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Dynamic behavior of real and stock markets with a varying degree of interaction

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Abstract

We develop a macroeconomic behavioral model in order to analyze the interactions between real and financial markets. The real subsystem is represented by a simple Keynesian income-expenditure model, while the financial subsystem is represented by an equilibrium stock market with heterogeneous speculators, i.e., chartists and fundamentalists. The interactions between the two markets are modeled in the following way: the aggregate demand depends, among other variables, also on the stock market price, while the fundamental value used by speculators in their decisional process depends on real economic conditions. In our model we introduce a parameter that represents the degree of interaction. With the aid of analytical and numerical tools we show that an increasing degree of interaction between markets tends to locally stabilize the system. This stabilization occurs via a sequence of period-halving bifurcations. Globally, we find that the stabilization process implies multistability, i.e., the coexistence of different kinds of attractors.

Keywords: Interacting markets; bifurcation; stabilization; complex dynamics; multistability.

JEL classification: D84, E12, E32, G02, G12

1 Introduction

Instabilities are known, both empirically and theoretically, to be features of all markets: the product markets, the labor market, and the financial markets.

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Over the last twenty years, many stock market models have been proposed in order to study the dynamics of financial markets (see Hommes (2013)). According to such models, even in absence of stochastic shocks, the interaction between heterogeneous speculators accounts for the dynamics of financial markets. Those models, when endowed with stochastic shocks, are able to replicate some important phenomena, such as bubbles and crashes, excess volatility and volatility clustering. However, in this kind of models authors have restricted their attention to the representation and the dynamics of financial markets only and the existing feedbacks between the real and financial markets are completely neglected. An exception is represented, for instance, by Charpe et al. (2011), Scheffknecht and Geiger (2011), Lengnick and Wohltmann (2013), and Westerhoff (2012). Charpe et al. (2011) propose an integrated macro model, using a Tobin-like portfolio approach, and consider the interaction of heterogeneous agents in the financial market in order to obtain financial market instability. They find that unorthodox fiscal and monetary policies are able to stabilize unstable macroeconomies. Scheffknecht and Geiger (2011) present a financial market model with leverage-constrained, heterogeneous agents integrated with a New Keynesian standard model; all agents are assumed to be boundedly rational. They show that a systematic reaction by central bank on financial market developments dampens macroeconomic volatility. Lengnick and Wohltmann (2013) propose an agent-based model with financial markets interconnected with a New Keynesian model with bounded rationality and explore the consequences of transaction taxes. The results are endogenous development of business cycles and stock price bubbles. Finally, in Westerhoff (2012) the real economy is described via a Keynesian good market approach, while the set-up for the stock market includes heterogeneous speculators. More precisely, in Westerhoff (2012) the real economy is represented by an income-expenditure model in which expenditures depend also on the dynamics of the stock market price. On the other hand, the financial side is represented by a market where traders choose between two behavioral forecasting rules concerning the future development of the stock price: fundamentalism and chartism. The stock market, in turn, is linked to the good market since the stock market's fundamental value depends on national income. In Westerhoff (2012) the real market subsystem is described by a stable linear relation, while the financial sector is represented by a nonlinear relation, that is, by a cubic functional relation. In that way, the oscillating behavior is generated by the financial subsystem only. In Westerhoff (2012), it is shown that interactions between real economy and the stock market appear to be destabilizing, giving rise to chaotic dynamics through bifurcations.

In our paper we present a model which is inspired to the one in Westerhoff (2012), but which also displays some crucial differences with respect to it. The first difference is that the oscillating behavior is generated by the real subsystem. To be more precise, the nonlinearity of the real subsystem is due to the nonlinearity of the adjustment mechanism of the good market with respect to the excess demand. Another difference with respect to Westerhoff (2012) is the way we represent and analyze the interaction between the two markets. We assume in fact that economic agents base their decisions on a weighted average between an exogenous value and an endogenous value given by the current realization of economic variables, such as stock prices and income. In this way, the parameter describing the weighted average represents also the degree of interaction between the two markets. The extreme values of the weighting parameter correspond to the two cases considered in Westerhoff (2012), i.e., the isolated markets case and the interacting markets scenario. The last main difference with respect to Westerhoff (2012) is given by our assumption that the financial market speed of adjustment tends to infinity, generating a permanent stock market equilibrium. Such assumption is motivated by the fact that the functioning of financial markets is such that the mechanism of adjustment of their prices is much faster than the mechanism of adjustment of good market prices. As a consequence of that equilibrium assumption, in our model national income and stock prices are jointly determined by a one-dimensional nonlinear map. Analytical and numerical tools are used in order to find the mechanisms and the channels through which instabilities are transmitted between markets[1].

The main contribution of this paper to the existing literature is to focus on the role of real and financial feedback mechanisms, not only for the dynamics and stability of a single market, but for those of the economy as a whole. More precisely, our main finding, contrarily to Westerhoff (2012), is about the stabilizing role of an increasing degree of interaction. We believe such difference is due to the fact that, as explained above, in our model we assume that the speed of adjustment in the stock market approaches infinity, which implies that the stock market is always in equilibrium, while in Westerhoff (2012) the stock market may not be in equilibrium and therein a full market interaction decreases the stability parameter set.

The specific results that we obtain can be summarized as follows. We prove in Proposition 3.3 the existence of an absorbing interval attracting all forward orbits, which prevents the system from divergence. Moreover, we show the presence of chaotic dynamics in the sense of Li and Yorke (see Li and Yorke (1975) and Proposition 4.2 below). Finally, with the aid of numerical tools, we show that an increasing degree of interaction between markets tends to locally stabilize the system. This stabilization occurs via a sequence of period-halving bifurcations. Globally, we find that the stabilization process implies multistability, i.e., the coexistence of different kinds of attractors.

The remainder of the paper is organized as follows. In Section 2 we introduce the model. In Section 3 we present analytical and numerical local results for both isolated and interacting markets. In Section 4 we analytically investigate the first flip bifurcation and the existence of Li-Yorke chaos and we numerically show the bifurcations leading from odd-period cycles to a chaotic regime. In Section 5 we present some global scenarios with multistability phenomena. Finally, in Section 6 we draw some

[1] A Keynesian IS-LM model has recently been analyzed through modern dynamical system methods, such as averaging theory, in Guirao et al. (2012).

conclusions and discuss our results.

2 The model

2.1 The real market

Similarly to Westerhoff (2012), we consider a model with a Keynesian good market, interacting with a stock market, in a closed economy without public intervention. It is assumed that private and government expenditures depend on national income and on the performance in the stock market. The dynamic behavior in the real economy is described by an adjustment mechanism depending on the excess demand. If aggregate excess demand is positive (negative), production increases (decreases), that is, income Y_{t+1} in period $t+1$ is defined in the following way

$$Y_{t+1} = Y_t + \gamma g(Z_t - Y_t), \tag{2.1}$$

where g is an increasing function with $g(0) = 0$, Z_t is the aggregate demand in a closed economy, defined as

$$Z_t = C_t + I_t + G_t,$$

where C, I and G stand for consumption, investment and government expenditure, respectively, and $\gamma > 0$ is the real market speed of adjustment between demand and supply.
In order to conduct our analysis, denoting by $E_t = Z_t - Y_t$ the excess demand, we specify g as the following sigmoid function

$$g(E_t) = a_2 \left(\frac{a_1 + a_2}{a_1 e^{-E_t} + a_2} - 1 \right),$$

with a_1, a_2 positive parameters. With such a choice, g is increasing and $g(0) = 0$. Moreover, it is bounded from below by $-a_2$ and from above by a_1. This prevents the real market from diverging and thus creates a real oscillator. In fact, the presence of the two asymptotes does not allow too large variations in income. We stress that this particular analytical specification does not compromise the generality of the achievements. In fact, we found analogous results for other sigmoid functions passing through the origin.
As commonly assumed, we suppose that private and government expenditures increase with national income. Moreover, we assume that households, firms and government financial situation depends on the belief about the stock price performance $\widehat{P}_t$, defined as

$$\widehat{P}_t = (1 - \omega)\widetilde{P} + \omega P_t,$$

with $\omega \in [0,1]$, $\widetilde{P}$ the long-period fundamental value and P_t the current stock price. $\widehat{P}_t$ may be interpreted as a weighted average between the long-period fundamental value and the current stock price. In particular, when $\omega = 0$ the belief about the stock price performance is completely exogenous and coincides with the long-period fundamental value; this is the case of isolated markets considered in Westerhoff (2012). When instead $\omega = 1$ the belief about the stock price performance is completely endogenous and coincides with the current stock price; this is the case of interacting markets in Westerhoff (2012). On the basis of these considerations, we can write the relation between private and government expenditures and national income and stock price as

$$Z_t = C_t + I_t + G_t = a + bY_t + c\widehat{P}_t = a + bY_t + c[(1-\omega)\widetilde{P} + \omega P_t], \qquad (2.2)$$

where $a > 0$ defines autonomous expenditure, $b \in [0,1]$ is the marginal propensity to consume and invest from current income and $c \in [0,1]$ is the marginal propensity to consume and invest from current stock market wealth belief.
Inserting Z_t from (2.2) into (2.1), we obtain the dynamic equation of the real market

$$Y_{t+1} = Y_t + \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(a+bY_t+c[(1-\omega)\widetilde{P}+\omega P_t]-Y_t)} + a_2} - 1 \right). \qquad (2.3)$$

Notice that, with this formulation, the parameter ω may be interpreted as the degree of interaction between the real and financial variables.

2.2 The stock market

With respect to the stock market, we consider the trading behavior of two types of speculators: chartists and fundamentalists. The market maker determines excess demand and adjusts the stock price for the next period. Chartists may be either optimistic or pessimistic, depending on stock price performance: in a bull market chartists buy stocks, while in a bear market they sell stocks. Fundamentalists have an opposite behavior: believing that stock prices will return to their fundamental value, they buy stocks in undervalued markets and sell stocks in overvalued markets. The market maker behavior is described by a linear price adjustment mechanism:

$$P_{t+1} = P_t + \sigma(D_t^C + D_t^F), \qquad (2.4)$$

where $\sigma > 0$ is the market maker price adjustment parameter, and D_t^C and D_t^F are the speculative demands of chartists and fundamentalists, respectively. According to (2.4), the market maker increases (decreases) the stock price if excess demand $D_t^C + D_t^F$ is positive (negative).
Chartists' demand is given by

$$D_t^C = e(P_t - F_t^C), \qquad (2.5)$$

where $e > 0$ is the chartists' reactivity parameter and F_t^C is the fundamental value perceived by chartists. Similarly, the fundamentalists' demand behavior is formalized by

$$D_t^F = f(F_t^F - P_t), \tag{2.6}$$

where $f > 0$ is the fundamentalists' reactivity parameter and F_t^F is the perceived fundamental value by fundamentalists.

For simplicity, according to Westerhoff (2012), we assume a direct relationship between the national income and the perceived fundamental stock market values, both for chartists and fundamentalists. In particular, we suppose that speculators perceive the following relation between the fundamental value and a proxy of the national income $\widehat{Y}_t$

$$F_t^i = d_i \widehat{Y}_t,\ i \in \{C, F\}, \tag{2.7}$$

where $d_i > 0$, $i \in \{C, F\}$, are the parameters capturing the above described direct relationship. For simplicity, we assume that the relationship between the fundamental value and the national income perceived by fundamentalists and chartists is the same, that is, $d_C = d_F = d$, for some $d > 0$.

We also assume that speculators use as a proxy for the expected future levels of national income the weighted average of an exogenous national income value $\widetilde{Y}$ and the current national income Y_t, that is,

$$\widehat{Y}_t = (1 - \omega)\widetilde{Y} + \omega Y_t, \tag{2.8}$$

with $\omega \in [0, 1]$. In particular, when $\omega = 0$ the belief about the national income is completely exogenous and coincides with an exogenous national income value; this is the case of isolated markets considered in Westerhoff (2012). When instead $\omega = 1$ the belief about the national income is completely endogenous and coincides with the current national income; this is the case of interacting markets in Westerhoff (2012).

Inserting $\widehat{Y}_t$ from (2.8) into (2.7), we obtain an expression for the perceived fundamental value, which can then be inserted in the demand functions in (2.5) and (2.6). Inserting in turn such demands into (2.4), we finally get the dynamic equation of the stock market

$$P_{t+1} = P_t + \sigma \left[e\big(P_t - d[(1 - \omega)\widetilde{Y} + \omega Y_t]\big) + f\big(d[(1 - \omega)\widetilde{Y} + \omega Y_t] - P_t\big) \right]. \tag{2.9}$$

Since the functioning of financial markets is such that the mechanism of adjustment of their prices is much faster than the mechanism of adjustment of good market prices, we assume that the speed of adjustment in the stock market limits to infinity. In such way, we obtain the equilibrium condition in that market, i.e., rewriting (2.9) as

$$\frac{P_{t+1}-P_t}{\sigma} = e\big(P_t - d[(1-\omega)\widetilde{Y} + \omega Y_t]\big) + f\big(d[(1-\omega)\widetilde{Y} + \omega Y_t] - P_t\big),$$

when $\sigma \to \infty$, we obtain

$$e\big(P_t - d[(1-\omega)\widetilde{Y} + \omega Y_t]\big) + f\big(d[(1-\omega)\widetilde{Y} + \omega Y_t] - P_t\big) = 0. \tag{2.10}$$

This implies that at time t, given Y_t, P_t assumes a value such that the chartists' and fundamentalists' demand functions determine an excess demand equal to zero.
Indeed, from (2.10) we get the following explicit formulation for P_t

$$P_t = d[(1-\omega)\widetilde{Y} + \omega Y_t], \tag{2.11}$$

in which the current stock price is proportional to weighted average between the belief about long-period national income and its current value. Inserting P_t from (2.11) into (2.3), we get

$$Y_{t+1} = Y_t + \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-\left(a + bY_t + c\left[(1-\omega)\widetilde{P} + \omega\left(d[(1-\omega)\widetilde{Y} + \omega Y_t]\right)\right] - Y_t\right)} + a_2} - 1 \right), \tag{2.12}$$

which is the integrated equation we are going to study in the next sections.
We stress that the case with isolated markets (that is, when $\omega = 0$) has been considered, for a slightly different model, in full details in the companion paper Naimzada and Pireddu (2013). In what follows, we will recall just some of the main results in Naimzada and Pireddu (2013), addressing the interested reader to that paper for a more complete analysis of the isolated market scenario.

3 Local analysis

In this section we discuss the existence of the steady state and we analyze its local stability.
In view of the subsequent analysis, it is expedient to introduce the map $F : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$F(Y) = Y + \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-\left(a + bY + c\left[(1-\omega)\widetilde{P} + \omega\left(d[(1-\omega)\widetilde{Y} + \omega Y]\right)\right] - Y\right)} + a_2} - 1 \right), \tag{3.1}$$

associated to the dynamic equation in (2.12).

In the following result we show the existence of a unique steady state.

Proposition 3.1 *The dynamical system generated by the map F in* (3.1) *has the unique steady state*

$$Y^*(\omega) = \frac{a + c[(1-\omega)\widetilde{P} + d\omega(1-\omega)\widetilde{Y}]}{1 - b - cd\omega^2}. \tag{3.2}$$

The corresponding steady state stock price is given by

$$P^*(\omega) = d[(1-\omega)\widetilde{Y} + \omega Y^*] = d\Big[(1-\omega)\widetilde{Y} + \omega \frac{a + c[(1-\omega)\widetilde{P} + d\omega(1-\omega)\widetilde{Y}]}{1 - b - cd\omega^2}\Big].$$

Both $Y^*(\omega)$ and $P^*(\omega)$ are positive if[2] $\omega < \sqrt{\frac{1-b}{cd}}$.

Proof. The expression for $Y^*(\omega)$ can be immediately found by solving the fixed point equation $F(Y) = Y$. The formulation of $P^*(\omega)$ follows then by inserting $Y^*(\omega)$ into (2.11). □

As suggested by the notation introduced in the proposition above, the steady state vector $(Y^*(\omega), P^*(\omega))$ depends, among others, on the parameter ω. We stress that when $\omega = 0$ it coincides with the isolated market steady state vector in Westerhoff (2012), while for $\omega = 1$ we find the same interacting market steady state vector in Westerhoff (2012). Moreover, notice that, when $\omega = 0$, we find $Y^*(0) = \frac{a+c\widetilde{P}}{1-b}$, where we recognize in $\frac{1}{1-b}$ the Keynesian multiplier and in $a + c\widetilde{P}$ the autonomous aggregate expenditures (see Ferguson and Lim (2003)).
As stated in the next result, the precise relationship between the steady state value for Y^* in the two extreme cases $\omega = 0$ and $\omega = 1$ crucially depends on the value of the autonomous component of expenditures $\widetilde{P}$. Indeed we have the following:

Corollary 3.1 *It holds that $Y^*(0) > Y^*(1)$ if and only if $\widetilde{P} > \frac{ad}{1-b-cd}$.*

A final remark about Proposition 3.1 concerns the positivity of $Y^*(\omega)$. In fact, since in our subsequent analysis on the role of the degree of interaction between the markets we consider ω varying in $[0, 1]$, we need the threshold on ω for the positivity of $Y^*(\omega)$ to be above 1, i.e., $\sqrt{\frac{1-b}{cd}} > 1$,[3] condition which is fulfilled when c and d are relatively small.

Proposition 3.2 *Setting $\overline{\gamma} = \frac{2(a_1+a_2)}{(1-b)a_1a_2}$ and $\overline{\overline{\gamma}} = \frac{2(a_1+a_2)}{(1-b-cd)a_1a_2}$, the steady state Y^* in* (3.2) *is stable in the following cases:*

- *for every $\omega \in [0, 1]$, if $\gamma < \overline{\gamma}$;*
- *for $\omega \in [R, 1]$, where $R = \sqrt{\frac{1}{cd}\left(1 - b - \frac{2(a_1+a_2)}{\gamma a_1 a_2}\right)}$, if $\overline{\gamma} < \gamma < \overline{\overline{\gamma}}$.*

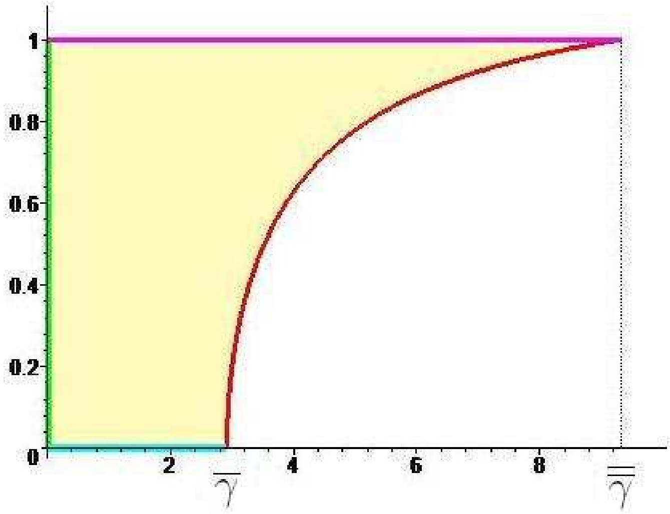

Figure 1: We represent in yellow the stability region in the (γ, ω)-plane, together with the two threshold values $\overline{\gamma}$ and $\overline{\overline{\gamma}}$ on the γ-axis.

Proof. The result follows by solving the chain of inequalities $-1 < F'(Y^*) < 1$ with respect to γ. In fact, it is immediate to find that $F'(Y^*) = 1 - \frac{\gamma a_1 a_2 (1-b-cd\omega^2)}{a_1+a_2}$ and thus, in order to get the stability of the steady state, we only need to impose $F'(Y^*) > -1$, taking into account that $\omega \in [0, 1]$. □

The stability region in the (γ, ω)-plane is represented in Figure 1. Notice in particular that Y^* is never stable if $\gamma > \overline{\overline{\gamma}}$ and that, when $\omega = 0$, Y^* is stable for $\gamma < \overline{\gamma}$. We also stress that the only parameters that do not influence the stability of Y^*, but just its position, are $\widetilde{Y}$, $\widetilde{P}$ and a, i.e., autonomous expenditure.
Proposition 3.2 represents our main achievement as it states that, under the condition that γ is not too large, an increasing degree of interaction between real and financial markets has a stabilizing effect, as in the case of Figure 2. This is probably due to the fact that, in our formalization, we assume that the stock market is an equilibrium market. A future research shall concern the case in which both real and financial markets may not be always in equilibrium.

We now describe some further dynamical features for the map F in the next result. More precisely, we show the existence of an absorbing interval and thus, differently from the linear case, in which local instability implies diverging trajectories (see Ferguson and Lim (2003)), in our framework local instability may imply periodic

[2] In the following analysis we will implicitly assume $\omega < \sqrt{\frac{1-b}{cd}}$, so that the objects we consider make economic sense.

[3] Notice that the same condition implies that, in Proposition 3.2, $\overline{\overline{\gamma}} > \overline{\gamma} > 0$, as desired.

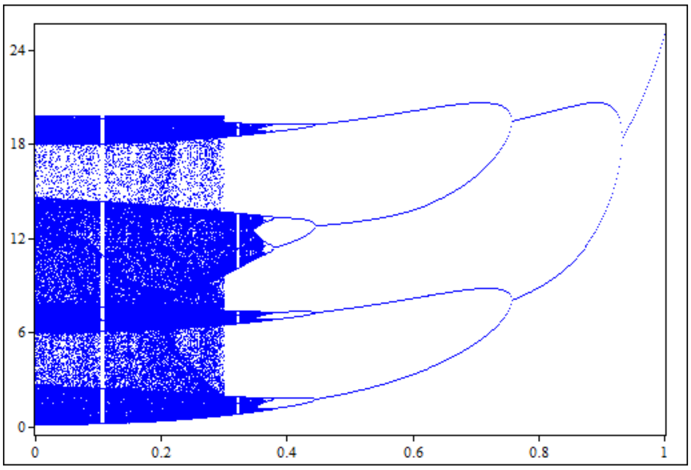

Figure 2: The bifurcation diagram w.r.t. ω for the map F with $a = 5$, $b = 0.2$, $c = 1$, $d = 0.6$, $a_1 = 3$, $a_2 = 2, \gamma = 6, \widetilde{Y} = \widetilde{P} = 1$. Since γ is not too large, an increasing degree of interaction between real and financial markets has a stabilizing effect.

and chaotic orbits.

Proposition 3.3 *If the map F in (3.1) is increasing, then the generated dynamical system is globally stable. Else*[4]*, call m and M the unique positive local minimum point and local maximum point of the map F, respectively, and set $m' := F(m)$ and $M' := F(M)$. Then the compact interval $I = [m', M']$ is "globally absorbing", i.e., for all $\bar{x} \in \mathbb{R}_+$ there exists $\bar{n} \in \mathbb{N}$ such that $F^{\bar{n}}(\bar{x}) \in I$ and for any $x \in I$, $F^n(x) \in I$, for all $n \in \mathbb{N}$.*

Proof. Let us assume at first that F is increasing and show that, given a generic starting point $\bar{x}$ in $\mathbb{R}_+$, its forward trajectory will tend to Y^*. Since $F(0) > 0$ and Y^* is the unique fixed point of F, then by continuity, $F(x) > x$, for every $x < Y^*$, and $F(x) < x$, for every $x > Y^*$. Hence, if $0 \le \bar{x} < Y^*$, then $F^n(\bar{x})$ will tend increasingly

[4]It is possible to show that no other scenarios may arise for the map F. In fact

$$F(0) = \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(a + c[(1-\omega)\widetilde{P} + \omega d(1-\omega)\widetilde{Y}])} + a_2} - 1 \right) > \gamma a_2 \left(\frac{a_1 + a_2}{a_1 + a_2} - 1 \right) = 0$$

and

$$F'(0) = 1 - \frac{\gamma a_1 a_2 (a_1 + a_2) e^{-(a + c[(1-\omega)\widetilde{P} + \omega d(1-\omega)\widetilde{Y}])}(1 - b - c\omega^2 d)}{(a_1 e^{-(a + c[(1-\omega)\widetilde{P} + \omega d(1-\omega)\widetilde{Y}])} + a_2)^2},$$

which is positive for any $a + c[(1-\omega)\widetilde{P} + \omega d(1-\omega)\widetilde{Y}]$ large enough. In particular, in all the pictures in the present paper, we have $a + c[(1-\omega)\widetilde{P} + \omega d(1-\omega)\widetilde{Y}] > a = 5$. Thus, in the cases considered, F is positive and locally increasing in a right neighborhood of 0. If it is not globally increasing, F has a local maximum point, followed by the steady state and then by a local minimum point, and after that F grows monotonically to infinity.

towards Y^* as $n \to \infty$, while if $\bar{x} > Y^*$, then $F^n(\bar{x})$ will tend decreasingly towards Y^* as $n \to \infty$.
If the map F is not increasing, let us consider a generic starting point $\bar{x}$ in $\mathbb{R}_+$ and show that its trajectory will eventually remain in I. If $\bar{x} \in I$, then by construction its forward orbit will be trapped inside I, as well. Let us now proceed with the two remaining cases, i.e., $\bar{x} < m'$ and $\bar{x} > M'$. Since $Y^* \in I$ and by continuity $F(x) > x$, for every $x < Y^*$, and $F(x) < x$, for every $x > Y^*$, if $\bar{x} < m'$, then its iterates will approach I in a strictly increasing way, while if $\bar{x} > M'$, then its iterates will approach I in a strictly decreasing way. Once that a forward iterate of $\bar{x}$ lies in I, then by construction all its subsequent iterates will be trapped inside I, as well. This concludes the proof. □

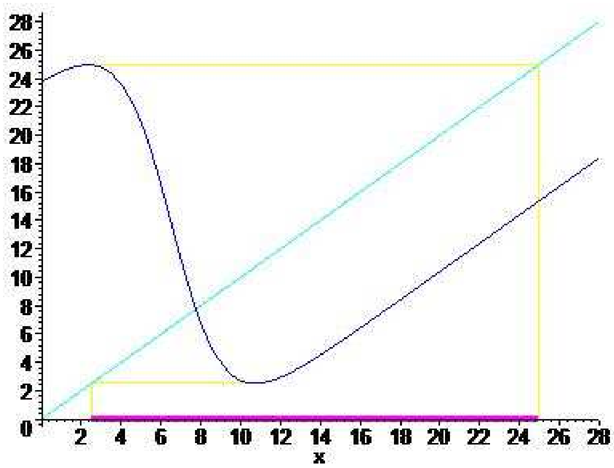

Figure 3: The interval highlighted on the x-axis is the absorbing interval I from Proposition 3.3. Notice that, in the case depicted, $F'(Y^*) < -1$ and thus the steady state is locally unstable.

As we shall see in what follows, in the absorbing interval many different dynamic behaviors may arise. We depicted in Figure 3 the absorbing interval in the case the map F is not increasing, and in fact in that graph it holds $F'(Y^*) < -1$. Notice that, in Proposition 3.3, the first scenario (i.e., F strictly increasing) may be seen as a limit case of the second framework, in which the absorbing set collapses into a unique point, that is, the steady state.

4 Bifurcations and chaotic dynamics

In the present section we will show that the stabilization of the system with an increasing degree of interaction is achieved through reversed period-doubling bifurcations, i.e., period-halving bifurcations, and that the emergence of complex dynamics when the degree of interaction is sufficiently small.

We start by proving that the map F undergoes a flip-bifurcation at the unique steady state Y^*.

Proposition 4.1 *For the map F in* (3.1)*, a flip bifurcation occurs around $Y = Y^*$ when $\frac{\gamma a_1 a_2 (1-b-cd\omega^2)}{a_1+a_2} = 2$, that is, for $\omega = R = \sqrt{\frac{1}{cd}\left(1 - b - \frac{2(a_1+a_2)}{\gamma a_1 a_2}\right)}$.*

Proof. According to the proof of Proposition 3.2, the steady state $Y = Y^*$ is stable when $F'(Y^*) > -1$. Then, the map F satisfies the canonical conditions required for a flip bifurcation (see Hale and Koçak (1991)) and the desired conclusion follows. Indeed, when $F'(Y^*) = -1$, i.e., for $\omega = R$, then Y^* is a non-hyperbolic fixed point; when $\omega > R$ it is attracting and finally, when $\omega < R$, it is repelling. □

In Proposition 4.2 we now show the existence of chaos in the sense of Li-Yorke, as described in conditions $(T1)$ and $(T2)$ in Theorem 1 in Li and Yorke (1975) (from now on, Th1 LY). In fact, in the proof of Proposition 4.2 we will use that well known result. We recall that if the map F in Th1 LY has a period-three orbit, then the hypotheses are satisfied and that result applies. Moreover, as observed in Li and Yorke (1975), Th1 LY can be generalized to the case in which $F : J \to \mathbb{R}$ is a continuous function that does not map the interval J onto itself.

Proposition 4.2 *Let F be the map in* (3.1)*. Fix $a = 5$, $b = 0.2$, $c = 1, d = 0.55$, $a_1 = 3$, $a_2 = 1.2, \gamma = 8$, $\widetilde{Y} = \widetilde{P} = 1$, $\omega = 0.3$, and set $J = [17.898, 23.230]$. Then for any point $x \in J$ it holds that $y = F(x)$, $z = F^2(x)$ and $w = F^3(x)$ satisfy $w \geq x > y > z$ and thus Conditions $(T1)$ and $(T2)$ in Th1 LY do hold true. In particular[5], for any $x \in \mathrm{int}(J)$ it holds that $F^3(x) > x$, while for $x \in \partial(J)$ it holds that $F^3(x) = x$, that is, the extreme points of J have period three.*

Proof of Proposition 4.2: We show that the chain of inequalities $w \geq x > y > z$ is satisfied on J by plotting in Figure 4 the graphs of the identity map in blue, of F in red, of F^2 in green and of F^3 in cyan. A direct inspection of that picture shows that it is possible to apply Th1 LY on J and thus we immediately get the desired conclusions. □

Notice that in the statement of Proposition 4.2 we have fixed some particular parameter values. However, we stress that the result is robust, as the same conclusions hold for several different sets of parameter values, as well. Moreover, once that a result analogous to Proposition 4.2 is proven for a certain parameter configuration, by continuity, the same conclusions still hold, suitably modifying the interval J, also for small variations in those parameters. Hence, Proposition 4.2 actually allows to infer the existence of Li-Yorke chaos for the map F when ω lies in a neighborhood of 0.3 and for some suitable values of the other parameters.

[5]Given an interval I, we denote its interior by $\mathrm{int}(I)$ and its boundary by $\partial(I)$.

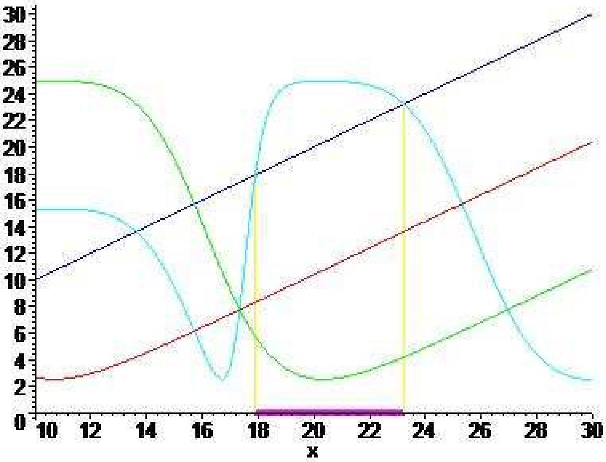

Figure 4: The graph of the identity map (in blue) and of the first three iterates of the map F (in red, green and cyan, respectively) for $a = 5$, $b = 0.2$, $c = 1$, $d = 0.55$, $a_1 = 3$, $a_2 = 1.2$, $\gamma = 8$, $\widetilde{Y} = \widetilde{P} = 1$ and $\omega = 0.3$. The interval highlighted on the $x-$axis is J from Proposition 4.2.

We draw in Figure 5 the bifurcation diagram with respect to ω for the map F for the same parameter values in the statement of Proposition 4.2 (except for ω now varying on $[0, 1]$), in order to show that when $\omega = 0.3$ the computer simulations confirm the presence of chaos, theoretically proven in Proposition 4.2.
Notice also that, differently from the more common period-doubling scenario, in Figure 5 the route to chaos occurs via a destabilization of an odd-period orbit, in particular of an orbit of period 7. In Figure 6 we show a similar framework, where however the destabilization occurs for a period-three orbit.

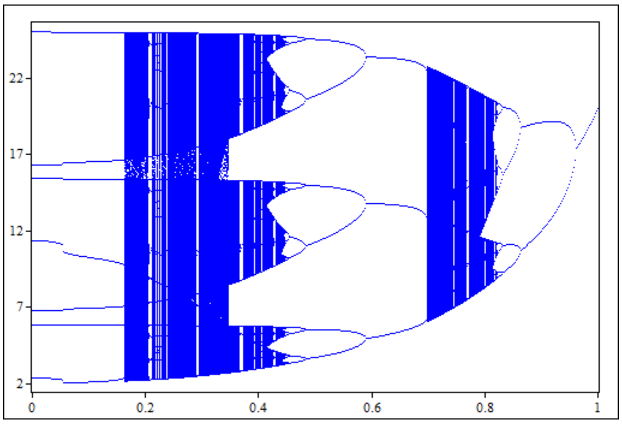

Figure 5: The bifurcation diagram w.r.t. ω for the map F with $a = 5$, $b = 0.2$, $c = 1$, $d = 0.55$, $a_1 = 3$, $a_2 = 1.2, \gamma = 8$, $\widetilde{Y} = \widetilde{P} = 1$.

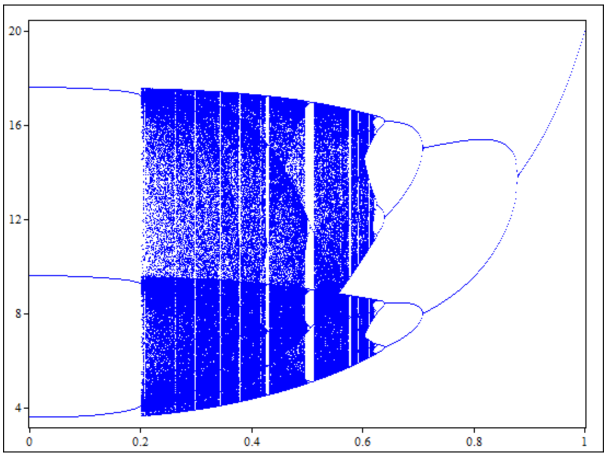

Figure 6: The bifurcation diagram w.r.t. ω for the map F with $a = 5$, $b = 0.2$, $c = 1$, $d = 0.55$, $a_1 = 4$, $a_2 = 2, \gamma = 4$, $\widetilde{Y} = \widetilde{P} = 1$.

5 Global analysis

In the previous sections, we presented some theoretical results about the dynamics of the real and financial interacting markets. We saw that different types of dynamics can occur, such as the existence of absorbing intervals, stable steady states, periodic cycles and chaotic behavior. In this section we investigate, using bifurcation diagrams, the global behavior of the system as the parameter ω increases.

In particular, in Figure 7 we illustrate a numerical example in which, from a chaotic attractor to a stable steady state, a multistability scenario occurs. Indeed we show that there exists a sufficiently large ω for which we have the coexistence of a stable eight-orbit with different kind of attractors, i.e., chaotic or periodic attractors.

In Figure 8 we present a magnification of Figure 7, in which we better show that

coexistence phenomenon.
We stress that such coexisting phenomena are not present with isolated markets.

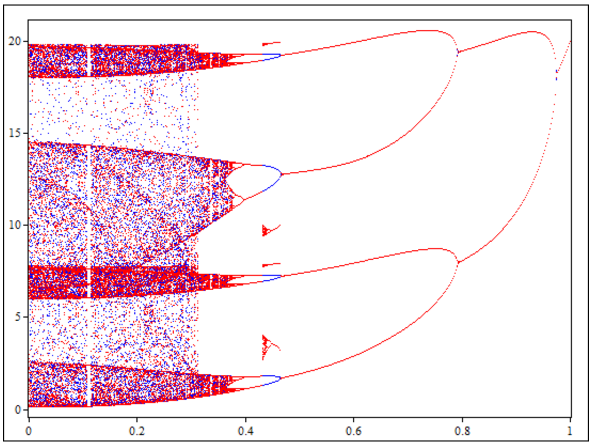

Figure 7: The bifurcation diagram w.r.t. ω for the map F with $a = 5$, $b = 0.2$, $c = 1$, $d = 0.55$, $a_1 = 3$, $a_2 = 2, \gamma = 6, \widetilde{Y} = \widetilde{P} = 1$, which highlights a multistability phenomenon characterized by the coexistence of chaotic and periodic attractors with a period-eight orbit.

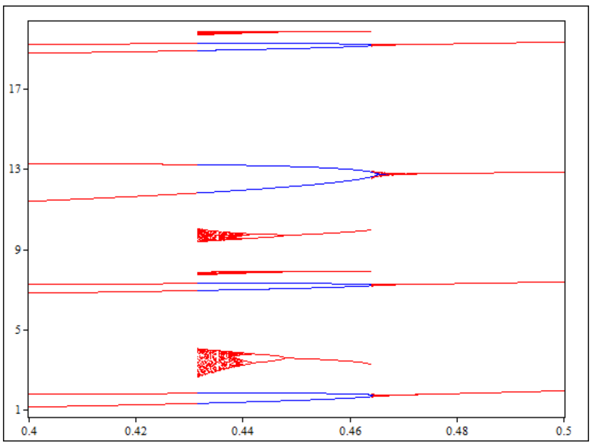

Figure 8: A magnification of Figure 7 for $\omega \in [0.4, 0.5]$, in order to better show the coexistence of chaotic and periodic attractors with a period-eight orbit.

6 Conclusions

In our paper we presented a model with real and financial interacting markets, where the oscillating behavior is generated by the real subsystem. The nonlinearity of the real market comes from the nonlinearity of the adjustment mechanism with respect to excess demand. The interaction between the two markets is described by a parameter representing the degree of interaction.
In this framework we proved, according to the parameter values, the presence of global stability or the existence of an absorbing interval attracting all forward orbits. In such way, all diverging to infinity behaviors are ruled out. The main result of the paper, differently from Westerhoff (2012), concerns the stabilizing role of an increasing degree of interaction between the two markets. Moreover, we numerically (and analytically just for the last step, that is, from a period-two cycle to the fixed point) showed that the passage from complicated dynamics to a stable steady state is due to a sequence of period-halving bifurcations. The existence of chaos in the sense of Li-Yorke has been proved as well. Finally we numerically showed the coexistence of an eight-cycle and different kinds of attractors.
We stress that some preliminary simulations of a model with interacting real and financial oscillators seem to suggest very different dynamic behaviors with respect to the ones observed in the present paper.

Here we proposed a simple model, in order to improve our knowledge about the role of interactions between real and financial markets. Of course, it may be extended in various directions, for instance, relaxing the equilibrium condition in the stock market, dealing with nonlinearities in both markets, or introducing heterogeneity in the fundamental values, in order to analyze its stabilizing/destabilizing effect.

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