing (SSSM) random sequence.

The PPP is a generalization of the LOP where price aggregates are used instead of individual prices. These price aggregates are (spatial) *price indexes* which for our purpose it is sufficient to define as time-invariant functions of the prices of the two countries that are to be compared and of two vectors of weights:

$$(2.2) \qquad {}_aP_{b,t} = f(\boldsymbol{p}_{a,t}, \boldsymbol{p}_{b,t}, \boldsymbol{w}_{a,t}, \boldsymbol{w}_{b,t}),$$

where $\boldsymbol{p}_{l,t}$ is a price vector for the goods in Ω, $\boldsymbol{w}_{l,t}$ is some positive weight vector that determines the importance of each good in country l, and $f : \mathbb{R}^{+4k} \mapsto \mathbb{R}^+$, with $k := \text{card}(\Omega)$, a (measurable) function. Generally the weights $\boldsymbol{w}_{l,t}$ are either (real or imputed) quantities or expenditure shares.[6]

DEFINITION 2 (Purchasing Power Parity – PPP)   For a given pair of countries a and b the purchasing power parity holds for a subset of goods $\Omega \subseteq \{1, \ldots, N\}$ if ${}_aP_{b,t}$ is a SSSM random sequence.

2.2. *Sufficient condition for PPP-preserving real exchange rates*

The next proposition provides a sufficient conditions under which PPP is a consequence of the LOP.

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[4]For a rigourous introduction to the theory of mixing refer to Davidson (1994, Ch.13-14).

[5]In fact the constancy of the mean of a random sequence is not a sufficient condition for the constancy of the mean of its nonlinear transformations.

[6]For a compact introduction to the axiomatic theory of index numbers the reader may refer to the survey by Balk (1995).

PROPOSITION 1 (Sufficient condition for PPP)   *Suppose that, for the set of goods* Ω *and the countries a and b, the LOP (Definition 1) holds. If the spatial price index (2.2) has form:*

$$_aP_{b,t} = g(\boldsymbol{p}_{a,t} \oslash \boldsymbol{p}_{b,t}),$$

where $g : \mathbb{R}^{+k} \mapsto \mathbb{R}^+$ *is a (measurable) time-invariant function and* $\oslash$ *denotes element-wise division, then in the countries a and b the PPP (Definition 2) holds for the goods in* Ω.

PROOF: The proof is trivial since measurable time-invariant finite-lag functions of strictly stationary strong mixing sequences are strictly stationary strong mixing sequences (Davidson, 1994, Th. 14.1).   *Q.E.D.*

In applied works real exchange rates are usually built using the ratio of the CPI of the two countries of interest expressed in the same currency. Thus, using our notation, the formula for the CPI is:

$$(2.3) \qquad CPI_{l,t} := (\boldsymbol{p}_{l,t} \oslash \boldsymbol{p}_{l,0})^\top \boldsymbol{w}_l = \sum_{n \in \Omega} \frac{p_{n,l,t}}{p_{n,l,0}} w_{n,l}, \quad \text{with } l \in \{a,b\}, \mathbf{1}^\top \boldsymbol{w}_l = 1.$$

The vector of weights $\boldsymbol{w}_l$ usually depends also on time, but since it changes rather slowly over time we will assume from now on that it is time-invariant. Note that the results of this subsection (i.e. the non-stationarity of CPI-based real exchange rates) is *a fortiori* true if we let $\boldsymbol{w}_l$ be time-dependent.

The CPI-based real exchange rate is then defined as

$$(2.4) \qquad _aP_{b,t}^{CPI} := \frac{CPI_{a,t}}{CPI_{b,t}} = \frac{\sum_{n \in \Omega} p_{n,a,t} w_{n,a}}{\sum_{n \in \Omega} p_{n,b,t} w_{n,b}},$$

where the CPI are expressed in the same currency and, without loss of generality, we set $t = 0$ as base year (i.e. $p_{n,l,0} = 1$, $\forall n,l$). Unfortunately (2.4) cannot be cast into the form of Proposition 1. In fact, by rewriting (2.4) as

$$_aP_{b,t}^{CPI} = \sum_{n \in \Omega} \frac{p_{n,a,t}}{p_{n,b,t}} \cdot \left(\frac{p_{n,b,t} w_{n,a}}{\sum_{m \in \Omega} p_{m,b,t} w_{m,b}} \right),$$

and noticing that second factor in the product on the right hand side is time-dependent, it is clear that the condition of Proposition 1 is not met.

2.3. *Counterexamples for CPI-based real exchange rates*

We have seen that CPI-based real exchange rates cannot be cast into the form of Proposition 1, but this does not prove that they are not SSSM under the LOP (or *PPP-preserving*), as the proposition provides only a sufficient condition.

In this subsection we use two counterexamples to show that CPI-based real exchange rates are not PPP-preserving.

Our counterexamples are based on a set Ω with just two goods:

$$(2.5) \qquad {}_aP^{CPI}_{b,t} = \frac{\alpha_1 p_{1,a,t} + \alpha_2 p_{2,a,t}}{\beta_1 p_{1,b,t} + \beta_2 p_{2,b,t}},$$

with $\alpha_1, \beta_1 \in (0,1)$, $\alpha_2 = 1 - \alpha_1$, $\beta_2 = 1 - \beta_1$ and where we assumed without loss of generality that $p_{n,l,0} = 1$ for $n = \{1,2\}$, $l = \{a,b\}$.

COUNTEREXAMPLE 1 Let us consider the case of the deterministic (strong) LOP, in which the prices of the same good across two countries are identical:

$$p_{1,a,t} = p_{1,b,t}, \qquad p_{2,a,t} = p_{2,b,t}, \quad \forall t \in \{1,2,\ldots,T\}.$$

The only way to assure that ${}_aP^{CPI}_{b,t}$ is not time-dependent, whatever the time-paths of $p_{1,l,t}$ and $p_{2,l,t}$ may be, is to set $\alpha_1 = \beta_1$.

Generalising to the case of the weak LOP, take $p_{n,a,t} = c_n p_{n,b,t}$ for $n = 1,2$ and c_n positive constants (i.e. the prices of the same good are proportional across the two countries). Now, the only condition[7] that ensures that ${}_aP^{CPI}_{b,t}$ is constant for every time-path of the prices is

$$(2.6) \qquad \beta_1 = \frac{\alpha_1 c_1}{\alpha_1 c_1 + \alpha_2 c_2}, \qquad \beta_2 = \frac{\alpha_2 c_2}{\alpha_1 c_1 + \alpha_2 c_2}.$$

In order to provide a form to the time-path of the prices, we can assume, for instance, that prices grow at a constant (continuous time) rate. This implies the exponential growth:

$$p_{n,l,t} = \exp(r_{n,l}t) \qquad r_{n,l} \in \mathbb{R}, \quad n = 1,2, \quad l = \{a,b\}.$$

Therefore, in case the deterministic (strong) LOP holds, the real exchange rate (2.5) becomes

$${}_aP^{CPI}_{b,t} = \frac{\alpha_1 e^{r_1 t} + \alpha_2 e^{r_2 t}}{\beta_1 e^{r_1 t} + \beta_2 e^{r_2 t}},$$

with $r_1 \neq r_2$[8]. Deriving with respect to time t we obtain:

$$\frac{(\alpha_1 - \beta_1)e^{(r_1+r_2)t}(r_1 - r_2)}{\left(\beta_1 e^{r_1 t} + \beta_2 e^{r_2 t}\right)^2}.$$

This derivative is zero and, thus, the real exchange rates constant, only when $\alpha_1 = \beta_1$.

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[7]Recall that $\beta_1 \in [0,1]$ and $\beta_2 = 1 - \beta_1$.
[8]This excludes the trivial case in which all prices have identical rates of growth

In the case of the (deterministic) weak LOP, we have

$$_aP_{b,t}^{CPI} = \frac{\alpha_1 c_1 e^{r_1 t} + \alpha_2 c_2 e^{r_2 t}}{\beta_1 e^{r_1 t} + \beta_2 e^{r_2 t}},$$

with $r_1 \neq r_2$ and c_1, c_2 positive constants. The derivative with respect to the time t is now

$$\frac{(a_1 b_2 c_1 - a_2 b_1 c_2)e^{(r_1+r_2)t}(r_1 - r_2)}{\left(b_1 e^{r_1 t} + b_2 e^{r_2 t}\right)^2},$$

and it is zero only if the identities (2.6) hold.

$$\diamondsuit$$

With Counterexample 1 we have shown that in a deterministic setting there is only one choice of the weight vector such that the PPP is a consequence of the LOP, and this choice is not necessarily the one pursued by national statistical institutes for the construction of real CPI.

In the next counterexample we adopt a more realistic assumption on the dynamic behaviour of prices.

COUNTEREXAMPLE 2   In empirical analyses the dynamics of the logarithm of the prices is generally well approximated by integrated processes of order either one or two. Therefore, we consider the case in which the log of each price follows a Gaussian random walk with drift[9] (possibly plus noise):

$$(2.7)\qquad\begin{aligned}
\log p_{n,b,t} &= \mu_{n,t}\\
\log p_{n,a,t} &= \nu_n + \mu_{n,t} + \eta_{n,t}, & \eta_{n,t} &\sim \text{NID}(0, \tau_n^2),\\
\mu_{n,t} &= \delta_n t + \sum_{i=1}^{t} \varepsilon_{n,i}, & \varepsilon_{n,t} &\sim \text{NID}(0, \sigma_n^2)
\end{aligned}$$

where $\varepsilon_{n,i}$ and $\eta_{n,i}$ are mutually independent, and also independent over the goods dimension n. This assumption entails the weak law of one price, since the log-prices of the same good in the two countries are cointegrated:

$$\log p_{n,a,t} - \log p_{n,b,t} = \nu_n + \eta_{n,t} \sim \text{SSSM}$$

or, equivalently,

$$\frac{p_{n,a,t}}{p_{n,b,t}} = \exp(\nu_n)\exp(\eta_{n,t}) \sim \text{SSSM},$$

which matches Definition 1 exactly.

Notice that the constant-growth price process of Counterexample 1 is a special case of (2.7) obtained by setting all the variances τ_n^2, σ_n^2 equal to zero.

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[9]In the following $\text{NID}(\mu, \sigma^2)$ stands for normally independently distributed with mean μ and variance σ^2.

From the discussion in Counterexample 1 it should be clear that the case of equal weights ($\beta_n = \alpha_n$) provides the most favourable setting for the index defined in (2.5) to be PPP-preserving, al least when the strong LOP holds, i.e. when $\delta_n = 0$, $n = 1, 2$. We, therefore, limit our attention to the case of equal weights and, using (2.7), rewrite (2.5) as

$$\frac{\alpha_1 \exp(\nu_1 + \mu_{1,t} + \eta_{1,t}) + \alpha_2 \exp(\nu_2 + \mu_{2,t} + \eta_{2,t})}{\alpha_1 \exp(\mu_{1,t}) + \alpha_2 \exp(\mu_{2,t})}.$$

By multiplying and dividing the numerator by $\alpha_1 \exp(\nu_1 + \mu_{1,t} + \eta_{1,t})$ and the denominator by $\alpha_1 \exp(\mu_{1,t})$, we obtain:

$$(2.8) \qquad \exp(\nu_1 + \eta_{1,t}) \frac{1 + \alpha \exp(\nu + \mu_t + \eta_t)}{1 + \alpha \exp(\mu_t)},$$

with $\alpha := \alpha_2/\alpha_1$, $\nu := \nu_2 - \nu_1$, $\eta_t := \eta_{2,t} - \eta_{1,t} \sim \text{NID}(0, \tau^2)$, $\tau := \tau_1 + \tau_2$, and

$$\mu_t := \mu_{2,t} - \mu_{1,t} = \delta t + \sum_{i=1}^{t} \varepsilon_i,$$

where $\delta := \delta_2 - \delta_1$, $\varepsilon_t := \varepsilon_{2,t} - \varepsilon_{1,t} \sim \text{NID}(0, \sigma^2)$, $\sigma^2 := \sigma_1^2 + \sigma_2^2$.

The process (2.8) is the product of a SSSM sequence and a process whose stochastic behaviour is to be investigated. Following Ullah (2004) we can derive the first moment of the log of process (2.8) as follows:

LEMMA 1 (Ullah 2004, Section 2.2) *Let $h : \mathbb{R}^u \mapsto \mathbb{R}^v$ be an analytic function of the normal random vector $\boldsymbol{y}$ with mean vector $\boldsymbol{m}$ and covariance matrix $\boldsymbol{S}$ such that $\mathbb{E}\, h(\boldsymbol{y})$ exists, then*

$$\mathbb{E}\, h(\boldsymbol{y}) = h(\mathbb{D}) \cdot \boldsymbol{1},$$

where $\mathbb{D}$ is the derivative operator $\mathbb{D} = \boldsymbol{m} + \boldsymbol{S}(\partial/\partial \boldsymbol{m})$.

Notice that the k-th power of the operator $\mathbb{D}$ that will be used throughout the section must be applied as $\mathbb{D}^{k-1}(\mathbb{D} \cdot \boldsymbol{1})$ and not as $(\mathbb{D}^{k-1}\mathbb{D}) \cdot \boldsymbol{1}$. Furthermore, even when $\boldsymbol{\mu} = \boldsymbol{0}$ the operator has to be applied to the symbolic mean $\boldsymbol{\mu}$ that will be set to zero at the end of the computations. For more details and examples refer to Ullah (2004).

Taking the log of (2.8) we obtain:

$$(2.9) \qquad \nu_1 + \eta_{1,t} + \log\left(1 + \alpha \exp(\nu + \mu_t + \eta_t)\right) - \log\left(1 + \alpha \exp(\mu_t)\right).$$

In order to assess if the expectation of this expression is time invariant or not, we need to compute the expectations of the last two addends, which are nonlinear functions of two normal random quantities:

$$(2.10) \qquad (\nu + \mu_t + \eta_t) \sim \text{N}\left(\nu + \delta t, \sigma^2 t + \tau^2\right), \qquad \mu_t \sim \text{N}\left(\delta t, \sigma^2 t\right).$$

Since $\log(1 + \alpha \exp(y))$ is analytic we can expand it around zero, and applying Lemma 1 we can write:

$$(2.11) \qquad \mathbb{E} \log \big(1 + \alpha \exp(y)\big) = \log \big(1 + \alpha \exp(\mathbb{D})\big) = \sum_{i=0}^{\infty} c_i \mathbb{D}^i,$$

where the expansion coefficients c_i depend only on α. Table I reports the first five quantities c_i and $\mathbb{D}^i \cdot 1$ necessary to compute the expectation of $\log(1 + \alpha \exp(y))$ for a normal random variable y with mean m and variance s^2.

TABLE I

FIRST FIVE QUANTITIES FOR THE EXPANSION OF $\mathbb{E} \log \big(1 + \alpha \exp(y)\big)$ WITH y NORMAL RANDOM VARIABLE WITH MEAN m AND VARIANCE s^2.

i	c_i	$\mathbb{D}^i \cdot 1$
0	$\log(1+\alpha)$	1
1	$\frac{\alpha}{1+\alpha}$	m
2	$\frac{\alpha}{2(1+\alpha)^2}$	$m + s^2$
3	$\frac{(\alpha - \alpha^2)}{6(1+\alpha)^3}$	$m^2 + ms^2 + s^2$
4	$\frac{(\alpha - 4\alpha^2 + \alpha^3)}{24(1+\alpha)^4}$	$m^3 + m^2 s^2 + 3ms^2 + s^4$

In order to apply Lemma 1 to the third and fourth addend of (2.9), we should consider that the function h is identical in both cases, only the means and variances of the two random variables are different:

$$
\mathbb{E} \left[\log \big(1 + \alpha \exp(\nu + \mu_t + \eta_t)\big) - \log \big(1 + \alpha \exp(\mu_t)\big) \right] =
$$

(2.12)

$$
= \log \big(1 + \alpha \exp(\mathbb{D}_1)\big) - \log \big(1 + \alpha \exp(\mathbb{D}_2)\big) = \sum_{i=0}^{\infty} c_i (\mathbb{D}_1 - \mathbb{D}_2) \cdot 1,
$$

where $\mathbb{D}_1$ and $\mathbb{D}_2$ are the derivative operators for the two normal variables (2.10), respectively.

Limiting the computations to the first five terms of the expansion we get the results in Table II.

TABLE II

FIRST FOUR TERMS OF THE EXPANSIONS (2.12).

i	$(\mathbb{D}_1^i \cdot 1) - (\mathbb{D}_2^i \cdot 1)$
0	0
1	ν
2	$\nu + \tau^2$
3	$\nu^2 + \nu\tau^2 + \tau^2 + t(2\nu\delta + \nu\sigma^2 + \delta\tau^2)$
4	$\nu^3 + 3\nu\tau^2 + \nu^2\tau^2 + \tau^4 + t \left[\nu^2 \big(3\delta + \sigma^2\big) + 3\delta\tau^2 + 2\sigma^2\tau^2 + \nu \big(3\sigma^2 + 2\delta\tau^2\big) \right]$
	$+ t^2 \left[\nu\delta \big(3\delta + 2\sigma^2\big) + \delta^2\tau^2 \right]$

While the terms of order 0,1 and 2 are time-invariant, from the term of order 3 on it is clear that the expectation of (2.9) depends on the time parameter t.

Therefore the processes described by (2.8) and (2.9) are not strictly stationary. In Table III the coefficients of orders $i = 1, 2, 3, 4$ are computed for few values of $\alpha = \alpha_2/\alpha_1$: the c_i sequence is the same for $\alpha = x$ and $\alpha = 1/x$.

TABLE III

COEFFICIENTS OF THE EXPANSION (2.12) FOR DIFFERENT VALUES OF α.

c_i	$\alpha = 10^{\pm 1}$	$\alpha = 2^{\pm 1}$	$\alpha = 1$
c_1	0.0909	0.3333	0.5000
c_2	0.0413	0.1111	0.1250
c_3	0.0113	0.0123	0.0000
c_4	0.0017	-0.0031	-0.0052

Notice that if we set $\nu = \tau = 0$, which corresponds to the case of the deterministic strong LOP (i.e. identical prices of the same good across the two countries), we obtain the result of Counterexample 1 as all terms of the expansion become time-invariant.

We have shown that the mean of the process (2.9) is not time-invariant, and, using analogous arguments, we can show that also the first difference of that process has a time-dependent mean: indeed, it is straightforward to see that the expansion of the expectation of the first difference of (2.9) has the form

$$\sum_{i=0}^{\infty} c_i \big(\mathbb{D}_1 - \mathbb{D}_2 - (\mathbb{D}_1^{\mathrm{lag}} - \mathbb{D}_2^{\mathrm{lag}}) \big) \cdot 1$$

where $(\mathbb{D}_1^{\mathrm{lag}} - \mathbb{D}_2^{\mathrm{lag}}) \cdot 1$ is as in Table II, but with $t - 1$ substituting t. The term of order $i = 3$ becomes time-invariant, but the terms of order $i \geq 4$ are still time-dependent.

$\diamond$

We proved that the mean of the process (2.9) and of its first difference are not time invariant when log-prices are pairwise cointegrated Gaussian random walks. Similar results could be obtained under more general distributional assumptions (cf. Ullah, 2004, Section 2.3). This finding can be summarised in the following proposition.

PROPOSITION 2 *Assume that, for all goods in a given set Ω, the log of prices of two identical goods across two different countries are cointegrated. Then the real exchange rate computed as ratio of the two CPI (for the goods in Ω) expressed in the same currency is neither stationary nor integrated.*

The two counterexamples we have provided show that, unless one is willing to make unrealistic assumptions on the weights, if real exchange rates are computed as the ratio of CPI indexes, then the fact that the LOP holds for individual goods is not sufficient to guarantee that the same applies for PPP at aggregate level. Again, we can summarise this result in the following proposition.

PROPOSITION 3   *When real exchange rates are computed as ratio of two CPI expressed in the same currency, the LOP is not sufficient for PPP.*

REMARK 1   Proposition 2 implies that both unit root tests and stationarity tests should reject their null hypotheses. The frequency of rejection depends on the power properties of the tests against this atypical alternative (cf. Section 3).

REMARK 2   From Proposition 2 and Remark 1 it is clear that rejecting the null of a unit root for the (log) real exchange rates does not imply that PPP holds. Thus, the empirical validation of PPP should be preferably based on short-memory stationarity tests, such as the KPSS (Kwiatkowski et al., 1992), rather than on unit root tests. The reason is that mean reversion is just one among the many alternatives against which unit root tests may have power.

REMARK 3   Proposition 3 states that the ratios of CPI indexes are not suitable for testing the PPP as they do not preserve the PPP even when it should hold. Pelagatti (2010) shows that neither the GEKS system (Gini, 1924; Eltetö and Köves, 1964; Szulc, 1964) used by the International Comparison Program, nor the well known Geary-Khamis method (Geary, 1958; Khamis, 1972) are able to transfer the cointegration properties of the individual (log) price pairs to the spatial price indexes.

3. SIMULATION EXPERIMENT

In the previous section we have shown that constructing a CPI-based real exchange rate, even if the LOP holds on individual prices, determines an atypical data generating process that is neither stationary nor integrated. In order to understand how PPP tests behave under this condition, we analyse the power of the ADF (Dickey and Fuller, 1979; Hasza and Fuller, 1979) and the KPSS (Kwiatkowski et al., 1992), under the LOP for different sample sizes and CPI weighting schemes. Although we generate price sample paths randomly, the moments of the data generating process are matched to those estimated on real world data.

We base our simulation experiment on the harmonised CPI data of 32 European countries as published by Eurostat. For each country Eurostat makes available on its web site the monthly price indexes of 91 COICOP[10] categories and the relative vector of weights used to compute the harmonised CPI. In order to determine the moments of the reference price time series we used French prices as base prices and Belgian prices as comparison prices. The reason for this choice is twofold: i) France and Belgium have index prices for all the COICOP categories, while in other countries some items are missing, ii) due to their proximity, economic and cultural integration, France and Belgium are likely to be countries where the LOP holds.

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[10]COICOP stands for Classification of Individual Consumption by Purpose.

Let $p_{n,l,t}$ be the annual[11] price index of category n, country l, in year t and $\boldsymbol{p}_{l,t} = (p_{1,l,t}, \ldots, p_{91,l,t})^{\top}$. We estimate the drift and the covariance matrix of the log-price increments for France as

$$\hat{\boldsymbol{\delta}}_F := \frac{1}{10} \sum_{t=2001}^{2010} \Delta \log \boldsymbol{p}_{F,t}, \qquad \hat{\boldsymbol{\Sigma}}_F := \frac{1}{9} \sum_{t=2001}^{2010} (\Delta \log \boldsymbol{p}_{F,t} - \hat{\boldsymbol{\delta}}_F)(\Delta \log \boldsymbol{p}_{F,t} - \hat{\boldsymbol{\delta}}_F)^{\top}.$$

Then, we estimate the mean and the covariance matrix of the difference of log-prices in Belgium with respect to France as

$$\hat{\boldsymbol{\delta}}_{BF} := \frac{1}{11} \sum_{t=2000}^{2010} (\log p_{B,t} - \log p_{F,t}),$$

$$\hat{\boldsymbol{\Sigma}}_{BF} := \frac{1}{11} \sum_{t=2000}^{2010} (\log p_{B,t} - \log p_{F,t} - \hat{\boldsymbol{\delta}}_{BF})(\log p_{B,t} - \log p_{F,t} - \hat{\boldsymbol{\delta}}_{BF})^{\top}.$$

The random paths of the prices in the simulation experiments are generated as $p_{l,0} = \boldsymbol{1}$,

$$\log \boldsymbol{p}_{0,t} = \hat{\boldsymbol{\delta}}_F + \log \boldsymbol{p}_{0,t-1} + \boldsymbol{\varepsilon}_t, \qquad \boldsymbol{\varepsilon}_t \sim \mathrm{NID}(0, \hat{\boldsymbol{\Sigma}}_F)$$

$$\log \boldsymbol{p}_{l,t} = \hat{\boldsymbol{\delta}}_{BF} + \log \boldsymbol{p}_{0,t} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim \mathrm{NID}(0, \hat{\boldsymbol{\Sigma}}_{BF}),$$

for $t = 1, \ldots, T$, where $l = 0$ represents the base country (France) and $l = 1, \ldots, 31$ the comparison countries. The power of the two tests is computed as the relative frequency of rejections at a nominal 5% level when applied to 10,000 replications of the *log-real exchange rate* process

$$rer_t := \log\left(\boldsymbol{w}_l^{\top} \boldsymbol{p}_{l,t}\right) - \log\left(\boldsymbol{w}_0^{\top} \boldsymbol{p}_{0,t}\right), \qquad t = 1, \ldots, T$$

where $\boldsymbol{w}_l$ is the CPI weights vector[12] of country l.

We construct the simulation experiment such that the LOP holds for each category n, since the log-prices of the same category across two countries are cointegrated with cointegrating vector $\{1, -1\}$. As explained below, in our analysis the difference in weights between countries plays an important role. We compute it with the mean absolute difference:

$$\mathrm{MAD}(\boldsymbol{w}_l, \boldsymbol{w}_0) := \frac{1}{91} \sum_{n=1}^{91} |w_{n,l} - w_{n,0}|.$$

The ADF and KPSS tests applied to the $\{rer_t\}$ sample paths have the following characteristics: for the ADF test, a constant term and four lags of the

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[11]We annualised the monthly indexes through geometric means. This makes the simulation experiment closer to the typical empirical analysis found in papers dealing with the empirical validation of the PPP; furthermore, seasonality issues are eliminated.

[12]Eurostat uses weights that sum to 1,000. We used the year 2005 weights for the whole simulation experiment.

differentiated dependent variable have been included; for the KPSS statistic the long-run-variance estimation has been implemented using a Bartlett kernel with bandwidth $12(T/100)^{1/4}$. We tried different configurations of the ADF lags and KPSS bandwidths, but the results were very similar to those described below.

In Figure 1 we report the number of rejections of the two tests for a sample size of $T = 100$ yearly observations; the country weights are ordered by the MAD distance with respect to France's weights. As stressed in the previous section,
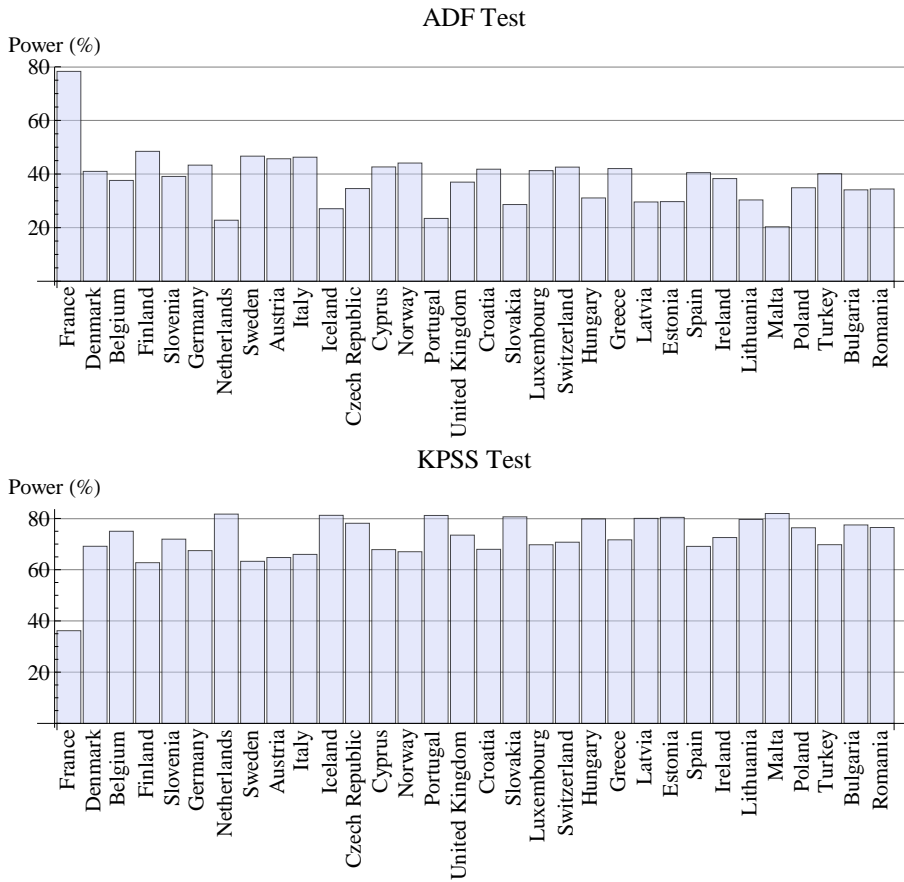

FIGURE 1.— Power (% of rejections) of the ADF and KPSS tests for $T = 100$. Countries are increasingly ordered by the CPI weights distances with respect to France.

the real exchange rate process tends to "look more stationary" when the CPI weights of the base country are equal to those of the comparison country. This is also confirmed by the frequency of rejections of the two tests: highest for the

ADF and lowest for the KPSS in the equal-weights case (French weights/French weights).

Figure 1 suggests two conclusions: a) the frequency of rejections of the ADF (KPSS) test decreases (increases) in the MAD. b) with the exception of the case of equal weights the KPSS performs better than the ADF test.

These conclusions needs to be further qualified as the relationship between the power of the test and the MAD is not clearly monotonic. This is not surprising, as the MAD provides only information on the distance between the two weight vectors, but cannot reveal anything about the relative direction or rotation between the same vectors. In order to isolate the relationship between the distance between two weight vectors and the frequency of rejection, we take French weights for the base country and the convex combination of French and Danish (the closest) weights

$$(3.1) \qquad \boldsymbol{w}_\omega = (1 - \omega)\boldsymbol{w}_F + \omega\boldsymbol{w}_{DK}, \qquad \omega \in [0, 1]$$

for the comparison country. Using this approach, the power of the tests can be evaluated with respect to ω, which can be seen as the amount of contamination between French and Danish weights (i.e., $\omega = 0$ means no contamination, the comparison country has the same weights as France, $\omega = 1$ means that the comparison country has Danish weights). It is straightforward to check that

$$\mathrm{MAD}(\boldsymbol{w}_\omega, \boldsymbol{w}_F) = \omega\mathrm{MAD}(\boldsymbol{w}_{DK}, \boldsymbol{w}_F),$$

so that ω represents the fraction of distance covered in moving from France in the direction of Denmark, as far as average consumption habits are concerned.
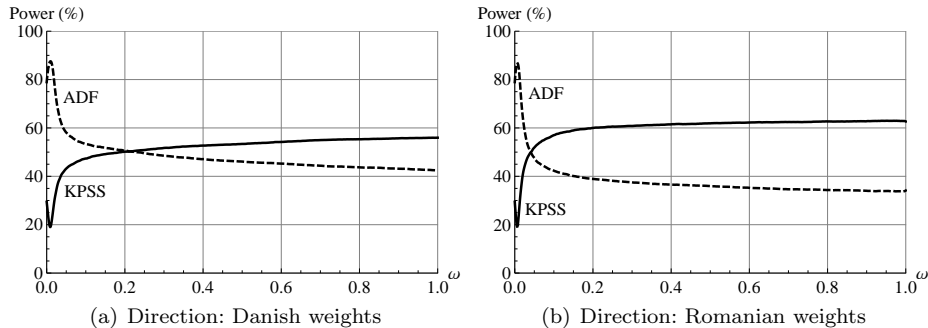

(a) Direction: Danish weights   (b) Direction: Romanian weights

FIGURE 2.— Power (% of rejections) of the ADF and KPSS tests as function of the weights contamination coefficient ω defined in equation (3.1).

Figure 2 represents the powers of the two tests as functions of the value ω for samples of $T = 100$ observations; the same analysis is also depicted using the Romanian weights which, according to the MAD metric, are the most distant

from the French ones. In both cases the ADF (KPSS) reaches its maximum (minimum) power in a neighbourhood of $\omega = 0$ and then decreases (increases) monotonically in ω. The fact that the extremum is not reached exactly at $\omega = 0$, but in a neighbourhood thereof, should not surprise, in fact, as shown in the previous section, if the mean of the inter-country difference of the log-prices of each good is zero, then the condition that makes the data generating process "look stationary" is the identity of the weights in the CPI ratio. However, when this expectation is not zero for all the log-price differences, the weights that achive the "most stationary looking" process depend on these mean values (cf. equation (2.6) and Table II).

Figure 3 depicts the power of the ADF and KPSS tests as function of the sample size T. The left panel shows the equal weights case (French weight/French weights), while the right panel covers the case of maximal MAD distance with respect to France (Romanian weights/French weights). Both tests seem to be consistent against the form of non-integrated non-stationarity implied by the CPI-based log-real exchange rate process, as the proportion of rejections increases with T for both tests in both cases. As expected, the ADF power function

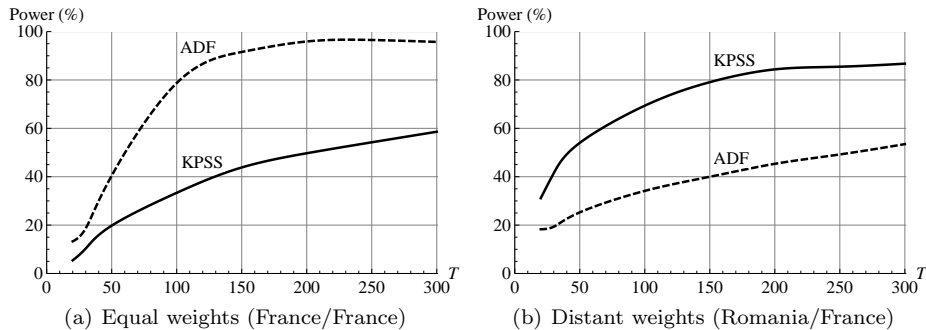(a) Equal weights (France/France)   (b) Distant weights (Romania/France)

FIGURE 3.— Power (% of rejections) of ADF and KPSS tests as function of the sample size T under different weighting schemes of the CPI.

is above the KPSS' and increases more quickly in the equal weights case, while the opposite is true for the distant weights case.

Finally, Figure 4 depicts the distribution (deciles) of the sample coefficient $\hat{\rho}_T$ of an AR(1) model fitted by least squares to the $\{rer_t\}$ sample paths, as a function of the sample size T. Recall that the closer to one is the coefficient, the higher is the persistence of the process and the longer the implied half-life. Again, as expected from the discussion in the previous section, the persistence is much lower in the equal-weights case than in the distant-weights setup. In the first case we have a median AR(1) coefficient that ranges from 0.04 (half-life = 0.21 years) to 0.15 (half-life = 0.37 years), while in the second case the median ranges between 0.48 (half-life = 0.94 years) to 0.86 (half-life = 4.6 years). Notice
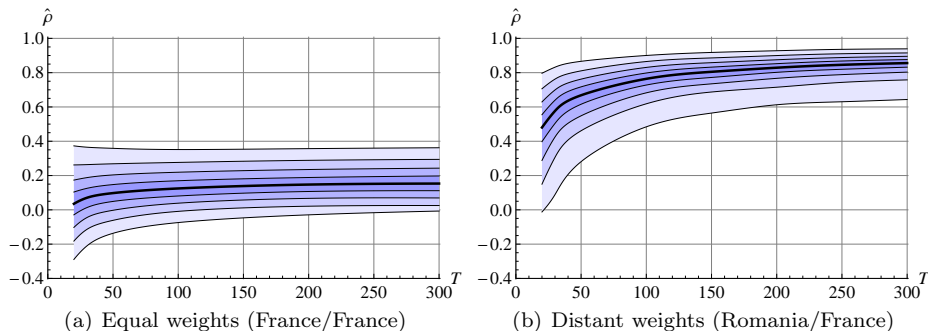
(a) Equal weights (France/France)   (b) Distant weights (Romania/France)

FIGURE 4.— Deciles of the empirical distribution of the AR(1) coefficient estimates $\hat{\rho}$ as function of sample size T.

that these autoregressive coefficients, and so the corresponding half-lives, tend to be smaller than those found in real data as in our data generating process the random walk shocks and the price differences are generated as white noise sequences, while some persistence in both quantities is generally found in real data.

4. CONCLUSIONS

It is well known that there are several reasons that prevent individual prices to respect the LOP, therefore contributing to the explanation of what is known as the PPP puzzle. In this paper we have shown that if the empirical validation of the PPP is based on real exchange rates built as ratio of CPI expressed in a common currency, then it is not possible to find evidence of the PPP even when the LOP holds on individual prices. The reason for this is that finding mean reversion in real exchange rates, does not depend only on individual prices, but also on the formula used to aggregate this information. We have proved by means of counterexamples that the ratio of CPI does not preserve PPP, and we have given a sufficient condition for building price indexes that let the LOP translate into PPP.

We have constructed a simulation experiment that investigated the behaviour of the ADF and KPSS tests when applied to the ratio of CPI under the LOP. Our results show that both tests seem to be consistent against the type of non-integrated non-stationary process that describe the evolution of CPI-based real exchange rates. When the weights of the two CPI are equal, the real exchange rate behaves almost as a stationary process and, in finite samples, the KPSS' power is rather low. On the contrary, when the weighting schemes of the two CPI indexes are different the real exchange rate is closer to an integrated process and the power of the ADF test scarce. In this latter case, which is the most frequent in

real world applications, we find a distribution of the persistence of real exchange rates that is compatible with those generally reported in empirical works.

Given that international price comparisons are based on indexes that do not preserve the PPP, we suggest economists interested in PPP testing to work directly on micro data as in Crucini and Shintani (2008), or to use indexes that comply with the requirements set out in this paper.

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