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CONGLOMERABILITY AND REPRESENTATIONS

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Abstract. We prove results concerning the representation of a given distribution by means of a given random quantity. The existence of a solution to this problem is related to the notion of conglomerability, originally introduced by Dubins. We show that this property has many interesting applications in probability as well as in analysis. Based on it we prove versions of the extremal representation theorem of Choquet and of Skhorohod theorem.

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1. Introduction

Let S and Ω be non empty sets, $\mathscr{H}$ a family of real valued functions on S and X a map from Ω to S. In this paper we consider several problems involving the equation

$$\int h dm = \int h(X) d\mu \qquad h \in \mathscr{H} \tag{1}$$

in which m is a given positive, real valued, finitely additive set function defined on some ring of subsets of S. When (1) is solved by a positive, finitely additive set function μ defined on a ring of subsets of Ω, we speak of m as the *distribution* of X and of X as the *representation* of m. These properties should be interpreted as defined relatively to a given family $\mathscr{H}$ which is at the same time our model of the information available and a constraint to the problem examined. In the general case addressed in this work we will avoid assuming that S is a metric or a topological space, or that X is measurable in some appropriate sense.

A classical problem fitting into (1) is the one faced by a statistician who, based on the outcome of an experiment in the form of a distribution m on S, has to conclude whether the observed data originate from some a priori model X or not. In similar situations a statistician may perhaps consider restrictions to the representing measure, such as μ being countably additivite or absolutely continuous with respect to some reference measure. The traditional representation problem of Skhorohod is another variant of the one discussed here in which μ is given and X is the unknown. Another question related, in a less obvious way, to (1) is the problem raised long ago by Lester Dubins [14] of whether a finitely additive set function m on a product space is strategic, a special form of the disintegration property. The answer to Dubins' problem depends on a special condition, conglomerability (apparently due to de Finetti) that received little attention in probability since Dubins'paper, with the notable exception of the work of Schervisch, Seidenfeld and Kadane [25].

The conglomerability property, we believe, may be formulated in more general terms than those in which it was originally stated and it may be applied to more ambitious problems in probability and analysis than those for which it had been originally devised. The first problem in which conglomerability turns out to be a crucial property is that of representing linear functionals as integrals whenever the underlying space is just an arbitrary vector space possessing no additional property. In this generality a direct integral representation is hardly possible – if meaningful at all. There is however the possibility that a linear operator T map the original vector space into an auxiliary function space on which such a representation obtains. In fact we prove in Theorem 1 that this is the case if and only if the given linear functional is T conglomerative. This simple result admits though a large number of implications of which the existence of a solution to (1) is just a case in point.

Conglomerability may be nicely restated as a geometric property characterizing two sets of linear functionals on a vector space. As such, it covers several situations of interest in analysis: a compact set is conglomerative with respect to the set of its extreme points; in a Banach space with the Radon Nikodým property a closed bounded set is conglomerative relatively to its strongly exposed points.

Theorem 2, the most important result in this paper, proves that if Φ and Ψ are two sets of positive functionals on a vector lattice then Φ is Ψ conglomerative if and only if each $\phi \in \Phi$ is the barycenter of a (countably additive) measure supported by Ψ. By exploiting the possibility of representing evaluations as positive linear functionals, this conclusion is then extended in Theorem 3 to obtain a generalization of the original result of Choquet [10].

We also provide applications to probability. In (1) with S separable we use conglomerability to explicitly construct one function X that represents *any* distribution relatively to the family $\mathscr{H}$ of continuous functions on S. This in turn implies that if a given classical probability space supports a random quantity uniformly distributed on the unit interval then every countably additive distribution m admits a representation X supported by that same space, a situation related to the problem of Skhorohod. In addition, we prove that there are stochastic processes, such as Brownian motion, which can assume whatever family of finite dimensional distributions on $\mathbb{R}$ upon an appropriate choice of the underlying probability.

In the closing section we apply our approach to prove that any convex functions on $\mathbb{R}$ decomposes into the sum of a piecewise linear component and an integral part, a representation curiously near to the one popular in mathematical finance as a model for option prices.

All proofs are quite simple and, despite the focus on countable additivity, they are obtained by exploiting the theory of the finitely additive integral in which the measurability constraint is much less burdensome. We hope to disprove thus, at least partially, the harsh judgment of Bourgin [7, p. 173] that *"an integral representation theory based on finitely additive measures is virtually useless"*.

2. Notation

Throughout the paper Ω and S are arbitrary, non empty sets and $\mathscr{A}$ and Σ rings of subsets of Ω and S, respectively. The symbol $\mathfrak{F}(\Omega, S)$ (with $\mathfrak{F}(\Omega, \mathbb{R}) = \mathfrak{F}(\Omega)$) is used to denote the family of all functions mapping Ω into S and $\mathfrak{F}$ is replaced with $\mathfrak{L}$, $\mathscr{C}$ or $\mathscr{C}_K$ when the functions considered are linear, continuous or continuous with compact support, respectively. If $f \in \mathfrak{F}(\Omega, S)$ and $A \subset \Omega$ the symbols $f|A$ and $f[A]$ designate the restriction of f to A and the image of A under f. More generally if $\mathscr{H} \subset \mathfrak{F}(\Omega)$ and $A \subset \Omega$ we denote the set $\{h(\omega) : h \in \mathscr{H}, \omega \in A\}$ as $\mathscr{H}[A]$. A sublattice $\mathscr{H}$ of some function space $\mathfrak{F}(S)$ is Stonean (or has the Stone property), if $h \in \mathscr{H}$ implies $h \wedge 1 \in \mathscr{H}$, where $1 \in \mathfrak{F}(S)$ indicates the function assigning the value 1 to all $s \in S$.

$\mathscr{S}(\mathscr{A})$ and $\mathfrak{B}(\mathscr{A})$ indicate the families of $\mathscr{A}$ simple functions on Ω and its closure in the topology of uniform convergence. By $fa(\mathscr{A})$, $ba(\mathscr{A})$ and $ca(\mathscr{A})$ we designate the spaces of real valued, finitely additive set functions on $\mathscr{A}$, the subspace of $fa(\mathscr{A})$ consisting of elements which are bounded relatively to the variation norm and the subspace of $ba(\mathscr{A})$ consisting of countably additive set functions, respectively. $fa(\Omega)$ is preferred to $fa(2^{\Omega})$. A pair $(\mathscr{A}, \lambda)$ with $\mathscr{A}$ a ring of subsets of Ω and $f \in fa(\mathscr{A})_+$ is called a measurable structure on Ω. We refer to $(\Omega, \mathscr{A}, P)$ as a classical probability space when $\mathscr{A}$ is a σ algebra of subsets of Ω and P is countably additive on $\mathscr{A}$ and we say that it supports $X \in \mathfrak{F}(\Omega, S)$ whenever S is a topological space and $X^{-1}(B) \in \mathscr{A}$ for every $B \subset S$ open.

We recall a few definitions and facts relative to the finitely additive integral, the main references being [15] and [4]. If $\lambda \in fa(\mathscr{A})_+$ then we say that $X \in \mathfrak{F}(\Omega)$ is λ-measurable if and only if there exists a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ such that X_n λ-converges to X, i.e. such that

$$\lim_n \lambda^*(|X_n - X| > c) = 0 \quad \text{for every} \quad c > 0 \tag{2}$$

where the set function λ^* and its conjugate λ_* are defined (with the convention $\inf \varnothing = \infty$) as

$$\lambda^*(E) = \inf_{\{A\in\mathscr{A}:E\subset A\}} \lambda(A) \quad \text{and} \quad \lambda_*(E) = \sup_{\{B\in\mathscr{A}:B\subset E\}} \lambda(B) \qquad E \subset \Omega. \tag{3}$$

We say that X is λ-integrable (or that $X \in L^1(\lambda)$) if there is a sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ that λ-converges to X and such that $\langle X_n \rangle_{n\in\mathbb{N}}$ is Cauchy in $L^1(\lambda)$. The integral of X with respect to λ is denoted by $\int X d\lambda$ but at times as $\int_\Omega X(\omega)d\lambda(\omega)$ when reference to the underlying space is important. We shall use the following fact: if $A, B \subset \Omega$, $\lambda \in fa(\mathscr{A})_+$ and $f \in L^1(\lambda)$, then

$$\mathbb{1}_A \le f \le \mathbb{1}_B \quad \text{implies} \quad \lambda^*(A) \le \int f d\lambda \le \lambda_*(B). \tag{4}$$

Associated with $\lambda \in fa(\mathscr{A})_+$ and $X \in \mathfrak{F}(\Omega)$ are the following, important collections:

$$D(X, \lambda) = \big\{t \in \mathbb{R} : \lim_n \lambda_*(X > t - 2^{-n}) = \lim_n \lambda_*(X > t + 2^{-n})\big\} \tag{5a}$$

$$\mathscr{R}_0(X, \lambda) = \big\{\{X > t\} : t \in D(X, \lambda)\big\} \tag{5b}$$

$$\mathscr{A}(\lambda) = \big\{E \subset \Omega : \lambda^*(E) = \lambda_*(E)\big\}. \tag{5c}$$

It is easily seen that $\mathscr{A}(\lambda)$ consists of subsets B of Ω which are λ-measurable, i.e. such that the corresponding indicator function $\mathbb{1}_B$ is a λ-measurable function, and that it forms a ring containing $\mathscr{A}$. We refer to $\mathscr{A}(\lambda)$ as the λ completion of $\mathscr{A}$. Moreover, there clearly exists a unique extension of λ to $\mathscr{A}(\lambda)$ and $X \in \mathfrak{F}(\Omega)$ is λ-measurable if and only if it is measurable with respect to such extension. Abusing notation, we will always denote with the same letter, λ, also its extension to $\mathscr{A}(\lambda)$. A sequence $\langle X_n \rangle_{n\in\mathbb{N}}$ in $L^1(\lambda)$ converges to X in norm if and only if it λ-converges to X and is Cauchy in the norm of $L^1(\lambda)$, [15, III.3.6]. Following [20], if $X \in \mathfrak{F}(\Omega, S)$ and S is a topological space, we say that X is λ-tight if for all $\varepsilon > 0$ there exists $K \subset S$ compact such that $\lambda^*(X \notin K) < \varepsilon$.

3. Finitely Additive Preliminaries

We characterize here measurability and integrability in a convenient way. Some of the following facts are intuitive and well known under countable additivity. We fix $\lambda \in fa(\mathscr{A})_+$.

Lemma 1. *$X \in \mathfrak{F}(\Omega)$ is λ-measurable if and only if it is λ-tight and either one of the following equivalent properties hold: (i) $\lambda_*(X > s) \ge \lambda^*(X \ge t)$ for all $s < t$, (ii) $\mathscr{R}_0(X, \lambda) \subset \mathscr{A}(\lambda)$, (iii) the set $\big\{t \in \mathbb{R} : \{X > t\} \in \mathscr{A}(\lambda)\big\}$ is dense in $\mathbb{R}$.*

Proof. If X is λ-measurable it is λ-tight, [21, p. 190]. Choose $\langle f_k\rangle_{k\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ λ-converging to X and $A_k^\eta\in\mathscr{A}$ such that $\{|X-f_k|\geq\eta\}\subset A_k^\eta$ and $\lambda(A_k^\eta)\leq\lambda^*(|X-f_k|\geq\eta)+2^{-k}$. Then,

$$\{X\geq s+2\eta\}\subset\{f_k\geq s+\eta\text{ or }|f_k-X|>\eta\}\subset\{f_k\geq s+\eta\}\cup A_k^\eta\subset\{X>s\}\cup A_k^\eta$$

so that $\lambda^*(X\geq s+2\eta)\leq\lambda\big(\{f_k\geq s+\eta\}\cup A_k^\eta\big)\leq\lambda_*(X>s)+\lambda\big(A_k^\eta\big)$. Assume (i). If $t\in D(X,\lambda)$ then $\lambda_*(X>t)=\lim_n\lambda_*(X>t-2^{-n})\geq\lambda^*(X\geq t)\geq\lambda^*(X>t)$. The implication $(ii)\Rightarrow(iii)$ is obvious. Assume that (iii) holds. Let $t_0^n=-2^n$, $t_{I_n+1}^n=2^n$ and choose $\{t_1^n\leq\ldots\leq t_{I_n}^n\}\subset[-2^n,2^n]$ to be such that $\{X>t_i^n\}\in\mathscr{A}(\lambda)$ for $i=1,\ldots,I_n$ and $\sup_{0\leq i\leq I_n}|t_i^n-t_{i+1}^n|<2^{-n}$. Define

$$X_n=\sum_{i=1}^{I_n-1}t_i^n\mathbb{1}_{\{t_i^n<X\leq t_{i+1}^n\}}\in\mathscr{S}\big(\mathscr{A}(\lambda)\big). \tag{6}$$

Then $\{|X-X_n|\geq 2^{-n}\}\subset\{|X|>2^{n-1}\}$ so that X_n λ-converges to X whenever X is λ-tight. □

Lemma 2. *$X\in L^1(\lambda)$ if and only if $\int_0^\infty\lambda_*(|X|>t)dt=\int_0^\infty\lambda^*(|X|>t)dt<\infty$. Then,*

$$\int Xd\lambda=\int_0^\infty\lambda_*(X>t)dt-\int_{-\infty}^0\lambda_*(X<\tau)d\tau. \tag{7}$$

Proof. Assume $\int\lambda_*(|X|>t)dt=\int\lambda^*(|X|>t)dt<\infty$. Then X is λ-tight and the set of $t\in\mathbb{R}$ for which $\{|X|>t\}\in\mathscr{A}(\lambda)$ is dense so that $|X|$ is λ-measurable. As in (6) we can construct an increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A}(\lambda))$ such that $0\leq f_n\leq|X|$ and λ-converges to $|X|$. But then,

$$\infty>\int_0^\infty\lambda_*(|X|>t)dt\geq\lim_n\int_0^\infty\lambda(f_n>t)dt=\lim_n\int f_nd\lambda=\int|X|d\lambda \tag{8}$$

as $\langle f_n\rangle_{n\in\mathbb{N}}$ is Cauchy in $L^1(\lambda)$. Assume conversely that $X\in L^1(\lambda)$ and take $b>a>0$. If $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ converges to X in $L^1(\lambda)$ and $0<\varepsilon<a$, then

$$\begin{aligned}\int_{a+\varepsilon}^{b+\varepsilon}\lambda^*(X>t)dt&\leq\int_a^b\lambda(f_n>t)dt+(b-a)\lambda^*(|X-f_n|>\varepsilon)\\&\leq\int_{a-\varepsilon}^{b-\varepsilon}\lambda_*(X>t)dt+2(b-a)\lambda^*(|X-f_n|>\varepsilon)\end{aligned}$$

by [4, 3.2.8.(iii)]. Thus, $\int_a^b\lambda_*(X>t)dt=\int_a^b\lambda^*(X>t)dt$ and

$$\int_a^b\lambda^*(X>t)dt=\lim_n\int_a^b\lambda(f_n>t)dt=\lim_n\int(b\wedge f_n-a)^+d\lambda=\int(b\wedge X-a)^+d\lambda.$$

Thus $\int_0^\infty\lambda_*(X>t)dt=\int_0^\infty\lambda^*(X>t)dt=\int X^+d\lambda$ so that $\int\lambda_*(|X|>t)dt<\infty$ and (7) holds. □

Proving uniqueness of the finitely additive set function generating a given class of integrals requires to identify a minimal measurable structure associated with a given family of functions, a not entirely standard problem under finite additivity. We make this idea precise based on an appropriate notion of order. If $(\mathscr{A},\lambda)$ and $(\mathscr{B},\xi)$ are measurable structures on the same underlying space then (upon identifying ξ with its extension to $\mathscr{B}(\xi)$) we define the partial order $\preceq$ by writing

$$(\mathscr{A},\lambda)\preceq(\mathscr{B},\xi)\quad\text{whenever}\quad\mathscr{A}\subset\mathscr{B}(\xi)\quad\text{and}\quad\xi|\mathscr{A}=\lambda \tag{9}$$

Speaking of a minimal measurable structure always refers to the above defined partial order.

Lemma 3. *Let* $\mathscr{H}$ *be a Stonean vector sublattice of* $\mathfrak{F}(\Omega)$ *and* $\phi \in \mathfrak{L}(\mathscr{H})_+$. *The family* $\mathfrak{M}(\phi)$ *of measurable structures* $(\mathscr{A}, \lambda)$ *on* Ω *satisfying*

$$\mathscr{H} \subset L^1(\lambda) \quad \text{and} \quad \int h d\lambda = \phi(h) \qquad h \in \mathscr{H}, \tag{10}$$

is either empty or contains a minimal element $(\mathscr{R}_\phi, \lambda_\phi)$.

Proof. Assume that $(\mathscr{A}, \lambda) \in \mathfrak{M}(\phi)$ and denote by $\mathscr{R}(\mathscr{H}, \lambda)$ the smallest ring containing

$$\mathscr{R}_0(\mathscr{H}, \lambda) = \big\{\{h > t\} : h \in \mathscr{H}_+,\ t \in D(h, \lambda),\ t > 0\big\}. \tag{11}$$

Write $\lambda_{\mathscr{H}} = \lambda|\mathscr{R}(\mathscr{H}, \lambda)$. $\varnothing \in \mathscr{R}_0(\mathscr{H}, \lambda)$, as $\mathscr{H}$ is Stonean. We claim that $(\mathscr{R}(\mathscr{H}, \lambda), \lambda_{\mathscr{H}})$ is a minimal element in the collection $\mathfrak{M}(\phi)$.

Suppose that $(\mathscr{B}, \xi) \in \mathfrak{M}(\phi)$. Fix $h \in \mathscr{H}_+$ and consider the classical inequality

$$\mathbb{1}_{\{h>a\}} \geq \frac{h \wedge b - h \wedge a}{b - a} \geq \mathbb{1}_{\{h \geq b\}} \qquad h \in \mathscr{H}, b > a > 0. \tag{12}$$

By the Stone property, the inner term belongs to $\mathscr{H}$, so that $\infty > \lambda_*(h > a) \geq \xi^*(h \geq b)$, by (4). Choosing a and b conveniently and interchanging λ with ξ we establish that $D(h, \lambda) \cap (0, \infty) = D(h, \xi) \cap (0, \infty)$ and that

$$\lambda^*(h \geq t) = \xi^*(h \geq t) = \xi_*(h > t) = \lambda_*(h > t) \qquad t \in D(h, \lambda),\ t > 0$$

so that $\mathscr{R}_0(\mathscr{H}, \lambda) \subset \mathscr{B}(\xi)$. For $i = 1, 2$ pick $h_i \in \mathscr{H}_+$, $t_i \in D(h_i, \lambda)$ and $t_i > 0$. Fix $t_1 \wedge t_2 \geq \eta > 0$, define $h_\eta = \big(h_1 - (t_1 - \eta)\big)^+ \vee \big(h_2 - (t_2 - \eta)\big)^+$ and, since the sets $D(h_\eta, \lambda)$ are dense in $\mathbb{R}$, choose

$$\delta \in (0, t_1 \wedge t_2] \cap \mathbb{Q} \cap \bigcap_{\eta \in \mathbb{Q} \cap (0, t_1 \wedge t_2]} D(h_\eta, \lambda).$$

Then $\delta \in D(h_\delta, \lambda)$, $h_\delta \in \mathscr{H}_+$ and $\{h_1 > t_1\} \cup \{h_2 > t_2\} = \{h_\delta > \delta\}$. In other words $\mathscr{R}_0(\mathscr{H}, \lambda)$ is closed with respect to union and, by a similar argument, to intersection as well. Because λ and ξ are additive and coincide on $\mathscr{R}_0(\mathscr{H}, \lambda)$ they also coincide on $\mathscr{R}(\mathscr{H}, \lambda)$, [4, Theorem 3.5.1]. Let $h \in \mathscr{H}_+$ and $t > s$. Then h is $\lambda_{\mathscr{H}}$-tight because $h \in L^1(\lambda)$. If $s < 0$ then $\lambda_{\mathscr{H}*}(h > s) \geq \lambda^*_{\mathscr{H}}(h \geq t)$. Otherwise there are $t', s' \in D(h, \lambda)$ with $t > t' > s' > s$ and therefore

$$\lambda_{\mathscr{H}*}(h > s) \geq \lambda_{\mathscr{H}}(h > s') \geq \lambda_{\mathscr{H}}(h > t') \geq \lambda^*_{\mathscr{H}}(h \geq t).$$

By Lemma 1 h is thus $\lambda_{\mathscr{H}}$-measurable and therefore $\int h d\lambda_{\mathscr{H}} = \int h d\lambda$. □

The minimal structure $(\mathscr{R}(\mathscr{H}, \lambda), \lambda_{\mathscr{H}})$ constructed explicitly above will generally depend on λ. However, since $D(h, \lambda)$ is dense, the σ ring generated by $\mathscr{R}(\mathscr{H}, \lambda)$ corresponds to the usual notion of the σ-ring generated by the family $\mathscr{H}$.

Lemma 4. *Let* $g \in \mathfrak{F}(\Omega)_+$ *be* λ*-measurable and define the ring* $\mathscr{R}_g = \big\{A \in \mathscr{A}(\lambda) : g\mathbb{1}_A \in L^1(\lambda)\big\}$. *There exists a unique* $\lambda_g \in fa(\mathscr{R}_g)_+$ *such that*

$$\int f \lambda_g = \int fg d\lambda \qquad f \in \mathfrak{B}(\lambda),\ fg \in L^1(\lambda). \tag{13}$$

Proof. (13) implies $\lambda_g(A) = \int \mathbb{1}_A g d\lambda$ for every $A \in \mathscr{R}_g$ and thus uniqueness. In proving (13) we may assume that $f \in \mathfrak{B}(\lambda)_+$. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be an increasing sequence in $\mathscr{S}(\mathscr{A}(\lambda))$ such that $0 \le f_n \le f$ and f_n converges to f uniformly, obtained as in (6). Then f_n is λ- and λ_g-convergent to f. Moreover, f_n and $f_n g$ are Cauchy sequence in $L^1(\lambda_g)$ and $L^1(\lambda)$. □

The preceding results may be exploited to prove the existence of distributions.

Proposition 1. *Let $\mu \in fa(\mathscr{A})_+$, $X \in \mathfrak{F}(\Omega, S)$ and $\mathscr{H}$ a Stonean vector sublattice of $\mathfrak{F}(S)$. There exists a minimal measurable structure $(\mathscr{R}, m)$ on $X[\Omega]$ satisfying*

$$h \in L^1(m) \quad \text{and} \quad \int h(X)d\mu = \int hdm \qquad h \in \mathscr{H},\ h(X) \in L^1(\mu). \tag{14}$$

Moreover, m is countably additive whenever: (i) μ is countably additive or (ii) S is a topological space, $\mathscr{H} \subset \mathscr{C}(S)$ and either (a) X is μ-tight or (b) $\mathscr{H} \subset \mathscr{C}_K(S)$.

Proof. Upon replacing $\mathscr{H}$ with $\{h \in \mathscr{H} : h(X) \in L^1(\mu)\}$ and noting that the latter is itself a Stonean sublattice of $\mathfrak{F}(S)$ there is no loss of generality in assuming $\mathscr{H}[X] \subset L^1(\mu)$. Define $\phi \in \mathfrak{L}(\mathscr{H})_+$ implicitly via $\phi(h) = \int h(X)d\mu$ and $\mathscr{R}(\mathscr{H}[X], \mu)$ as in (11). Let

$$\bar{\mathscr{R}} = \big\{B \subset X[\Omega] : X^{-1}(B) \in \mathscr{R}(\mathscr{H}[X], \mu)\big\} \quad \text{and} \quad \bar{m}(B) = \mu(X \in B) \qquad B \in \bar{\mathscr{R}} \tag{15}$$

Then $(\bar{\mathscr{R}}, \bar{m})$ is a measurable structure on $X[\Omega]$ and $D(h(X), \mu) = D(h, \bar{m})$ for every $h \in \mathscr{H}$. Lemma 1 implies that $h \in \mathscr{H}$ is $\bar{m}$-measurable; by Lemma 2

$$\begin{aligned}\int h(X)d\mu &= \int_{D(h(X),\mu)\cap(0,\infty)} \mu(h(X) > t)dt - \int_{D(h(X),\mu)\cap(-\infty,0]} \mu(h(X) < t)dt \\ &= \int_{D(h,\bar{m})\cap(0,\infty)} \bar{m}(h > t)dt - \int_{D(h,\bar{m})\cap(-\infty,0]} \bar{m}(h < t)dt \\ &= \int hd\bar{m}\end{aligned}$$

so that $\phi(h) = \int hd\bar{m}$. By Lemma 3 there is a minimal measurable structure with this property.

ϕ is a Daniell integral when μ is countably additive. To prove the same under (ii) we follow Karandikar [20] and [21] quite closely. We only need to consider case (a), as the restriction to compact sets is obvious under (b). Let the sequence $\langle h_k \rangle_{k\in\mathbb{N}}$ in $\mathscr{H}$ decrease to 0. For each $n \in \mathbb{N}$, let $A_n \in \mathscr{A}$, $A_n \subset \{X \in K_n\}$ and $\mu(A_n^c) < 2^{-n}$, for some $K_n \subset S$ compact. Then, $h_k(X)\mathbb{1}_{A_n}$ converges uniformly to 0 and, by absolute continuity of the finitely additive integral [15, III.2.15],

$$\lim_k \int h_k(X)d\mu = \lim_k \lim_n \int h_k(X)\mathbb{1}_{A_n}d\mu = \lim_n \lim_k \int h_k(X)\mathbb{1}_{A_n}d\mu = 0$$

which proves the Daniell property. □

When $\mathscr{H}$ is Stonean the minimal measurable structure on $X[\Omega]$ of Proposition 1 may be written as $(\mathscr{R}^X_{\mathscr{H}}, \mu^X_{\mathscr{H}})$. Letting $\mathcal{H}$ be the directed set of all Stonean vector sublattices $\mathscr{H}$ of $\mathfrak{F}(S)$ we obtain

$$\mu^X_{\mathscr{H}'}\big|\mathscr{R}^X_{\mathscr{H}} = \mu^X_{\mathscr{H}} \qquad \mathscr{H}, \mathscr{H}' \in \mathcal{H},\ \mathscr{H} \subset \mathscr{H}' \tag{16}$$

so that, in the language of [8], $\big(\mu^X_{\mathscr{H}} : \mathscr{H} \in \mathcal{H}\big)$ is a finitely additive martingale.

Claim (*ii*) was originally formulated, for the case $S = \Omega = \mathbb{R}$, by Dubins and Savage [13, p. 190] who refer to m as the *"conventional companion"* and to the tightness condition as μ not being *"partially remote"*. Karandikar [20] revived their proof and extended it to the case $S = \mathbb{R}^d$ in [21].

4. Integral Representation of Linear Functionals

We obtain in this section a general theorem concerning the integral representation of linear functionals on vector spaces. Several results in the next sections will follow from this claim.

Theorem 1. *Let $\mathscr{H}$ be a vector space and $\phi \in \mathfrak{L}(\mathscr{H})$. Assume that $T \in \mathfrak{L}\big(\mathscr{H}, \mathfrak{F}(\Omega)\big)$ satisfies*

$$\forall h \in \mathscr{H},\ \exists h' \in \mathscr{H} \quad \text{such that} \quad |Th| \le Th' \tag{17}$$

and write $L = \big\{f \in \mathfrak{F}(\Omega) : |f| \le Th \text{ for some } h \in \mathscr{H}\big\}$. The condition

$$\phi(h) < 0 \quad \text{implies} \quad \inf_\omega (Th)(\omega) < 0 \qquad h \in \mathscr{H} \tag{18}$$

is necessary and sufficient for the existence of (i) $F^\perp \in \mathfrak{L}(L)_+$ with $F^\perp[L \cap \mathfrak{B}(\Omega)] = \{0\}$ and (ii) a measurable structure $(\mathscr{R}, \mu)$ on Ω such that $L \subset L^1(\mu)$ and

$$\phi(h) = F^\perp\big(Th\big) + \int Th d\mu \qquad h \in \mathscr{H}. \tag{19}$$

Proof. $T[\mathscr{H}]$ is a majorizing subspace of the vector lattice L, by (17). Under (18), writing

$$F\big(Th\big) = \phi(h) \qquad h \in \mathscr{H} \tag{20}$$

implicitly defines a positive linear functional on $T[\mathscr{H}]$. By [1, Theorem 1.32], F extends as a positive linear functional (still denoted by F) to the whole of L. For each $\alpha \subset \mathscr{H}$ finite, let $h_\alpha \in \mathscr{H}$ be such that $Th_\alpha \ge \bigvee_{h \in \alpha} |Th|$, $\Omega_\alpha = \{Th_\alpha \ne 0\}$ and define $I_\alpha \in \mathfrak{F}\big(L, \mathfrak{F}(\Omega_\alpha)\big)$ by letting

$$I_\alpha(f)(\omega) = \frac{f(\omega)}{Th_\alpha(\omega)} \qquad f \in L,\ \omega \in \Omega_\alpha.$$

Let also

$$L_\alpha = \{f \in L : |f| \le c\, Th_\alpha \text{ for some } c > 0\} \quad \text{and} \quad H_\alpha = I_\alpha[L_\alpha]. \tag{21}$$

H_α is a sublattice of $\mathfrak{B}(\Omega_\alpha)$ containing the constants; $f, g \in L_\alpha$ and $I_\alpha(f) \ge I_\alpha(g)$ imply $f \ge g$. Thus, upon writing

$$U_\alpha\big(I_\alpha(f)\big) = F(f) \qquad f \in L_\alpha \tag{22}$$

we obtain yet another positive, linear functional U_α on H_α. [9, Theorem 1] implies

$$U_\alpha\big(I_\alpha(f)\big) = \int I_\alpha(f) d\bar{m}_\alpha \qquad f \in L_\alpha \tag{23}$$

for some $\bar{m}_\alpha \in ba(\Omega_\alpha)_+$. Let $m_\alpha(A) = \bar{m}_\alpha(A \cap \Omega_\alpha)$ for each $A \subset \Omega$. By Lemma 4, we can write (with the convention $0/0 = 0$)

$$F(f) = \int \frac{f}{Th_\alpha} \mathbb{1}_{\Omega_\alpha} dm_\alpha = \int f d\bar{\mu}_\alpha \qquad f \in L_\alpha \cap \mathfrak{B}(\Omega) \tag{24}$$

with $\bar{\mu}_\alpha = m_{\alpha,g}$ defined as in (13) with $g = \mathbb{1}_{\Omega_\alpha}/Th_\alpha$. Given that $L_\alpha \cap \mathfrak{B}(\Omega)$ is a Stonean lattice, we deduce from Lemma 3 the existence of a minimal measurable structure $(\mathscr{R}_\alpha, \mu_\alpha)$ supporting the representation (24). Define $\mathscr{R} = \bigcup_\alpha \mathscr{R}_\alpha$ and $\mu(A) = \lim_\alpha \mu_\alpha(A)$ for all $A \in \mathscr{R}$. $\alpha \subset \alpha'$ implies $L_\alpha \subset L_{\alpha'}$, $(\mathscr{R}_\alpha, \mu_\alpha) \preceq (\mathscr{R}_{\alpha'}, \mu_{\alpha'})$ as well as the martingale restriction

$$\mu_\alpha = \mu_{\alpha'}|\mathscr{R}_\alpha = \mu|\mathscr{R}_\alpha \qquad \alpha \subset \alpha'. \tag{25}$$

But then for each $f \in L_\alpha$ with $f \geq 0$,

$$\begin{aligned} F(f) &= \lim_k F(f \wedge k) + \lim_k F((f-k)^+) \\ &= \lim_k \int (f \wedge k) d\mu + F^\perp(f) \\ &= \int f d\mu + F^\perp(f) \end{aligned} \tag{26}$$

where we have set $F^\perp(f) = \lim_k F((f-k)^+)$ and the inequality $\mu^*(f > k) \leq k^{-1} \int f \wedge k d\mu \leq k^{-1} F(f)$ induces the conclusion that $f \wedge k$ is μ-convergent to f and is Cauchy in $L^1(\mu)$. $\int |f| d\mu \leq F(|f|)$ follows from (26) and implies $L \subset L^1(\mu)$. (32) is a consequence of (20) and (26). Necessity is obvious as the right hand side of (32) defines a positive linear functional on L. □

Condition (17) is trivially true when $\mathscr{H}$ is a directed vector space – i.e. an ordered vector space which is directed by its own partial order – and T a positive map. An immediate corollary is the following representation of positive linear functionals on vector lattices that may fail to be Stonean.

Corollary 1. *Let $\mathscr{H}$ be a vector sublattice of $\mathfrak{F}(\Omega)$, $L = \{f \in \mathfrak{F}(\Omega) : |f| \leq h$ for some $h \in \mathscr{H}\}$ and $\phi \in \mathfrak{L}(\mathscr{H})_+$. There exists $\phi^\perp \in \mathfrak{L}(L)_+$ with $\phi^\perp[L \cap \mathfrak{B}(\Omega)] = \{0\}$ and a measurable structure $(\mathscr{R}, \mu)$ on Ω such that*

$$L \subset L^1(\mu) \quad \textit{and} \quad \phi(h) = \phi^\perp(h) + \int h d\mu \qquad h \in \mathscr{H}. \tag{27}$$

If $\mathscr{H}$ is Stonean then $\phi^\perp$ is unique and $(\mathscr{R}, \mu)$ can be chosen to be minimal.

Proof. Take T to be the identity map in Theorem 1. If $\mathscr{H}$ is Stonean then $\phi^\perp$ is unique since, from (27), one obtains $\int h d\mu = \lim_k \int (h \wedge k) d\mu = \lim_k \phi(h \wedge k)$ for each $h \in \mathscr{H}_+$. □

(18) is a minimal condition and simply requires that ϕ and T do not rank the elements of $\mathscr{H}$ in a totally opposite way. It will appear in various forms in later sections and, following Dubins, we shall refer to it by saying that ϕ is T *conglomerative*. To make the connection with the work of Dubins [14] more transparent we establish the following version of the problem considered by him:

Corollary 2. *Let $(\mathscr{B}, \lambda)$ be a measurable structure on $S \times \Omega$ and $\mathscr{H}$ a Stonean sublattice of $L^1(\lambda)$. Let $\{\sigma_\omega : \omega \in \Omega\} \subset \mathfrak{L}(\mathscr{H})_+$. The condition*

$$\int h d\lambda < 0 \quad \textit{implies} \quad \inf_\omega \sigma_\omega(h) < 0 \qquad h \in \mathscr{H} \tag{28}$$

is equivalent to the existence of a measurable structure $(\mathscr{R},\gamma)$ *on* Ω *such that*

$$\int h d\lambda = \int \sigma_\omega(h) d\gamma \qquad h \in \mathscr{H}. \tag{29}$$

Proof. Apply Theorem 1 with $T \in \mathfrak{F}\big(\mathscr{H}, \mathfrak{F}(\Omega)\big)$ defined as $Th(\omega) = \sigma_\omega(h)$. □

Dubins considered the case with $\mathscr{H} = \mathfrak{B}(S)$ and Ω a partition of S. The family σ indexed by Ω is a *strategy* in his terminology and a probability λ admitting the representation (29) is called *strategic*. Theorem 1 in [14] states that λ is σ strategic if and only if it is σ conglomerative.

An inductive limit version of Theorem 4 can also be easily proved.

Corollary 3. *For each* α *in a directed set* $\mathfrak{A}$*, let* $\mathscr{H}_\alpha$ *be a vector space,* $\phi_\alpha \in \mathfrak{L}(\mathscr{H}_\alpha)$ *and let* $T_\alpha \in \mathfrak{L}(\mathscr{H}_\alpha, \mathfrak{L}(\Omega))$ *satisfy* (17)*. For each* $\alpha, \beta \in \mathfrak{A}$ *with* $\alpha \le \beta$ *let it be defined a map* $\chi_{\alpha\beta} \in \mathfrak{L}(\mathscr{H}_\alpha, \mathscr{H}_\beta)$ *satisfying* $\chi_{\alpha\alpha} = id$, $\chi_{\alpha\gamma} = \chi_{\alpha\beta} \cdot \chi_{\beta\gamma}$ *for* $\alpha \le \beta \le \gamma$ *and*

$$\phi_\alpha = \phi_\beta \cdot \chi_{\alpha\beta} \text{ and } T_\alpha = T_\beta \cdot \chi_{\alpha\beta} \qquad \alpha \le \beta \tag{30}$$

Write $L = \{f \in \mathfrak{F}(\Omega) : |f| \le T_\alpha h_\alpha \text{ for some } \alpha \in \mathfrak{A},\ h_\alpha \in \mathscr{H}_\alpha\}$*. The condition*

$$\phi_\alpha(h_\alpha) < 0 \quad \text{implies} \quad \inf_\omega T_\alpha h_\alpha(\omega) < 0 \qquad h_\alpha \in \mathscr{H}_\alpha,\ \alpha \in \mathfrak{A} \tag{31}$$

is necessary and sufficient for the existence of (i) $F^\perp \in \mathfrak{L}(L)_+$ *with* $F^\perp[L \cap \mathfrak{B}(\Omega)] = \{0\}$ *and (ii) a measurable structure* $(\mathscr{R},\mu)$ *on* Ω *such that* $L \subset L^1(\mu)$ *and*

$$\phi_\alpha(h_\alpha) = F^\perp\big(T_\alpha h_\alpha\big) + \int T_\alpha h_\alpha d\mu \qquad \alpha \in \mathfrak{A},\ h_\alpha \in \mathscr{H}_\alpha. \tag{32}$$

Proof. If $(\mathscr{H}, \langle\chi_\alpha\rangle_{\alpha\in\mathfrak{A}})$ denotes the inductive limit of the directed family $\langle\mathscr{H}_\alpha, \chi_{\alpha\beta}\rangle_{\alpha\in\mathfrak{A}}$ then, given (30), it is possible to define $\phi \in \mathfrak{L}(\mathscr{H})$ and $T \in \mathfrak{L}(\mathscr{H}, \mathfrak{F}(\Omega))$ by letting

$$\phi(\chi_\alpha h_\alpha) = \phi_\alpha(h_\alpha) \quad \text{and} \quad T(\chi_\alpha h_\alpha) = T_\alpha(h_\alpha) \qquad \alpha \in \mathfrak{A},\ h_\alpha \in \mathscr{H}_\alpha. \tag{33}$$

It is then clear that T satisfies (17) and that (31) is equivalent to ϕ being T conglomerative. □

5. Extreme Points and Choquet Representation Theorem

Conglomerability may be formulated geometrically in terms of the $\mathfrak{X}$-topology, [15, V.3.2]. Denote by $\overline{\text{con}}^{\mathfrak{X}}(F)$ the $\mathfrak{X}$-closed conical hull of the set $F \subset \mathfrak{F}(\mathfrak{X})$. The function $e_x \in \mathfrak{F}(F)$ satisfying $e_x(f) = f(x)$ for each $f \in F$ is an evaluation on F.

Theorem 2. *Let* $\mathfrak{X}$ *be a vector lattice and let* $\Psi \subset \mathfrak{L}(\mathfrak{X})_+$ *satisfy* $\sup_{\psi\in\Psi} \psi(x) < \infty$ *for every* $x \in \mathfrak{X}$*. Denote by* $\mathscr{R}$ *the* σ *ring on* Ψ *generated by the evaluations* $\{e_x : x \in \mathfrak{X}\}$ *on* Ψ*. Then,*

$$\Phi \subset \overline{\text{con}}^{\mathfrak{X}}(\Psi) \tag{34}$$

if and only if each $\phi \in \Phi$ *admits the representation*

$$\phi(x) = \int_\Psi \psi(x) d\mu_\phi(\psi) \qquad x \in \mathfrak{X} \tag{35}$$

with $\mu_\phi \in fa(\mathscr{R})_+$ *countably additive and such that the evaluations on* Ψ *are* μ_ϕ*-integrable.*

Proof. By ordinary separation theorems $\phi \notin \overline{\text{con}}^{\mathfrak{X}}(\Psi)$ is equivalent to $\phi(x) < 0 \leq \inf_{\psi\in\Psi}\psi(x)$ for some $x \in \mathfrak{X}$ and contrasts with (35). Thus if $\phi \in \overline{\text{con}}^{\mathfrak{X}}(\Psi)$ then for each $x \in \mathfrak{X}$ there is $\psi_x \in \Psi$ such that $\phi(x) < 0$ implies $\psi_x(x) < 0$. Define $T \in \mathfrak{L}\big(\mathfrak{X}, \mathfrak{F}(\mathfrak{X})\big)$ and $U \in \mathfrak{F}\big(\mathfrak{X}, \Psi\big)$ implicitly by letting

$$Tx(y) = U(y)(x) = \psi_y(x) \qquad x, y \in \mathfrak{X}. \tag{36}$$

Then $\phi \in \Phi$ is T conglomerative, $T[\mathfrak{X}] \subset \mathfrak{B}(\mathfrak{X})$, as $\sup_{y\in\mathfrak{X}}|Tx(y)| \leq \sup_{\psi\in\Psi}|\psi(x)| < \infty$, and (17) follows from $|Tx|(y) = |\psi_y(x)| \leq \psi_y(|x|) = T|x|(y)$. Theorem 1 applies and (32) translates into

$$\phi(x) = \int_{\mathfrak{X}} Tx(y)d\bar{m}_\phi(y) = \int_{\mathfrak{X}} U(y)(x)d\bar{m}_\phi(y) = \int e_x(U)d\bar{m}_\phi \qquad x \in \mathfrak{X} \tag{37}$$

for some $\bar{m}_\phi \in fa(\mathfrak{X})_+$ such that $\{e_x(U) : x \in \mathfrak{X}\} \subset L^1(\bar{m}_\phi)$. Denote by $\mathfrak{Y}$ the space $\mathfrak{L}(\mathfrak{X})$ endowed with the $\mathfrak{X}$-topology and let $(\hat{\mathscr{R}}_\phi, \hat{\mu}_\phi)$ be the minimal measurable structure on Ψ that satisfies (14) with $X = U$, $\mathscr{H} = \mathscr{C}(\mathfrak{Y})$, $S = \mathfrak{L}(\mathfrak{X})$ and $\Omega = \mathfrak{X}$. From the inclusion $e_x \in \mathscr{C}(\mathfrak{Y})$ we conclude:

$$\phi(x) = \int e_x(U)d\bar{m}_\phi = \int_\Psi e_x(\psi)d\hat{\mu}_\phi(\psi) = \int_\Psi \psi(x)d\hat{\mu}_\phi(\psi)$$

Since Ψ is relatively compact in the $\mathfrak{X}$-topology, [15, V.4.1], U is $\bar{m}_\phi$-tight. By Proposition 1.(ii).(a), $\hat{\mu}_\phi$ is then countably additive and can be uniquely extended as a countably additive set function $\bar{\mu}_\phi$ on the generated σ ring, $\bar{\mathscr{R}}_\phi$. By the remarks following Lemma 3, $\bar{\mathscr{R}}_\phi$ contains the σ ring generated by $\{h \in \mathscr{C}(\mathfrak{Y}) : h(U) \in L^1(\mu_\phi)\}$ and, *a fortiori*, $\mathscr{R}$. It is then enough to set $\mu_\phi = \bar{\mu}_\phi|\mathscr{R}$. $\square$

Upon rewriting condition (18) in Theorem 1 in the form $\phi \in \overline{\text{con}}^{\mathscr{H}}(\{T_\omega : \omega \in \Omega\})$ it is apparent that (34) is just a geometric reformulation of the notion of conglomerability. Theorem 2 thus asserts that Φ is Ψ conglomerative if and only if each $\phi \in \Phi$ is the generalized barycenter of some set function μ_ϕ concentrated on Ψ where the qualification *generalized*, suggested by Dynkin [16], refers to μ_ϕ not being a probability.

The conical hull and the $\mathfrak{X}$-closure make the inclusion (34) a very weak restriction. While Theorem 2 extends thus well known results which typically involve the strong closure of the convex hull, we highlight that the set function intervening in the barycentrical representation (35) need not be unique here as the collection of evaluations is too poor to deduce some minimality criterion. Moreover, the lattice structure imposed on $\mathfrak{X}$ and the restriction to positive functionals are essential in avoiding additional topological assumptions such as strong compactness (see e.g. [23, p. 14]), separability or the Radon-Nikodým property (as in [17, p. 355] or [18, corollary 5.3])[1].

We now prove a version of the preceding Theorem 2 in which every notion of order is abandoned.

Theorem 3. *Let $U, V \subset \mathfrak{Y}$ be non empty sets and let $\mathscr{H} \subset \mathfrak{F}(\mathfrak{Y})$ separate points of $\mathfrak{Y}$ and satisfy*

$$\sup_{v\in V}|h(v)| < \infty \quad \textit{for all} \quad h \in \mathscr{H} \quad \textit{and} \quad \sup_{y\in\mathfrak{Y}}|h_0(y)| = \infty \quad \textit{for some} \quad h_0 \in \mathscr{H}. \tag{38}$$

Write $\sigma(\mathscr{H}|V)$ for the σ algebra generated by $\mathscr{H}|V = \{h|V : h \in \mathscr{H}\}$. The following are equivalent:

[1]Another version of Choquet theorem valid under weak topological assumptions is the one proved by Dynkin [16, Theorem 3.1], based on a separability condition (for measures) and the notion of H-sufficiency.

(i) each $u \in U$ admits a net $\langle \pi_\alpha^u \rangle_{\alpha \in \mathfrak{A}}$ of convex weights on V such that

$$h(u) = \lim_\alpha \sum_v h(v)\pi_\alpha^u(v) \qquad h \in \mathscr{H} \tag{39}$$

(ii) each $u \in U$ admits a probability $\mu_u \in ca(\Sigma)$ such that $\mathscr{H}|V \subset L^1(\mu_u)$ and

$$h(u) = \int_V h(v)d\mu_u(v) \qquad h \in \mathscr{H} \tag{40}$$

If $\mathscr{H}$ is a Stonean lattice, the probability μ_u associated with $u \in U$ via (40) *is necessarily unique.*

Proof. (i) follows from (ii) and ordinary rules of integration. For the converse, write

$$\mathfrak{X} = \left\{ b_0 + \sum_{n=1}^N b_n h_n : N \in \mathbb{N},\ b_0, b_1, \ldots, b_N \in \mathbb{R},\ h_1, \ldots, h_N \in \mathscr{H} \right\} \tag{41}$$

to denote the set of affine transforms of elements of $\mathscr{H}$. Define the function $\kappa \in \mathfrak{F}\big(\mathfrak{Y}, \mathfrak{L}(\mathfrak{X})_+\big)$ by

$$(\kappa y)x = b_0 + \sum_{n=1}^N b_n h_n(y) \qquad y \in \mathfrak{Y},\ x = b_0 + \sum_{n=1}^N b_n h_n \in \mathfrak{X} \tag{42}$$

and write $\Psi = \kappa[V]$ and $\Phi = \kappa[U]$. Observe that (39) is equivalent to the inclusion

$$\Phi \subset \overline{\text{co}}^{\mathfrak{X}}(\Psi) \tag{43}$$

We deduce from (38) that $\emptyset \notin \overline{\text{co}}^{\mathfrak{X}}(\kappa z + \Psi)$ for some $z \in \mathfrak{Y}$. Let $f = \kappa z$. By ordinary separation theorems, there exists $x_0 \in \mathfrak{X}$ such that $(f + \Psi)[x_0] \geq 1$. Fix $\phi \in \Phi$. As in the proof of Theorem 2, the inclusion $f + \phi \in \overline{\text{co}}^{\mathfrak{X}}(f + \Psi)$ implies the existence of an operator $T_\phi \in \mathfrak{L}\big(\mathfrak{X}, \mathfrak{F}(\mathfrak{X})\big)$ as in (36) with respect to which the set $f + \phi$ is congolomerative. To show that T_ϕ satisfies (17) observe that

$$|T_\phi x(y)| = |(f + \psi_y)(x)| \leq \sup_{\psi \in \Psi} |(f + \psi)(x)| (f + \psi_y)(x_0) = T_\phi \tilde{x}(y)$$

with $\tilde{x} = x_0 \sup_{\psi \in \Psi} |(f + \psi)(x)|$. We are then in the position to apply (35) and obtain, for $\phi = \kappa u$,

$$x(u + z) = (\phi + f)(x) = \int (\psi + f)(x) d\mu_\phi = \int x(v + z) d\mu_u \qquad x \in \mathfrak{X} \tag{44}$$

with $\mu_u = \mu_\phi \in fa(\mathscr{R})_+$ countably additive and $\mathscr{R}$ the σ ring on V generated by $\mathscr{H}$. Given that $\mathfrak{X}$ contains the constants, we deduce $\|\mu_u\| = 1$ and that

$$h(u) = \int h(v) d\mu_u \qquad h \in \mathscr{H}. \tag{45}$$

$\mathscr{R}$ may be replaced with $\sigma(\mathscr{H}|V)$ and μ_u with its unique extension to it defined by $\mu_u(V) = 1$. □

Without an order structure the conical condition (34) is replaced with the more restrictive inclusion (43), similar to that considered by several other authors. In particular, if $\mathfrak{Y}$ is a linear space and $\mathscr{H}$ consists of affine functions, then (39) is equivalent to the more explicit and familiar condition $U \subset \overline{\text{co}}^{\mathscr{H}}(V)$. Most papers on Choquet theorem take $\mathfrak{Y}$ to a be a locally convex, topological vector space (often just a Banach space) and $\mathscr{H}$ its dual $\mathfrak{Y}^*$. In this special case Theorem 3 may be rephrased into the following minor extension of a result of Edgar [17, p. 355]:

Corollary 4 (Edgar)**.** *Let V be a bounded set of a locally convex space. Then $u \in \overline{\text{co}}(V)$ if and only if u may be represented as the barycenter of a countably additive probability supported by V.*

In particular, in Banach spaces possessing the Radon Nikodým property (and thus the Krein Milman property [7, 3.3.6]) the validity of the preceding Corollary extends to closed, convex and bounded (but not necessarily compact) sets. By a theorem of Phelps [7, 3.5.4], a set with such properties also admits a barycentrical representation with respect to the set of its strongly exposed points. Eventually in Banach spaces, in which a well developed theory of vector integration is available, the integral in (40) may be interpreted as the Bochner integral

$$u = \int v d\mu_u \tag{46}$$

of the identity on V, as remarked in [24].

6. Finitely Additive Representations

Theorem 4. *Let $m \in fa(\Sigma)_+$, $\mathscr{H}$ a Stonean vector sublattice of $L^1(m)$ and $X \in \mathfrak{F}(\Omega, S)$. There is equivalence between the condition*

$$\int hdm < 0 \quad \textit{implies} \quad \inf_\omega h\big(X(\omega)\big) < 0 \qquad h \in \mathscr{H} \tag{47}$$

and the existence of a minimal measurable structure $(\mathscr{R}, \mu)$ on Ω satisfying

$$h(X) \in L^1(\mu) \quad \textit{and} \quad \int hdm = \int h(X)d\mu \qquad h \in \mathscr{H}. \tag{48}$$

and either one of these properties is implicit in $m^*(X[\Omega]^c) = 0$. Moreover, if m is countably additive, Σ a σ ring and $m_*(X[\Omega]^c) = 0$ then μ is countably additive.

Proof. (47) is equivalent to (18) with

$$\phi(h) = \int hdm \quad \text{and} \quad Th = h(X) \qquad h \in \mathscr{H}$$

Thus, (48) follows from (32) after noting that, in the present setting, $\phi(h) = \lim_k \phi(h \wedge k)$ for every $h \in \mathscr{H}_+$. If $\langle B_n \rangle_{n\in\mathbb{N}}$ is a decreasing sequence in Σ with $X[\Omega]^c \subset B_n$ and $m(B_n) \le 2^{-n}$ and if $h \in \mathscr{H}$ then

$$\int hdm = \lim_n \int h\mathbb{1}_{B_n^c} dm \ge \inf_{s\in X[\Omega]} h(s) \lim_n m(B_n^c) \ge \inf_\omega h(X(\omega)) \lim_n m(B_n^c)$$

Suppose that m is countably additive, Σ a σ ring and that $m_*(X[\Omega]^c) = 0$. Let $\langle h_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathscr{H}$ with $h_n(X)$ decreasing monotonically to 0. If $g \in \mathscr{H}$ and $t \in D(g, m) \cap D(g(X), \mu)$, then from (15)

$$\{g > t\} \in \Sigma(m),\ \{g(X) > t\} \in \mathscr{R}(\mu) \quad \text{and} \quad m(g > t) = \mu(g(X) > t). \tag{49}$$

We thus have a dense subset of $t > 0$ for which the following holds:

$$m(h_{n+1} - h_n > t) = \mu(h_{n+1}(X) - h_n(X) > t) = 0 \qquad n = 1, 2, \dots$$

By countable additivity, the sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ is thus decreasing m a.s. to $h = \inf_n h_n$. If $\varepsilon > 0$ then $\{h > \varepsilon\} \subset X[\Omega]^c$ and $\{h > \varepsilon\} \in \Sigma$ so that, by assumption, $0 = m(h > 0)$. But then $\lim_n \int h_n(X)d\mu = \lim_n \int h_n dm = \int h dm \le 0$ and the functional ϕ above is a Daniell integral. □

The problem of finding a probability space and a random variable on it with a preassigned distribution is often part of more general problems, e.g. the Skorohod representation theorem, see section 7. We remark that n the absence of restrictions on μ, the existence of representations is guaranteed a very weak condition such as conglomerability. If, e.g., $X[\Omega]^c \in \Sigma$, then in order for X to represent m relatively to the whole of $L^1(m)$ it is necessary and sufficient that $m(X[\Omega]^c) = 0$. If m consists of sample frequencies, then this condition means that all the observations in the given sample must belong to the range of X.

Example 1. *Let $\Omega = S = \mathbb{N}$, let $\mathscr{A} = \Sigma$ be the algebra of finite-cofinite subsets of $\mathbb{N}$ and define implicitly the probability $\mu \in ba(\mathscr{A})$ by letting $\mu(N) = 0$ whenever N is finite. It is easily seen that $h \in L^1(\mu)$ if and only if $\langle h(n)\rangle_{n\in\mathbb{N}}$ is a convergent sequence and that then $\int h d\mu = \lim_n h(n)$. Define $X \in \mathfrak{F}(\Omega, S)$ by letting $X(n) = 2n + 1$ so that $X[\Omega]$ coincides with the set of odd numbers. It is obvious that, for $A, B \in \mathscr{A}$ the inclusions $A \subset X[\Omega] \subset B$ imply that A is finite while B is cofinite. Thus, $\mu_*(X[\Omega]) = 0$ while $\mu^*(X[\Omega]) = 1$. Moreover, $\inf_n h(X(n)) \le \int h d\mu$ so that condition* (34) *of Theorem 4 holds and in fact $\int h d\mu = \lim_n h(n) = \lim_n h(2n + 1) = \int h(X)d\mu$.*

Example 2. *Let $(\Omega, \mathscr{A}, P)$ be a classical probability space, $S = \mathbb{R}$ and let X be a normally distributed random quantity on Ω. Fix $m \in ba(\mathscr{B}(\mathbb{R}))_+$ arbitrarily and let $\mathscr{H} = \mathscr{C}(\mathbb{R}) \cap L^1(m)$. Given that $P(X \in B) > 0$ for every B open, we conclude that m is X conglomerative relatively to $\mathscr{H}$. In other words a normally distributed random quantity can assume any arbitrary distribution (relatively to the continuous functions) upon an accurate choice of the reference measure.*

It is possible to write $m = p + r$ with $p^*(X[\Omega]^c) = r_*(X[\Omega]) = 0$ (so that $p \perp r$) by simply letting

$$r(B) = \inf\left\{m(B \cap E) : E \in \Sigma,\ X[\Omega]^c \subset E\right\} \qquad B \in \Sigma. \tag{50}$$

From this remark we obtain,

Corollary 5. *Let $m \in fa(\Sigma)_+$ and $X \in \mathfrak{F}(\Omega, S)$. There exist $r \in fa(\Sigma)_+$ with $r_*(X[\Omega]) = 0$, a measurable structure $(\mathscr{R}, \mu)$ on Ω such that*

$$h(X) \in L^1(\mu) \quad \text{and} \quad \int h dm = \int h dr + \int h(X)d\mu \qquad h \in L^1(m). \tag{51}$$

The decomposition (51) *is unique if and only if $X[\Omega] \in \Sigma$.*

Proof. Existence is clear, given (50) and Theorem 4. If μ^X denotes the distribution of X induced by μ (relatively to $L^1(m)$), then $\mu^X_*(X[\Omega]^c) = 0$. If $X[\Omega] \in \Sigma$ and if $\hat{r}$ and $\hat{\mu}$ is another pair satisfying (51), then $|r - \hat{r}|(B) = |r - \hat{r}|(B \cap X[\Omega]) + \left|\mu^X - \hat{\mu}^X\right|(B \cap X[\Omega]^c) = 0$ for every $B \in \Sigma$. Conversely, assume that $X[\Omega] \notin \Sigma(m)$. Then, letting $\bar{\Sigma}$ be the minimal algebra containing Σ and $X[\Omega]$, there exist two extensions $\bar{m}^1$ and $\bar{m}^2$ to $\bar{\Sigma}$ with $\bar{m}^1(X[\Omega]) = m^*(X[\Omega])$ and $\bar{m}^2(X[\Omega]) = m_*(X[\Omega])$, see [4, Theorem 3.4.4]. Denote by $\bar{r}^i$ and $\bar{\mu}^i$ the set functions satisfying (51) associated with $\bar{m}^i$ for $i = 1, 2$ and let $r^i = \bar{r}^i|\Sigma$ and $\mu^i = \bar{\mu}^i|\mathscr{R}$ for $i = 1, 2$. Then, r^i and μ^i still satisfy (51) while $\mu^1(\Omega) = \bar{m}^1(X[\Omega]) = m^*(X[\Omega]) > m_*(X[\Omega]) = \mu^2(\Omega)$. □

The following is an extension of the notion of conventional companion.

Corollary 6. *Let $X \in \mathfrak{F}(\Omega, S)$, $\mu \in fa(\mathscr{A})_+$ and a Stonean vector sublattice $\mathscr{H} \subset \mathfrak{F}(S)$ satisfy $\mathscr{H}[X] \subset L^1(\mu)$. Let W be a non empty set and $Z \in \mathfrak{F}(W, S)$. The condition*

$$\int h(X)d\mu < 0 \quad \textit{implies} \quad \inf_w h(Z(w)) < 0 \qquad h \in \mathscr{H} \tag{52}$$

is equivalent to the existence of a minimal structure $(\mathscr{R}, \lambda)$ on W such that

$$\mathscr{H}[Z] \subset L^1(\lambda) \quad \textit{and} \quad \int h(X)d\mu = \int h(Z)d\lambda \qquad h \in \mathscr{H}. \tag{53}$$

Moreover, λ is countably additive whenever (i) μ is countably additive, (ii) X is μ-tight and $\mathscr{H} \subset \mathscr{C}(S)$ or (iii) $\mathscr{H} \subset \mathscr{C}_K(S)$.

In Theorem 4 the representing measure μ is completely unrestricted. A possible mitigation is to require that μ vanishes on a given collection $\mathscr{N}$ of subsets of Ω. Let henceforth $\mathscr{N}$ be an ideal of subsets of Ω, i.e. $N, M \in \mathscr{N}$ and $A \subset N$ imply $N \cup M, A \in \mathscr{N}$.

Theorem 5. *Let m, $\mathscr{H}$ and X be as in Theorem 4. The condition*

$$\int hdm < 0 \quad \textit{implies} \quad \sup_{N \in \mathscr{N}} \inf_{\omega \in N^c} h\big(X(\omega)\big) < 0 \tag{54}$$

is equivalent to the existence of a measurable structure $(\mathscr{R}, \mu)$ on Ω with $\mathscr{N} \subset \mathscr{R}$ which satisfies

$$\mu[\mathscr{N}] = \{0\}, \quad h(X) \in L^1(\mu) \quad \textit{and} \quad \int hdm = \int h(X)d\mu \qquad h \in \mathscr{H}. \tag{55}$$

Moreover, if m is a countably additive set function on a σ ring, $\mathscr{N}$ is closed with respect to countable unions and $m_*\big(X[N^c]^c\big) = 0$ for all $N \in \mathscr{N}$ then μ is countably additive.

Proof. Define the binary relation $\succeq$ on $\mathfrak{F}(\Omega)$ by letting for each $f, g \in \mathfrak{F}(\Omega)$

$$f \succeq g \quad \text{if and only if} \quad \sup_{N \in \mathscr{N}} \inf_{\omega \in N^c} (f - g)(\omega) \geq 0. \tag{56}$$

Since $\mathscr{N}$ is an ideal, $\succeq$ is a partial order and $f \geq g$ implies $f \succeq g$. Moreover, $f_i \succeq g_i$ for $i = 1, 2$ implies $f_1 \vee f_2 \succeq g_1 \vee g_2$. In fact, $f_1 \vee f_2 \succeq f_i \succeq g_i$ i.e. $f_1 \vee f_2 \geq g_i - \varepsilon$ outside of some $N_i \in \mathscr{N}$. Thus, $f_1 \vee f_2 \geq g_1 \vee g_2 - \varepsilon$ outside of $N_1 \cup N_2 \in \mathscr{N}$ which, by (56), is equivalent to $f_1 \vee f_2 \succeq g_1 \vee g_2$. But then, if $\widetilde{f}$ denotes the class of elements of $\mathfrak{F}(\Omega)$ equivalent to f, it is easy to see that, relatively to pointwise ordering, the set

$$\mathscr{F} = \Big\{ f \in \mathfrak{F}(\Omega) : f \in \widetilde{h(X)} \text{ for some } h \in \mathscr{H} \Big\} \tag{57}$$

is a Stonean vector sublattice of $\mathfrak{F}(\Omega)$. Writing

$$\phi(f) = \int hdm \qquad f \in \widetilde{h(X)},\ h \in \mathscr{H} \tag{58}$$

implicitly defines, via (54), a positive linear functional on $\mathscr{F}$ so that, by Corollary 1, we conclude that there exists a minimal measurable structure $(\mathscr{R}, \mu)$ satisfying

$$f \in L^1(\mu) \quad \text{and} \quad \phi(f) = \int f d\mu \qquad f \in \mathscr{F} \tag{59}$$

If $N \in \mathscr{N}$ then $\mathbb{1}_N \in \widetilde{0}$, $\mathbb{1}_N \in \mathscr{F}$ and $\mu(N) = 0$. Then (55) follows while the converse implication, is obvious.

The last claim will be proved, once again, by showing that under the stated conditions the functional ϕ defined in (58) is a Daniell integral over $\mathscr{F}$. In fact, let $\langle f_n \rangle_{n\in\mathbb{N}}$ be sequence in $\mathscr{F}$ decreasing pointwise to 0. Let $h_n \in \mathscr{H}$ be such that $f_n \in \widetilde{h_n(X)}$, $n = 1, 2, \ldots$. Define $g_n = \bigwedge_{1\le j\le n} h_j$. As shown above, $f_n \in \widetilde{g_n(X)}$. Let $g = \lim_n g_n$. Of course, $f_n \succeq g(X)$ so that, by the countable union property, $\{g(X) > \varepsilon\} \subset \bigcup_n \{g(X) \ge f_n + \varepsilon\} \in \mathscr{N}$ and $\{g > \varepsilon\} \subset X[\{g(X) \le \varepsilon\}]^c$. By assumption then, $m(g > \varepsilon) = 0$ and so

$$\lim_n \int f_n d\mu = \lim_n \int g_n(X) d\mu = \lim_n \int g_n dm = \int g dm = 0.$$

□

Example 3 (Example 2 continued)**.** *Let X be normally distributed on some classical probability space $(\Omega, \mathscr{A}, P)$ and let $\mathscr{N}$ consists of all P null sets. The set $X[N]$ has empty interior for each $N \in \mathscr{N}$, i.e. $\overline{X[N^c]} = \mathbb{R}$ so that*

$$\sup_{N\in\mathscr{N}} \inf_{\omega\in N^c} h(X(\omega)) = \sup_{N\in\mathscr{N}} \inf_{s\in X[N^c]} h(s) = \sup_{N\in\mathscr{N}} \inf_{s\in \overline{X[N^c]}} h(s) = \inf_{s\in\mathbb{R}} h(s) \qquad h \in \mathscr{C}(\mathbb{R}).$$

The conglomerative property (54) then holds for every $m \in fa(\mathscr{B}(\mathbb{R}))_+$ with $\mathscr{H} = \mathscr{C}(\mathbb{R})$. Thus, the representing measure Q under which X assumes the distribution m may chosen to vanish on P-null sets. If, in addition, m is countably additive then $Q \ll P$. Of course the same conclusion would be true of any variable possessing a strictly positive density over the whole of $\mathbb{R}$.

Theorem 4 may be extended to stochastic processes.

Theorem 6. *Let a family $\mathscr{I}$ of finite subsets of some set $\mathbf{I}$ be directed by inclusion. For each $t \in \mathbf{I}$, let $\mathscr{R}_t$ be a ring on some non empty set S_t and $X_t \in \mathfrak{F}(\Omega, S_t)$. For each $\alpha \in \mathscr{I}$ write*

$$S_\alpha = \bigtimes_{t\in\alpha} S_t, \; \mathscr{R}_\alpha = \bigotimes_{t\in\alpha} \mathscr{R}_t \quad \textit{and} \quad X_\alpha = \bigtimes_{t\in\alpha} X_t, \tag{60}$$

let π_α be the projection of S onto S_α and let $\mathscr{H}_\alpha$ a Stonean vector sublattice of $\mathfrak{F}(S_\alpha)$ such that

$$\left\{h \circ \pi_\alpha : h \in \mathscr{H}_\alpha\right\} \subset \left\{h' \circ \pi_\beta : h' \in \mathscr{H}_\beta\right\} \qquad \alpha \subset \beta. \tag{61}$$

Assume that the family $(m_\alpha : \alpha \in \mathcal{I})$, $m_\alpha \in fa(\mathscr{R}_\alpha)_+$ is projective, i.e.

$$m_\beta(B \times S_{\beta\setminus\alpha}) = m_\alpha(B) \qquad \alpha \subset \beta, \; B \in \Sigma_\alpha. \tag{62}$$

The condition

$$\int h_\alpha dm_\alpha < 0 \quad \textit{implies} \quad \inf_\omega h_\alpha(X_\alpha(\omega)) < 0 \qquad \alpha \in \mathcal{I}, \; h_\alpha \in \mathscr{H}_\alpha \cap L^1(m_\alpha) \tag{63}$$

is equivalent to the existence of a minimal measurable structure $(\mathscr{A}, \mu)$ *on* Ω *such that*

$$\int h_\alpha dm_\alpha = \int h_\alpha(X_\alpha)d\mu \qquad h_\alpha \in \mathscr{H}_\alpha \cap L^1(m_\alpha),\ \alpha \in \mathcal{I}. \tag{64}$$

If, for each $\alpha \in \mathscr{I}$*,* m_α *admits a countably additive extension* $\bar{m}_\alpha$ *to the generated* σ *algebra satisfying* $\bar{m}_{\alpha *}([X_\alpha]^c) = 0$ *then* μ *is countably additive.*

Proof. (64) follows from Corollary 3 once we identify ϕ_α with the m_α expectation, χ_α with the adjoint π_α^* of the projection π_α and upon writing $T_\alpha(h_\alpha) = h_\alpha(X_\alpha)$. In fact,

$$\int h_\alpha dm_\alpha = \int \pi_\alpha^* h_\alpha(X)d\mu = \int h_\alpha(\pi_\alpha(X))d\mu = \int h_\alpha(X_\alpha)d\mu$$

If each $\mathscr{H}_\alpha$ is Stonean then $\mathscr{H} = \bigcup_{\alpha\in\mathscr{I}} \pi_\alpha^*[\mathscr{H}_\alpha]$ is a Stonean vector sublattice of $\mathfrak{F}(S)$. From this we deduce the minimality property. □

The preceding result has immediate implications for Brownian motion.

Corollary 7. *Let* $X = (X_t : t \in \mathbb{R}_+)$ *be brownian motion on some filtered probability space,* $(\Omega, \mathscr{F}, (\mathscr{F}_t : t \in \mathbb{R}_+), P)$ *and let* $(m_{t_1,\ldots,t_n} : t_1, \ldots, t_n \in \mathbb{R}_+)$ *be a projective family with* $m_{t_1,\ldots,t_n} \in fa(\mathscr{B}(\mathbb{R}^n))$*. There exists a probability space* $(\Omega, \mathscr{A}, Q)$ *such that*

$$m_{t_1,\ldots,t_n}(B) = Q(X_{t_1}, \ldots, X_{t_n} \in B) \qquad B \in \mathscr{B}(\mathbb{R}^n). \tag{65}$$

Proof. It is enough to remark that if $\alpha = \{t_1 < \ldots < t_n\} \subset \mathbb{R}_+$, $\{s_1, \ldots, s_n\} \subset \mathbb{R}$ and if $B \subset \mathbb{R}^n$ is an open set containing $\{s_1, \ldots, s_n\}$ then there exist open sets $B_1, \ldots, B_n \subset \mathbb{R}$ such that $s'_i - s'_{i-1} \in B_i$ for $i = 1, \ldots, n$ (and $s'_0 = 0$) implies $\{s'_1, \ldots, s'_n\} \in B$. Given the property of normally distributed, independent increments, $P(X_{t_1}, \ldots, X_{t_n} \in B) > 0$. Thus, the conglomerability condition (63) is satisfied for all h_α continuous. This is enough to prove the claim. □

7. Applications to Probability

In this section we first investigate the classical problem of Skhorohod which has been studied by a number of authors too large to give exact references. We have been influenced by the work of Berti, Pratelli and Rigo [3]. The starting point is the construction of a universal representation for the case of a separable space.

Corollary 8. *Let* $U \in \mathfrak{F}(\Omega)$ *with* $\overline{U[\Omega]}$ *having non empty interior and let* S *be a separable, topological space. There exists a Borel function* $H \in \mathfrak{F}(\mathbb{R}, S)$ *with countable range and such that* $X = H(U)$ *represents* m *relatively to* $\mathscr{C}(S) \cap L^1(m)$ *for every* $m \in fa(\Sigma)_+$*.*

Proof. Given that $[a, b] \subset \overline{U[\Omega]}$ for some $a, b \in \mathbb{R}$ then, upon replacing U with a suitable continuous transformation, we can assume that $\overline{U[\Omega]} = [0, 1]$. Let S_0 be a countable, dense subset of S and $\iota \in \mathfrak{F}(\mathbb{N}, S_0)$ an enumeration of S_0. Define,

$$G(x) = \inf\{n \in \mathbb{N} : 1 - 2^{-n} \geq x\} \qquad x \in (0, 1) \quad \text{and} \quad H = \iota \circ G. \tag{66}$$

H maps $(0,1)$ onto S_0. Moreover, if we endow $\mathbb{N}$ with its power set then both G and ι are Borel – since $G^{-1}(n) = (1-2^{-(n-1)}, 1-2^{-n}]$ – and thus so is H. If $h \in \mathscr{C}(S) \cap L^1(m)$ and $\int hdm < 0$ then $\{h < 0\}$ is an open, non empty subset of S and as such it contains some element $\iota(n_h)$ of S_0. The set $B_h = \{\omega : U(\omega) \in G^{-1}(n_h)\}$ is non empty (as $\overline{U[\Omega]} = [0,1]$) and coincides with $\{\omega : X(\omega) = \iota(n_h)\}$. Thus, $B_h \subset \{h(X) < 0\}$ so that m is X conglomerative relatively to $\mathscr{C}(S) \cap L^1(m)$. □

Corollary 8 extends to the case of finite additivity and of a separable state space the classical idea of generating a random quantity with given distribution by applying to a uniformly distributed random quantity the inverse of the corresponding cumulative density function. Interestingly, we obtain that the *same* function X represents *all* possible distributions relatively to the class of continuous functions and for some suitable set function μ. Let us also mention the possibility of dropping the condition that S is separable by assuming that m is supported by a measurable, separable subset of S.

We highlight the advantage of doing without measurability. Constructing a function such as U in Corollary 8 is a rather trivial exercise as long as Ω has the right cardinality. Requiring that U is uniformly distributed on the unit interval under some classical probability measure P, as in the following Theorem 7, requires, in contrast, additional assumptions – see e.g. [2] where P is taken to be non atomic.

The following result is inspired by [3, theorem 3.1].

Theorem 7. *Let S be a normal, separable topological space and $(\Omega, \mathscr{A}, P)$ a classical probability space supporting a random quantity U uniformly distributed on $(0,1)$. Let either $m \in fa(\Sigma)_+$ be countably additive or S be compact. There exists a Borel function $g \in \mathfrak{F}((0,1), S)$ such that $X = g(U)$ is supported by $(\Omega, \mathscr{A}, P)$ and*

$$\int hdm = \int h(X)dP \qquad h \in \mathscr{C}(S) \cap L^1(m). \tag{67}$$

Proof. If S is compact then the restriction of m to $\mathscr{R}\big(\mathscr{C}(S) \cap L^1(m), m\big)$ is countably additive. Let H be the map defined in (66). Then, as was shown in the proof of Corollary 8, m is H conglomerative relatively to $\mathscr{C}(S)$ so that, by Theorem 4,

$$\int hdm = \int h(H)d\mu \qquad h \in \mathscr{C}(S) \cap L^1(m) \tag{68}$$

for some countably additive $\mu \in fa(\mathscr{R})_+$ and $\mathscr{R}$ an appropriate ring of subsets of $(0,1)$. We claim that $\sigma\mathscr{R} = \mathscr{B}((0,1))$. Recall that $\sigma\mathscr{R}$ is generated by sets of the form $\{h(H) > t\}$ which are Borel since h is continuous and H is Borel. Conversely, if $0 \le a \le b \le 1$ then the set $H[(a,b)]$ is a finite subset of S – and therefore closed. For any other finite subset F of $H[(a,b)^c]$ we can find a function $f \in \mathfrak{F}(S, [0,1])$ such that $f = 1$ on $H[(a,b)]$ and $f = 0$ on F. Thus $(a,b) \subset \{f(H) \ge 1\} \in \sigma\mathscr{R}$. Since $H[(0,1)]$ is countable we find a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of such functions each vanishing on a finite subset of $H[(a,b)^c]$ so that the intersection $\bigcap_n \{f_n(H) \ge 1\}$ is again an element of $\sigma\mathscr{R}$ and coincides with (a,b). In other words, we can assume that μ is defined on the Borel sets of $(0,1)$. From the classical Skhorohod theorem, we deduce the existence of an S valued random quantity Z supported by $((0,1),\mathscr{B}((0,1)),\Lambda)$ (with Λ the Lebesgue measure on $(0,1)$) and admitting μ as its distribution, i.e. $\Lambda^Z = \mu$. On its turn, $\Lambda = P^U$. A repeated application of Theorem 4 with $g = H \circ Z$ and $X = g(U)$ gives

$$\int hdm = \int h(H)d\mu = \int h(g)d\Lambda = \int h(X)dP \qquad h \in \mathscr{C}(S) \cap L^1(m).$$

Thus the random quantity X is supported by $(\Omega,\mathscr{A},P)$ and represents m relatively to $\mathscr{C}(S)$. □

8. Convex Functions

Eventually, we turn attention to convex functions. For $f \in \mathfrak{F}(\mathbb{R})$ we denote by D^+f and D^-f the right and left derivatives and by $f(x+)$ and $f(x-)$ the right and left limits at x, provided such quantities exist.

Theorem 8. *Let $X \in \mathfrak{F}(\Omega)$ and $\varphi \in \mathfrak{F}(\mathbb{R})$ and fix $x_0 \in \operatorname{arginf}_{x\in\overline{\mathbb{R}}} \varphi(x)$. For each $u, v \in \mathbb{R}$ define*

$$h_u^v(x) = (v - x \vee u)^+ \mathbb{1}_{\{x>x_0\}} - (v \wedge x - u)^+ \mathbb{1}_{\{x \le x_0\}} \qquad x \in \mathbb{R}. \tag{69}$$

Let $\mathscr{N}$ be an ideal of subsets of Ω and $\mathscr{R}_X$ the σ ring $\sigma(\{s < X \le t\} : s,t \in \mathbb{R})$. The following properties are mutually equivalent:

(i) φ is convex and $\{u < X < v\} \in \mathscr{N}$ implies $D^-\varphi(v) \le D^+\varphi(u)$;

(ii) there exist $y_0^+, y_0^- \in \mathbb{R}$ and a measurable structure $(\mathscr{R},\lambda)$ on Ω such that (a) $\mathscr{N} \subset \mathscr{R}$ and $\lambda[\mathscr{N}] = \{0\}$, (b) $\lim_n \lambda^*(|X - x_0| < 2^{-n}) = 0$ and (c) for all $v \ge u$

$$h_u^v(X) \in L^1(\lambda) \quad \text{and} \quad \varphi(v) - \varphi(u) = y_0^+ h_u^v(x_0+) + y_0^- h_u^v(x_0) + \int h_u^v(X)d\lambda \tag{70}$$

Moreover, if $x_0 \in \mathbb{R}$ and $\{|X - x_0| < c\} \in \mathscr{N}$ for some $c > 0$ then $y_0^+ = y_0^-$;

(iii) there exist $y_0^+, y_0^- \in \mathbb{R}$ and $\nu \in fa(\mathscr{R}_X)_+$ countably additive such that (a) $\nu(X \in A) = 0$ for A open and $X^{-1}(A) \in \mathscr{N}$ and (b) for all $v \ge u$

$$h_u^v(X) \in L^1(\nu) \quad \text{and} \quad \varphi(v) - \varphi(u) = y_0^+ h_u^v(x_0+) + y_0^- h_u^v(x_0-) + \int h_u^v(X)d\nu \tag{71}$$

Moreover, if $x_0 \in \mathbb{R}$ and $\{|X - x_0| < c\} \in \mathscr{N}$ for some $c > 0$ then $y_0^+ = y_0^-$.

Proof. (i)⇒(ii). Set conventionally

$$D^+\varphi(\infty) = D^-\varphi(\infty) = \lim_{x\to\infty} D^+\varphi(x) \quad \text{and} \quad D^+\varphi(-\infty) = D^-\varphi(-\infty) = \lim_{x\to-\infty} D^+\varphi(x)$$

and observe that necessarily $|D^+\varphi(x_0)|, |D^-\varphi(x_0)| < \infty$. Put $y_0^+ = D^+\varphi(x_0)$ and $y_0^- = D^-\varphi(x_0)$. If $x_0 \in \mathbb{R}$ and $\{u < X < v\} \in \mathscr{N}$ for some $u \le x_0 \le v$ then by assumption $D^+\varphi(x_0) \le D^-\varphi(v) \le D^+\varphi(u) \le D^-\varphi(x_0)$ so that $y_0^+ + y_0^- = 0$. Write $\mathcal{D} = \big\{t : D^-\varphi(t) = D^+\varphi(t)\big\} \cup \{x_0\}$ and define $A_u = \{u < X \le x_0\}$, $A^v = \{x_0 < X \le v\}$ and

$$\mathscr{R}_0 = \Big\{\big(A_u \cap N_u^c\big) \cup \big(A^v \cap N_v^c\big) \cup N : u,v \in \mathcal{D},\ N_u, N_v, N \in \mathscr{N}\Big\}. \tag{72}$$

It is clear that $\mathscr{R}_0$ contains $\mathscr{N}$ (upon taking $u = v = x_0$) as well as $\{A_u, A^v : u, v \in \mathcal{D}\}$. Moreover, it is routine to verify that $\mathscr{R}_0$ is closed with respect to union and intersection with

$$H_1 \cup H_2 = \big(A_{u_1 \wedge u_2} \cap N_u^c\big) \cup \big(A^{v_1 \vee v_2} \cap N_v^c\big) \cup N \tag{73a}$$

$$H_1 \cap H_2 = \big(A_{u_1 \vee u_2} \cap \hat{N}_u^c\big) \cup \big(A^{v_1 \wedge v_2} \cap \hat{N}_v^c\big) \cup \hat{N} \tag{73b}$$

whenever $H_i = \big(A_{u_i} \cap N_{u_i}^c\big) \cup \big(A^{v_i} \cap N_{v_i}^c\big) \cup N_i \in \mathscr{R}_0$ for $i = 1, 2$. Write $F(x) = \big[D^+\varphi(x \vee x_0) - y_0^+\big] + \big[D^-\varphi(x \wedge x_0) - y_0^-\big]$ and

$$\lambda_0(H) = F(v \vee x_0) - F(u \wedge x_0) \quad \text{when} \quad H = (A_u \cap N_u^c) \cup (A^v \cap N_v^c) \cup N \in \mathscr{R}_0. \tag{74}$$

To see that λ_0 is well defined observe that if $u_1 \wedge x_0 < u_2 \wedge x_0$ and

$$\big(A_{u_1} \cap N_{u_1}^c\big) \cup \big(A^{v_1} \cap N_{v_1}^c\big) \cup N_1 = \big(A_{u_2} \cap N_{u_2}^c\big) \cup \big(A^{v_2} \cap N_{v_2}^c\big) \cup N_2 \in \mathscr{R}_0$$

then $\{u_1 \wedge x_0 < X \le u_2 \wedge x_0\} \in \mathscr{N}$. Thus by (i) and the fact that $u_1, u_2 \in \mathcal{D}$ and that $u_1 < x_0$,

$$D^-\varphi(u_1 \wedge x_0) = D^-\varphi(u_2 \wedge x_0) \quad \text{i.e.} \quad F(u_1 \wedge x_0) = F(u_2 \wedge x_0)$$

and likewise $F(v_1 \vee x_0) = F(v_2 \vee x_0)$. In other words $\lambda_0 \in fa(\mathscr{R}_0)_+$ with $\lambda[\mathscr{N}] = \{0\}$. Moreover, if $H_1, H_2 \in \mathscr{R}_0$ then by (73)

$$\begin{aligned}
\lambda_0(H_1) + \lambda_0(H_2) &= F(v_1 \vee x_0) + F(v_2 \vee x_0) - F(u_1 \wedge x_0) - F(u_2 \wedge x_0) \\
&= F(v_1 \vee v_2 \vee x_0) + F((v_1 \wedge v_2) \vee x_0) - F((u_1 \vee u_2) \wedge x_0) - F(u_1 \wedge u_2 \wedge x_0) \\
&= \lambda_0(H_1 \cup H_2) + \lambda_0(H_1 \cap H_2)
\end{aligned}$$

i.e. λ_0 is strongly additive on $\mathscr{R}_0$. It follows from [4, 3.1.6 and 3.2.4] that λ_0 admits a unique extension $\lambda_1 \in fa(\mathscr{R}_1)_+$ to the generated ring $\mathscr{R}_1$. Let I be an interval with endpoints in $\mathbb{R} \cup \{x_0\}$. Given that $\mathcal{D}$ is dense in $\mathbb{R} \cup \{x_0\}$, $\lambda^*(X \in I) < \infty$. By [4, 3.4.1 and 3.4.4] we obtain a further extension $\lambda \in fa(\mathscr{R})_+$ to the ring $\mathscr{R} = \big\{A \subset \Omega : \lambda_1^*(A) < \infty\big\}$. Then $\{X \in I\} \in \mathscr{R}$ and $X\mathbb{1}_I(X)$ is λ-measurable whenever I is as above, by Lemma 1. Therefore,

$$\begin{aligned}
\int_{u \vee x_0}^{v \vee x_0} D^+\varphi(t)dt &= y_0^+ h_u^v(x_0+) + \int_{u \vee x_0}^{v \vee x_0} \mathbb{1}_{\mathcal{D}}[D^+\varphi(t) - D^+\varphi(x_0)]dt \\
&= y_0^+ h_u^v(x_0+) + \int_u^v \mathbb{1}_{\mathcal{D}} \lambda_1(x_0 < X \le t)dt \\
&= y_0^+ h_u^v(x_0+) + \int_u^v \lambda(x_0 < X \le t)dt \\
&= y_0^+ h_u^v(x_0+) + \int_{x_0}^{\infty} (v - u \vee X)^+ d\lambda \qquad \text{(by Lemma 2)}
\end{aligned}$$

and similarly $\int_{u \wedge x_0}^{v \wedge x_0} D^+\varphi(t)dt = y_0^- h_u^v(x_0) - \int_{-\infty}^{x_0} (v \wedge X - u)^+ d\lambda$. We conclude

$$\varphi(v) - \varphi(u) = \int_{u \vee x_0}^{v \vee x_0} D^+\varphi(t)dt + \int_{u \wedge x_0}^{v \wedge x_0} D^-\varphi(t)dt = y_0^+ h_u^v(x_0+) + y_0^- h_u^v(x_0) + \int h_u^v(X)d\lambda.$$

Fix an increasing $\langle u_n \rangle_{n\in\mathbb{N}}$ and a decreasing $\langle v_n \rangle_{n\in\mathbb{N}}$ sequence in $\mathcal{D}$ converging to x_0, with $u_n < u_{n+1} < x_0$ if $x_0 > -\infty$ and $v_n > v_{n+1} > x_0$ if $x_0 < \infty$. Then,

$$\lim_n \lambda^*(u_n < X < v_n) \leq \lim_n D^+\varphi(v_n) - D^+\varphi(x_0) - D^-\varphi(u_n) + D^-\varphi(x_0) = 0$$

so that $\lim_n \lambda^*(|X - x_0| < 2^{-n}) = 0$.

(ii)⇒*(iii)*. Let u_n and v_n be as above and define $h_u^v(x;n) = h_u^v(x)$ for $x \notin (u_n, v_n]$ or else

$$h_u^v(x;n) = h_u^v(u_n)\frac{u_{n+1}-x}{u_{n+1}-u_n} \quad \text{if} \quad u_n < x \leq u_{n+1}$$
$$h_u^v(x;n) = h_u^v(v_n)\frac{x-v_{n+1}}{v_n-v_{n+1}} \quad \text{if} \quad v_{n+1} < x \leq v_n$$

$h_u^v(\cdot;n)$ is a continuous function vanishing outside of the interval $[u \wedge v_{n+1}, v \vee u_{n+1}]$. Moreover: (a) $\left\{\left|h_u^v(x;n) - h_u^v(x)\right| > c\right\} \subset (u_n, v_n]$ so that $h_u^v(X;n)$ is λ-convergent to $h_u^v(X)$, (b) $\left|h_u^v(x;n)\right| \leq \left|h_u^v(x;n+1)\right| \leq \left|h_u^v(x)\right|$, (c) $\lim_n h_u^v(x;n) = h_u^v(x)$ for all $x \neq x_0$ and (d) $h_u^v(X;n)$ is λ-measurable and therefore an element of $L^1(\lambda)$. Let ν be the conventional companion of λ relatively to the family $\{h(X) : h \in \mathscr{C}_K(\mathbb{R})\}$. Observe that if $x_0 \in \mathbb{R}$ and $h_n \in \mathscr{C}_K(\mathbb{R})$ is such that $\mathbb{1}_{(u_n,v_n]} \geq h_n \geq \mathbb{1}_{(u_{n+1},v_{n+1}]}$, then

$$\nu^*(\{X = x_0\}) \leq \lim_n \int h_n(X)d\lambda \leq \lim_n \lambda(u_n < X \leq v_n) = 0$$

It follows that

$$\int h_u^v(X)d\lambda = \lim_n \int h_u^v(X;n)d\lambda = \lim_n \int h_u^v(X;n)d\nu = \int h_u^v(X)d\nu$$

Let $I \subset \mathbb{R}$ be an open interval with $X^{-1}(I) \in \mathscr{N}$ and $\langle g_n \rangle_{n\in\mathbb{N}}$ a sequence of non negative, continuous functions which increases to $\mathbb{1}_I$. It is then obvious that

$$0 = \lim_n \int g_n(X)d\lambda = \lim_n \int g_n(X)d\nu = \nu(X \in I).$$

The conclusion extends to open sets.

(iii)⇒*(i)*. If φ satisfies (71) it is clearly convex since the function $v \to h_u^v(x)$ is convex for every $u \leq v$. Assume that $u < v$ and $\{u < X < v\} \in \mathscr{N}$. Then, $\nu(u < X < v) = 0$ so that, for arbitrary $u < t < v$

$$\frac{\varphi(v)-\varphi(u)}{v-u} = \begin{cases} y_0^+ + \nu(x_0 \leq X < t) & \text{if} \quad v > u \geq x_0 \\ y_0^- + \nu(t \leq X < x_0) & \text{if} \quad x_0 \geq v > u \\ y_0^+ & \text{if} \quad v > x_0 > u \end{cases} \tag{75}$$

and (i) follows. □

The above result can be stated in a slightly different way:

Corollary 9. *Let $X \in \mathfrak{F}(\Omega)$ with $\overline{X[\Omega]} = \mathbb{R}$, $\varphi \in \mathfrak{F}(\mathbb{R})$ and define x_0 and h_u^v as in Theorem 8. φ is convex if and only if there exist $y_0^+, y_0^- \in \mathbb{R}$ and a countably additive, measurable structure $(\mathscr{R}, \nu)$*

on Ω *such that* $\nu(u < X < v) = 0$ *whenever* $D^+\varphi(v) \le D^-\varphi(u)$ *and*

$$h_u^v(X) \in L^1(\nu) \quad \text{and} \quad \varphi(v) - \varphi(u) = y_0^+ h_u^v(x_0+) + y_0^- h_u^v(x_0-) + \int h_u^v(X) d\nu \qquad v \ge u \tag{76}$$

where $y_0^+ = y_0^-$ *if* $x_0 \in \mathbb{R}$ *and* $D^+\varphi(x_0 + c) = D^-\varphi(x_0 - c)$ *for some* $c > 0$.

Proof. Define $\mathscr{N} = \big\{\{u < X < v\} : u, v \in \mathbb{R}, D^+\varphi(v) \le D^-\varphi(u)\big\}$. From $\overline{X[\Omega]} = \mathbb{R}$ follows that $\{u < X < v\} \in \mathscr{N}$ if and only if $D^+\varphi(v) \le D^-\varphi(u)$ and that $\mathscr{N}$ is an ideal of sets. Then (76) follows from Theorem 8.(*iii*). □

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