DEMS WORKING PAPER SERIES

Solidarity and efficiency in preference aggregation: a tale of two rules

Stergios Athanaglosou
No. 390 – November 2018

Solidarity and efficiency in preference aggregation: a tale of two rules

Stergios Athanasoglou[*]

December 2017; revised November 2018

Abstract

This paper is concerned with preference-aggregation rules satisfying desirable efficiency and solidarity requirements. We formulate weaker versions of existing solidarity axioms and show how they imply, in conjunction with strategy-proofness, the existence of reference outcomes holding privileged status. We propose a new class of rules, *fixed order status-quo rules*, that can be productively contrasted to their closest counterparts in the literature, status-quo rules based on the least upper bound of a lattice. Fixed order status-quo rules satisfy stronger efficiency requirements than lattice status-quo rules but have weaker, though still significant, solidarity properties. A subfamily based on lexicographic orders is analyzed further. Fixed order status-quo rules are characterized by strategy-proofness, strong efficiency, and a third axiom, unanimity-basedness.

Keywords: social choice, preference aggregation, solidarity, efficiency, strategy-proofness, status quo rules
JEL classifications: D71, C70

---

[*]Department of Economics, University of Milan - Bicocca, stergios.athanasoglou@unimib.it.

1 Introduction

Consider a group of agents submitting linear orderings over a set of alternatives. A *rule* is a function synthesizing the agents' individual preferences into a single social ordering. Originating in the seminal work of Arrow [3], this modeling framework corresponds to a variety of decision-making settings. For example, academic departments might wish to determine an ordered list of job-market candidates on the basis of the faculty members' preferences. As the availability of candidates is not known in advance, a single name will not do and an ordered list is called for.

Arrovian preference aggregation runs into difficulties when issues of interpersonal comparability come into play. Orderings are relatively complex mathematical objects and determining when one is better than another on the basis of an agent's preferences is not straightforward. This has complicated the formulation of efficiency criteria and impeded the study of strategic interaction within the model. For if we cannot readily compare two orderings how are we to know whether an agent will strategically misrepresent his preferences to obtain one over another?

To address this issue of comparability, Bossert and Sprumont [5] broke important new ground by employing a betweeness relation originally due to Grandmont [14]. Specifically, they defined the *prudent extension* of an agent's preferences over alternatives to her preferences over orderings. According to the prudent extension, an ordering is deemed at least as good as another if and only if, when considering the agent's own preferences, it unambiguously dominates it (hence the adjective "prudent"). This dominance occurs when *all* pairs of alternatives that the agents' preferences have in common with one ordering are *also* present in the other. This way of comparing orderings naturally lead to efficiency and strategy-proofness criteria that were subsequently used by Bossert and Sprumont to analyze, and in some cases axiomatize, three classes of rules (monotonic majority alteration, status-quo, and Condorcet-Kemeny).

Along with efficiency and strategy-proofness, fairness is a critical component of any mechanism. Within the preference aggregation setting, we follow Harless [10] and channel fairness as a concern for *solidarity*. Solidarity "embodies the idea of a common endeavor and shared outcome" (Harless [10], p. 74) and thus holds considerable appeal in the design of social policy. At a high level, solidarity requires that agents be affected in the same manner by events they cannot control. Different meanings given to the components of this qualitative definition lead to different formal concepts. Bossert and Sprumont considered the property of *population monotonicity* whereby agents are required to be unambiguously at least as well off if a subset of their peers choose to

depart the aggregation process. Along a similar vein, Harless [10] proposed an axiom of *welfare dominance*, a property originating in models of binary choice (Moulin [17]), stipulating that when an agent changes her preferences, all others must be affected in the same unambiguous way. If the outcome changes, either they all clearly benefit or they all clearly lose.

The starkness of unambiguous comparisons based on the prudent extension means that population monotonicity and welfare dominance are very strong properties. Indeed, when combined with mild efficiency requirements, welfare dominance delivers an impossibility result (Harless [10]). A more positive picture emerges if we apply population monotonicity, as efficiency and population monotonicity uniquely characterize a class of rules which Bossert and Sprumont named *status-quo rules* [5]. A similar characterization obtains when efficiency is coupled with a significant weakening of welfare dominance known as *adjacent welfare dominance* (which restricts attention to changes in preferences that involve a single adjacent pair of alternatives in the ordering of a single agent [10]). All of the above characterizations and impossibilties are consistent with findings in different models of social choice (e.g., Gordon [11, 12, 13], Harless [9]) in which solidarity principles significantly restrict the set of efficient aggregation procedures.

Status-quo rules are designed to Pareto-improve upon an exogenous reference ordering. They do so by considering agent preferences and taking the least upper bound of a suitably-defined lattice (for reasons that will become clear, from now on we refer to these rules as *lattice status-quo rules*). Lattice status-quo rules satisfy group strategy-proofness, but fail a stricter standard of efficiency known as *strong efficiency*. Strong efficiency ensures that when considering any binary comparison of alternatives, the rule respect the unanimous wishes of the population. And while violations of this property don't necessarily affect the rule's efficiency in a Pareto sense, they do occasionally lead to outcomes that are hard to square with intuitive notions of efficient aggregation. Pathological examples of such undesirable situations are easy to construct (see Athanasoglou [4] and Example 4 in the present paper).

Contribution. Considering the above discussion, this paper poses a simple question. Keeping our non-manipulability criterion fixed, what happens if we strengthen our efficiency requirements and partially relax our concern for solidarity? Do other rules emerge? If so, how are they related to lattice status-quo rules?

The efficiency improvement we employ in our inquiry is the strong efficiency criterion mentioned earlier. As regards solidarity, we introduce novel weaker versions of popu-

lation monotonicity and welfare dominance. *Weak population monotonicity* requires that the departure of an agent not make some of the remaining agents unambiguously better off while others unambiguously worse off. Similarly, *weak welfare dominance* requires that a change in preferences by one agent not leave some agents unambiguously better off while others unambiguously worse off. Middle-of-the-road situations in which preference changes/agent departures lead to new outcomes that are not directly comparable (in the prudent extension sense) to current ones are allowed.

Weak solidarity criteria can be interpreted as a weak version of the *envy-freeness* requirement often encountered in social choice. If they are met, no agent would strongly desire to trade places with another, were the aggregation outcome to change due to factors that are out of her control. Though milder than their strong counterparts, the constraints they impose on the aggregation process are not trivial; it is easy to show that, except for lattice status-quo rules, none of the other strategy-proof rules considered by Bossert and Sprumont satisfy them. In fact, when combined with strategy-proofness, weak solidarity implies the existence of a *reference* ordering enjoying privileged status in the aggregation procedure. This ordering is such that, if the preferences of at least one agent coincide with it, then the rule must pick it. If strong efficiency is added to the rule's requirements then a structured set of reference orderings, to accommodate a number of possible patterns of unanimous agreement, emerges.

Weakening the solidarity requirement gives rise to a different kind of status-quo rules, *fixed order status-quo rules*. These rules take into account an exogenous, appropriately defined *partial order* over orderings, and select the first-ranked element on this list consistent with strong efficiency.[1] The set of admissible partial orders over orderings is very broad, and encompasses all linear orders. By construction, fixed order status-quo rules are strongly efficient. They are also shown to satisfy group strategy-proofness, thus retaining the incentive properties of their lattice counterparts. As regards solidarity, they satisfy the stronger group versions of weak population monotonicity and weak welfare dominance.

We proceed to analyze further a compelling subfamily of fixed order status-quo rules in which the partial order over orderings has lexicographic structure. These rules imply an aggregation operator that, subject to respecting strong efficiency, stays as close as

---

[1]In fairness, it should be noted that a sub-class of fixed order status-quo rules were mentioned once previously in the literature, in the concluding section of Bossert and Storcken [6], where the authors briefly discuss the tightness of their impossibility results. Though familiar with the Bossert-Storcken paper, I became aware of this fact only after having identified these rules myself via numerical simulations.

possible, in the lexicographic sense, to a given reference ordering. Evidently, greater importance is placed on adhering to the recommendations of the status-quo regarding its top-ranked alternatives, a feature that may be desirable in several practical settings. An additional appealing property of lexicographic status quo rules is that they can be implemented with an efficient polynomial-time algorithm. This is not the norm for preference aggregation which tends to produce difficult combinatorial problems [7].

Finally, we show that fixed order status-quo rules are characterized by strategy-proofness, strong efficiency and a third axiom, *unanimity-basedness*. Unanimity-basedness is a technical condition imposing that the rule be robust to changes in preferences that do not alter the structure of unanimous agreement in the population. Lattice-status quo rules are also shown to satisfy it.

Relation to earlier literature. The interplay between solidarity and efficiency has been the focus of earlier work in social choice (e.g., Thomson [21], Ching and Thomson [8], Gordon [11]). Consistent to the preference aggregation setting, these early papers were concerned with models in which agents submit linear orderings over alternatives. Their main objective was to explore the compatibility of efficiency and solidarity in a variety of contexts. An important difference between them and the current work regards the outcome space: in [21, 8] agent preferences were mapped to the unit interval, while in [11] to an arbitrary set.

Along related lines, preference aggregation can be viewed as a special case of attribute-based social choice models (Nehring and Puppe [18], Gordon [13]). Here, the preference space involves a collection of binary attributes that may be logically related to one another. An implication of the results in [18, 13] is that, in the context of preference aggregation, solidarity is incompatible with very mild notions of efficiency such as voter sovereignty (i.e., onto-ness) and unanimity. To overcome this impossibility, Gordon [13] proposed a family of rules that is similar to those explored in the present work, *unanimity rules*, for all attribute domains in which they are well-defined. These rules consider a target outcome on whose binary attributes all agents vote separately. For a given attribute, the corresponding target outcome is changed only if all agents prefer its complement. When they are well-defined, these rules are characterized by a combination of appealing solidarity, incentive-compatibility, and efficiency axioms.

In line with the impossibility result of [18, 13] cited above, unanimity rules are not well-defined in the preference aggregation domain. The primary reason behind this negative result is the assumption of complete preferences over the outcome space. This element suggests a unifying theme to the recent literature on preference aggregation that

this paper contributes to. Bossert and Sprumont, Harless [10] and the current study all attempt to overcome, in different ways, the difficulties posed by complete preferences by (i) considering incomplete preferences over the outcome space and (ii) modifying the efficiency and solidarity axioms. All three are ultimately successful in producing possibility results.

Paper outline. The paper is organized as follows. Section 2 presents the formal model and analyzes the relevant notions of efficiency, solidarity, and strategy-proofness and their implications. Section 3 introduces lattice- and fixed order status-quo rules and establishes their formal properties. It also discusses a compelling subfamily of rules based on lexicographic orders. Section 4 offers a characterization of fixed order status-quo rules and Section 5 presents some concluding thoughts. The Appendix collects all proofs.

2 Model description

Let A denote a finite set of $m \geq 3$ alternatives and N a finite set of $n \geq 2$ agents. N is allowed to vary in the countable set $\mathcal{N}$.

Agents in N submit **linear orderings** over alternatives in A (i.e., complete, transitive, and antisymmetric binary relations) and the set of such linear orderings is denoted by $\mathcal{R}$. From now on we simply refer to elements of $\mathcal{R}$ as **orderings**. Given an ordering R and a pair of alternatives (a, b), we use the notation $(a, b) \in R$ to indicate that a is at least as good as b according to R. A preference profile $R_N = (R_1, R_2, ..., R_n)$ is an n-tuple of orderings, representing the preferences of all agents in N (R_i denotes the preferences of agent $i \in N$). Given a population $N \subset \mathcal{N}$, the corresponding set of preference profiles is given by $\mathcal{R}^N$. An **rule** is a function $f : \bigcup_{N \in \mathcal{N}} \mathcal{R}^N \mapsto \mathcal{R}$, assigning to each preference profile an ordering.

Consistent with Grandmont [14], for any $R, R', R'' \in \mathcal{R}$, we say that R'' is **between** R and R', and write $R'' \in [R, R']$, if and only if R'' agrees with R whenever the latter agrees with R'. That is, $R'' \in [R, R']$ if and only if $R'' \supseteq R \cap R'$. Bossert and Sprumont [5] defined the **prudent extension** of $R \in \mathcal{R}$ as the binary relation $\mathbf{R}$ over $\mathcal{R}$ given by

$$R'' \ \mathbf{R} \ R' \ \Leftrightarrow R'' \in [R, R'], \ \text{ for all } R'', R' \in \mathcal{R}.$$

Hence, for an agent holding preferences R, R'' is at least as good as R' if and only if R'' is between R and R', and it is strictly better if also $R'' \neq R'$. Since not all pairs

of orderings are comparable in this way, the relation **R** is a **partial order** (i.e., a reflexive, transitive, and antisymmetric binary relation) on $\mathcal{R}$. Indeed, the relation **R** ranks one ordering over another if and only if the former *unambiguously dominates* the latter. As such, it can be thought of as the most conservative relation over orderings that is consistent with an agent having preferences R.

2.1 Axioms

Let f denote a rule. We begin with a discussion of concepts relating to efficiency. Bossert and Sprumont used the prudent relation to propose the following property.

Efficiency. There do not exist $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$, and $R' \in \mathcal{R}$ such that $R' \in [R_i, f(R_N)]$ for all $i \in N$ and $R' \neq f(R_N)$.

Thus, a rule satisfies efficiency if it produces an ordering such that there exists no other ordering that all agents unambiguously prefer to it. It thus provides an analog of Pareto efficiency for the economic environment at hand.

An additional property that Bossert and Sprumont discussed is *strong efficiency* (they referred to it as "local unanimity" – we adopt the arguably better-suited name used by Harless [10]).

Strong efficiency. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, f(R_N) \supseteq \bigcap_{i \in N} R_i$.

Introduced by Arrow [3], strong efficiency applies to preference profiles in which there is unanimous agreement over individual binary comparisons. When such unanimous agreement is present, strong efficiency requires that the rule respect its wishes. As its name suggests, strong efficiency implies efficiency (see footnote 11 in Harless [10]).[2] The reverse direction can be easily seen not to hold [5, 4].

We now define our notion of non-manipulability. Given $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$ and $R_i' \in \mathcal{R}$, the notation $\left(R_i', R_{N \setminus i} \right)$ denotes the profile that is identical to R_N except for the preferences of agent $i \in N$ that are equal to R_i'.

Strategy-proofness. There do not exist $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, i \in N, R_i' \in \mathcal{R}$ such that $f(R_i', R_{N \setminus i}) \in [R_i, f(R_N)]$ and $f(R_i', R_{N \setminus i}) \neq f(R_N)$.

Strategy-proofness ensures that by misreporting one's preferences it is not possible to obtain an ordering that unambiguously dominates that produced under truthfulness. It imposes a minimal standard of non-manipulability.

---

[2]Note that Bossert and Sprumont [5] contains a typo in this regard, as it claims the two properties are independent.

7

Let us now discuss properties pertaining to solidarity. We begin with the strong solidarity properties discussed by Harless [10].

Welfare dominance. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$, $i \in N$, $R_i' \in \mathcal{R}$, either $f(R_N) \in \bigcap_{j \in N \setminus i} [R_j, f(R_i', R_{N \setminus i})]$ or $f(R_i', R_{N \setminus i}) \in \bigcap_{j \in N \setminus i} [R_j, f(R_N)]$.

Hence, *welfare dominance* requires that whenever an agent changes his preferences then –in the event of a change in outcome– all other agents be affected in the same *unequivocal* way: either they all unambiguously benefit or they all unambiguously lose. In particular, there is no room for before/after outcomes that are not comparable via Bossert and Sprumont's prudent extension.

As mentioned in the introduction, welfare dominance is a strong requirement that is incompatible with efficiency [10]. Given this impossibility result, we are interested in relaxations of welfare dominance that preserve concern for solidarity while allowing for some measure of efficiency. Harless [10] tackled this question by considering a version of welfare dominance in which before/after unambiguous comparisons are required only for a restricted domain of preference changes involving a single agent and a single adjacent pair of alternatives.

Adjacent welfare dominance. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$, $i \in N$, and $R_i' \in \mathcal{R}$ such that R_i' and R_i are adjacent, either $f(R_N) \in \bigcap_{j \in N \setminus i} [R_j, f(R_i', R_{N \setminus i})]$ or $f(R_i', R_{N \setminus i}) \in \bigcap_{j \in N \setminus i} [R_j, f(R_N)]$.

Harless showed that adjacent welfare dominance is compatible with efficiency but cannot be reconciled with strong efficiency. In addition, as mentioned in the introduction, he characterized lattice status-quo rules with these two properties.

We pursue a different relaxation of welfare dominance and look for criteria that leave the domain of preference changes intact but weaken the before/after dominance requirement. To this end, we introduce the following concept of solidarity.

Weak welfare dominance. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$, $i \in N$, $R_i' \in \mathcal{R}$, there do not exist $j, k \in N \setminus i$ such that $f(R_N) \in [R_j, f(R_i', R_{N \setminus i})]$ and $f(R_i', R_{N \setminus i}) \in [R_k, f(R_N)]$ and $f(R_N) \neq f(R_i', R_{N \setminus i})$.

Weak welfare dominance requires that a change in an agent's preferences not result in diametrically-opposed outcomes for some of the remaining agents. In particular, it cannot be the case that some agents unambiguously benefit from this change while others unambiguously lose. This element suggests an interpretation of weak welfare dominance as a form of *envy-freeness*: if an agent changes her preferences, none of the remaining

ones should strongly wish to switch identities with another. Note that middle-of-the-road situations in which the before/after outcomes are not comparable via the prudent extension are allowed.

Weak welfare dominance is clearly implied by welfare dominance. On the other hand, it is logically unrelated to adjacent welfare dominance. Which of the two properties is the more compelling is, to some extent, a subjective matter having to do with the relative importance we place on the two pillars of the solidarity principle. If we value the sharpness of the before/after dominance criterion more highly than the comprehensiveness of the before/after changes in preferences, then adjacent welfare dominance resonates more strongly than weak welfare dominance. Conversely, if we care more about covering all possible preference changes than we do about ensuring ultra-robust before/after comparisons, then we should prefer weak welfare dominance.[3]

Related solidarity properties, both strong and weak, emerge when agents enter and exit the population.

Population monotonicity. For all $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$, $f(R_{N \setminus i}) \in \bigcap_{j \in N \setminus i} [R_j, f(R_N)]$.[4]

Weak population monotonicity. For all $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $i \in N$, there do not exist $j, k \in N \setminus i$ such that $f(R_N) \in [R_j, f(R_{N \setminus i})]$ and $f(R_{N \setminus i}) \in [R_k, f(R_N)]$ and $f(R_N) \neq f(R_{N \setminus i})$.

Population monotonicity was first introduced by Thomson [19, 20] in a bargaining context. It stipulates that when an agent departs from the population, all remaining agents must find the new ordering unambiguouly at least as good as the previous one. The starkness of this before/after comparison renders population monotonocity a very strong requirement. Analogously to adjacent welfare dominance, population monotonicity is compatible with efficiency but cannot be reconciled with strong efficiency (Bossert and Sprumont [5]). Moreover, again echoing earlier findings, efficiency combined with population monotonicity characterize lattice status-quo rules [5].

As before, we propose a weakening of this solidarity property by relaxing the stringency of before/after comparisons. Weak population monotonicity imposes that, when an agent departs, no agents are left unambiguously worse off while others are left un-

---

[3]Especially since weak welfare dominance can be meaningfully strengthened to cover all possible changes in preferences by *groups* of agents (see next Section).

[4]Note how population monotonicity implies the stronger conclusion that for all $S \subset N$ we have: $f(R_{N \setminus S}) \in \bigcap_{j \in N \setminus S} [R_j, f(R_N)]$.

ambiguously better off. It is clearly implied by population monotonicity and the envy-freeness interpretation that we discussed in the case of weak welfare dominance carries over to this property too.

2.2 How discriminating are weak notions of solidarity?

The solidarity requirements just presented, while weaker than their stronger counterparts, retain a fair amount of bite. Indeed, with the exception of lattice status-quo rules, none of the other classes of strategy-proof rules examined by Bossert and Sprumont (i.e., Condorcet-Kemeny and monotonic majority alteration) satisfy them. Before showing why this is true definitions are in order.

We begin with Condorcet-Kemeny rules, originating in the influential writings of Condorcet in the 18th century. In modern times these rules were formalized by Kemeny [16] and axiomatized by Young and Levenglick [22]. Given two orderings $R, R' \in \mathcal{R}$, let $D(R, R') = (R \setminus R') \cup (R' \setminus R)$. The **Kemeny distance** between R and R', denoted by $\delta(R, R')$, is defined as $\delta(R, R') = |D(R, R')|$. Let $\succeq$ be an ordering on $\mathcal{R}$. Given $N \subset \mathcal{N}, R_N \in \mathcal{R}^N$, let

$$K(R_N) = \arg\min_{R \in \mathcal{R}} \sum_{i \in N} \delta(R, R_i).$$

The $\succeq$-**Condorcet-Kemeny rule** is defined as the aggregation rule which assigns to each $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$ the strict ordering belonging to $K(R_N)$ ranked first according to $\succeq$.

We now turn to monotonic majority alteration rules, introduced by Bossert and Sprumont. Given $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$, the **majority relation** $M(R_N)$ on A is a complete and reflexive relation such that, for all $(a, b) \in A \times A$, we have $a\ M(R_N)\ b$ if and only if

$$|\{i \in N : aR_ib\}| \geq |\{i \in N : bR_ia\}|.$$

The majority relation is antisymmetric when the number of agents is odd but may fail to be so for n even. Thus, a tie-breaking rule is needed: Given a *tournament* (i.e., a complete, transitive, and asymmetric relation) $\succeq$ on A, define $M_{\succeq}(R_N)$ on A to be a complete, reflexive and antisymmetric binary relation such that, for all $(a, b) \in A \times A$, we have $a\ M_{\succeq}(R_N)\ b$ if and only if

$$|\{i \in N : aR_ib\}| > |\{i \in N : bR_ia\}|,$$

$$\text{or} \quad |\{i \in N : aR_ib\}| = |\{i \in N : bR_ia\}| \quad \text{and} \quad a \succeq b.$$

Clearly, $M_\succeq(R_N)$ can fail to be transitive and thus may not always lead to an ordering. A **monotonic majority alteration rule** alters $M_\succeq(R_N)$ to obtain a transitive relation (and thus a unique ordering) in a way that is *agreement-monotonic* (for detailed definitions see Section 4 in [5]). Furthermore, if $M_\succeq(R_N)$ is itself an ordering, then the monotonic majority alteration rule must pick it.

Proposition 1 *No Condorcet-Kemeny or monotonic majority alteration rule satisfies either weak welfare dominance or weak population monotonicity.*

2.3   Implications of weak solidarity: emergence of a reference ordering

In this section we explore some important implications of the weak solidarity axioms we have just introduced. The main result we obtain is that weak solidarity in combination with strategy-proofness implies the existence of an ordering enjoying privileged status. This ordering may be interpreted as a *reference* outcome that the aggregation process must favor when determining the social preference.

Theorem 1 *Suppose rule f satisfies statety-proofness, weak welfare dominance and weak population monotonicity. There exists a **reference** ordering R^0 such that for all $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$, we have $f(R_N) = R^0$ whenever there exists $i \in N$ such that $R_i = R^0$.*

Theorem 1 demonstrates that weak solidarity combined with strategy-proofness imply the existence of an ordering R^0 that functions as a reference point for the rule. The defining characteristic of this ordering is that, as soon as there exists at least one agent whose preferences coincide with it, the rule must select it. Thus, two potentially very different profiles must yield the same aggregate outcome, provided they both include at least one agent with preferences R^0.

While Theorem 1 establishes an important invariance constraint on the set of strategy-proof rules satisfying weak solidarity, it does not offer guidance for profiles in which no agent holds preferences R^0. The following result shows how adding strong efficiency to the rule requirements helps, at least partially, in addressing this issue.

Proposition 2 *Suppose rule f satisfies statety-proofness, weak welfare dominance, weak population monotonicity and strong efficiency. Suppose U is a partial order on A such that*

(i) for all $R \in \mathcal{R}$ such that $R \supseteq U$ there exists $R' \in \mathcal{R}$ such that $R \cap R' = U$ ("cycle" partial orders), or

(ii) there exist exactly two orderings R^1, R^2 satisfying $R^1 \cap R^2 = U$ and for all $R, R' \in \mathcal{R}$ satisfying $R', R'' \supseteq U$ either $R' \in [R^1, R'']$ and $R'' \in [R', R^2]$, or $R'' \in [R^1, R']$ and $R' \in [R'', R^2]$ ("chain" partial orders).

There exists a reference ordering R^u such that, for all $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$ satisfying $\bigcap_{l \in N} R_l = U$, we have $f(R_N) = R^u$ whenever there exists $i \in N$ such that $R_i = R^u$.

Proposition 2 establishes that, when combined with strong efficiency, weak solidarity and strategy-proofness extend the existence of reference orderings to profiles in which no agents hold preferences R^0 and/or where picking R^0 would violate strong efficiency. These profiles have a specific kind of structure as they imply patterns of unanimous agreement that are "cyclic" or "chain-like". Figure 1 provides illustrative examples for the case $m = 4$.[5]
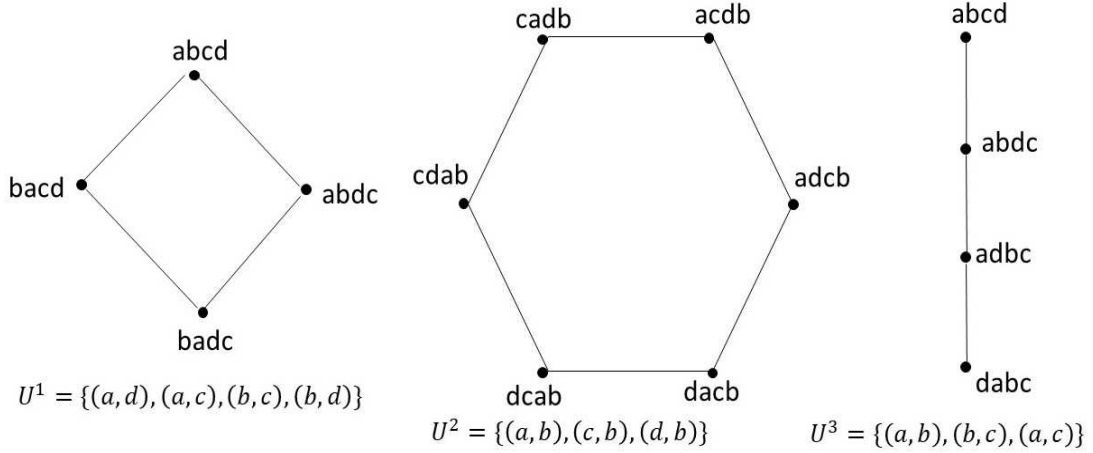

Figure 1: Let $m = 4$. Partial orders U^1, U^2 [resp., U^3] satisfy the assumptions of part (i) [resp., (ii)] of Proposition 2.

Conversely, Figure 2 provides two examples of partial orders that violate the assumptions of Proposition 2.

---

[5]To avoid excessive clutter we omit all pairs of identical alternatives in the statement of partial orders U. We continue this practice repeatedly throughout the text.
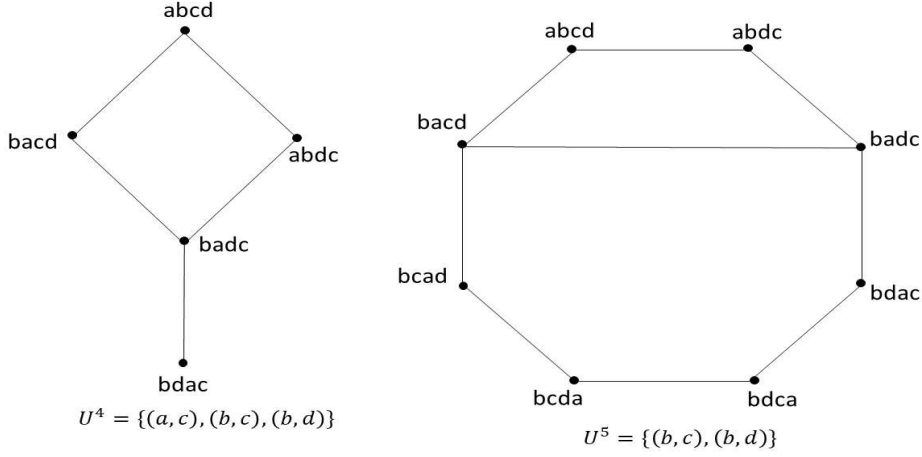
Figure 2: Let $m = 4$. Examples of partial orders violating the assumptions of Proposition 2.

Finally, when there are only $m = 3$ alternatives, reference orderings can be defined for all patterns of unanimous agreement.

Corollary 1 *Suppose $m = 3$ and rule f satisfies stategy-proofness, weak welfare dominance, weak population monotonicity and strong efficiency. For all partial orders U on A there exists a reference ordering R^u such that, for all $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$ satisfying $\bigcap_{l \in N} R_l = U$, we have $f(R_N) = R^u$ whenever there exists $i \in N$ such that $R_i = R^u$.*

Example 1. Let $A = \{a, b, c\}$ and suppose f satisfies the conditions of Theorem 1. There will exist an ordering R^0, say $R^0 = abc$, such that whenever at least one agent has preferences R^0, the rule must pick it. Now, suppose that f is also strongly efficient. Consider the partial order on A given by $U = \{(b, a)\}$. What Corollary 1 says is the following: There exists another ordering R^u, say $R^u = bca$, such that for all $N \subset \mathcal{N}$ and profiles R_N satisfying $\bigcap_{l \in N} R_l = \{(b, a)\}$, if there exists an agent $i \in N$ with preferences $R_i = R^u = bca$ then $f(R_N) = bca$. ∎

3 Status-quo rules

We begin this section with a formal definition of two rules that meet various efficiency and solidarity properties. The common thread running through them is the existence of an exogenous reference outcome that they seek to Pareto-improve upon.

3.1 Improving upon a status quo in two different ways

3.1.1 Lattice status-quo rules

These rules were introduced in Bossert and Sprumont [5] and refined and extended by Harless [10]. Given an ordering $R^0 \in \mathcal{R}$ and its prudent extension $\mathbf{R}^0$, Guilbaud and Rosenstiehl [15] proved that $(\mathcal{R}, \mathbf{R}^0)$ is a *lattice* so that every collection $\{R^1, R^2, ..., R^T\} \subseteq 2^{\mathcal{R}}$ has a unique *least upper bound*, i.e., a unique ordering $R \in \mathcal{R}$ such that

$$R \, \mathbf{R}^0 \, R^t, \ \text{ for all } t \in \{1, 2, ..., T\}, \tag{1}$$

and

$$\left[R' \, \mathbf{R}^0 \, R^t, \ \text{ for all } t \in \{1, 2, ..., T\} \right] \ \Rightarrow R' \, \mathbf{R}^0 \, R. \tag{2}$$

With this notion at hand, f is a **lattice status-quo rule** with reference ordering R^0 if, for all $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $f(R_N)$ equals the unique ordering satisfying conditions (1)-(2) when applied to the collection $\{R_i : \ i \in N\}$. [6]

3.1.2 Fixed order status-quo rules

We now turn to the second, novel class of rules. Before proceeding, we define a class of partial orders on $\mathcal{R}$ that will prove essential.

A partial order $\succeq$ on $\mathcal{R}$ is **conclusive** if, for every partial order U on A, there exists (a unique) $R^u \in \mathcal{R}$ such that (i) $R^u \supseteq U$ and (ii) $R^u \succeq R$ for all $R \in \mathcal{R}$ satisfying $R \supseteq U$.

We note an obvious corollary of the above definition.

Corollary 2 *All linear orders on $\mathcal{R}$ are conclusive partial orders on $\mathcal{R}$.*

Example 2. An example of a conclusive partial order on $\mathcal{R}$ is the following. Let $A = \{a, b, c\}$ and define $\succeq$ as per Figure 3. That is, (i) $abc \succeq R$ for all $R \in \mathcal{R}$, (ii) $cab \succeq bca, cba, acb$, (iii) $bac \succeq bca, cba$, and (iv) $cba \succeq bca$. The respective R^u orderings are summarized in Table 1. We see clearly how $\succeq$ is agnostic on the relative order of the pair $\{cab, bac\}$, as well as pairs $\{acb, cba\}$ and $\{acb, bca\}$.

By contrast, the partial order $\succeq$ such that only $abc \succeq R$ for all $R \in \mathcal{R}$, with no other binary comparisons specified, is clearly not conclusive. ∎

---

[6] While we are concerned with the strict version, it is worthwhile to note that Harless [10] extended lattice status-quo rules to account for status-quo *weak* orderings (i.e., where R^0 is a complete, reflexive, and transitive binary relation on A).

Example 2: Figure 3 and Table 1

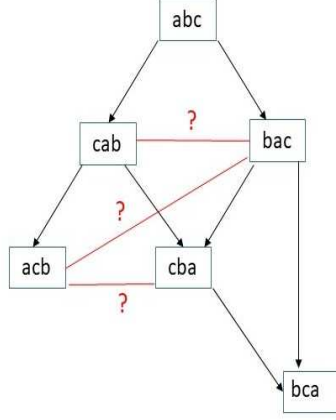| U	R^u
$\emptyset$	abc
all subsets of $\{(a,b),(a,c),(b,c)\}$	abc
$\{(c,a)\}$	cab
$\{(c,b)\}$	cab
$\{(c,a),(c,b)\}$	cab
$\{(b,a)\}$	bac
$\{(b,a),(b,c)\}$	bac
$\{(b,a),(c,a)\}$	cba

Figure 3: A schematic view of $\succeq$. Two orderings $R \neq \tilde{R}$ are connected by a directed path originating in R if and only $R \succeq \tilde{R}$. A question mark indicates undetermined binary comparisons.

Table 1: The associated R^u orderings for each partial order U on A. Pairs of identical alternatives are omitted in the statement of the U's. U's implying unanimous preferences are also omitted.

Example 3. A more subtle non-conclusive partial order is the following. Let $A = \{a,b,c\}$ and consider a partial order $\succeq$ on $\mathcal{R}$ such that for all pairs of orderings R^1 and R^2 we have $\delta(abc, R^1) < \delta(abc, R^2) \Rightarrow R^1 \succeq R^2$. Thus, $\succeq$ assigns first rank to ordering abc and orders the remaining elements of $\mathcal{R}$ in increasing Kemeny distance to it; the smaller the distance, the better the rank. Binary comparisons of orderings with equal Kemeny distance from abc are undetermined. Thus, for example, $bac \succeq cab$ and $\{bac \not\succeq acb, acb \not\succeq bac\}$. Consider now $U = \{(c,a)\}$, implying $\{R \in \mathcal{R} : R \supseteq U\} = \{cab, bca, cba\}$. Since $\delta(abc, cab) = \delta(abc, bca) = 2 < \delta(abc, cba) = 3$, there does not exist an ordering R^u satisfying both (i) $R^u \supseteq U$ and (ii) $R^u \succeq R$ for all $R \in \mathcal{R}$ such that $R \supseteq U$. ∎

Given a conclusive partial order $\succeq$ on $\mathcal{R}$, f is a **fixed order status-quo rule** associated with $\succeq$ if, for all $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$,

$$f(R_N) = R, \text{ where } R \supseteq \bigcap_{i \in N} R_i \text{ and } R \succeq R' \text{ for all } R' \in \mathcal{R} \text{ s.t. } R' \supseteq \bigcap_{i \in N} R_i. \tag{3}$$

Thus, for each profile R_N, a fixed order status-quo rule associated with $\succeq$ assigns to it the first-ranked ordering according to $\succeq$ that satisfies strong efficiency. Since $\succeq$ is conclusive, this ordering will be unique.

Fixed order status-quo rules can be interpreted in the following way. Suppose $\succeq$ represents a pre-existing consensus on the relative desirability of different outcomes. Moreover, assume the recommendations of $\succeq$ should always be heeded, unless they result in unanimous opposition on the part of the agents. As we are dealing with orderings, such opposition is interpreted as unanimous disagreement regarding individual binary comparisons. Thus, the reasoning goes, the decision-maker should consult $\succeq$ and pick its first-ranked element that does not meet such resistance. The fact that $\succeq$ is a conclusive partial order ensures that this element is unique and thus that the rule is well-defined.

3.2 Properties of status-quo rules

We begin this section by stating stronger versions of our strategy-proofness and solidarity axioms involving groups of agents.

Group strategy-proofness. There do not exist $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, S \subset N$ and $R'_S \in \mathcal{R}^S$ such that $f(R'_S, R_{N \setminus S}) \in \bigcap_{i \in S} [R_i, f(R_N)]$ and $f(R'_S, R_{N \setminus S}) \neq f(R_N)$.

Group weak welfare dominance. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, S \subset N, R'_S \in \mathcal{R}^S$, there do not exist $i, j \in N \setminus S$ such that $f(R_N) \in [R_i, f(R'_S, R_{N \setminus S})]$ and $f(R'_S, R_{N \setminus S}) \in [R_j, f(R_N)]$ and $f(R_N) \neq f(R'_S, R_{N \setminus S})$.

Group weak population monotonicity. For all $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, S \subset N$, there do not exist $j, k \in N \setminus S$ such that $f(R_N) \in [R_j, f(R_{N \setminus S})]$ and $f(R_{N \setminus S}) \in [R_k, f(R_N)]$ and $f(R_N) \neq f(R_{N \setminus S})$.

Note how there is no need to introduce group population monotonicity as it is implied by population monotonicity. By contrast, the group versions of weak welfare dominance and weak population monotonicity do not follow from the singleton ones.[7] We proceed to establish the properties of lattice and fixed order status-quo rules.

Proposition 3 *Lattice status-quo rules satisfy group strategy-proofness, group population monotonicity, adjacent welfare dominance, group weak welfare dominance, and efficiency.*

---

[7]Details available upon request.

Proposition 4 *Fixed order status-quo rules satisfy group strategy-proofness, group weak population monotonicity, group weak welfare dominance, and strong efficiency.*

It is worth nothing that the proof of Proposition 4 demonstrates that fixed order status-quo rules satisfy the following stronger version of group weak population monotonicity: for all $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $S \subset N$ there does not exist $j \in N \setminus S$ such that $f(R_N) \in [R_j, f(R_{N \setminus S})]$ and $f(R_N) \neq f(R_{N \setminus S})$. Thus, the departure of a subset of agents cannot leave any of the remaining agents unambiguously worse off.

3.3 A subfamily of fixed order status-quo rules

In this section we discuss an intuitive subfamily of fixed order status-quo rules. This subfamily is obtained by imposing additional structure on the set of conclusive partial orders.

Before doing so, we introduce the notion of a lexicographic order on $\mathcal{R}$. Given $R \in \mathcal{R}$, the **lexicographic order** $\succeq_R$ on $\mathcal{R}$ is defined as follows: for all $R^1, R^2 \in \mathcal{R}$ where $R^1 = a_1 a_2 .... a_m$ and $R^2 = b_1 b_2 .... b_m$, $R^1 \succeq_R R^2$ if and only if there exists $k \in \{1, 2, ..., m\}$ such that $a_l = b_l$ for all $l < k - 1$ and $(a_k, b_k) \in R$. Note that for all $R \in \mathcal{R}$, the relation $\succeq_R$ is complete, so that $\succeq_R$ is a linear order on $\mathcal{R}$.

Suppose f is a fixed order status quo rule associated with $\succeq$. The rule f is a **lexicographic fixed order status-quo** rule if there exists $R^0 \in \mathcal{R}$ such that $\succeq = \succeq_{R^0}$. For simplicity, we refer to f as a **lexicographic status-quo** rule with reference ordering R^0. Thus, lexicographic status-quo rules pick the strongly efficient ordering that is ranked first by linear order $\succeq_{R^0}$ on $\mathcal{R}$ or, put differently, is lexicographically closest to the reference ordering R^0.

Subject to respecting strong efficiency, lexicographic status-quo rules search for an ordering that strays as little as possible from the top-ranked recommendations of R^0. Evidently, greater importance is placed on maintaining agreement vis-a-vis the top-ranked alternatives of R^0 compared to the less desirable ones. This feature of the rule resonates in many practical settings. For instance, in the faculty recruitment example of the introduction, given the tightness of the academic job marker it is often the case that not more than two or three candidates will ever reject an offer. Thus, if the aggregation procedure must take carefully into account a reference ordering of the candidates, it is important that it do so by staying as close as possible to its top-ranked candidates.

Computational considerations. A compelling feature of lexicographic status-quo

rules is that they can be efficiently implemented in polynomial time. Computational tractability is often elusive in voting and social choice theory (Brandt et al. [7]) so it is worth briefly elaborating on the details.

Suppose f is a lexicographic status-quo rule with reference ordering R^0. Given $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$, define the directed graph $G = (V, E)$ where $V = A$ and $E = \left\{ (x, y) \in A \times A : x \neq y, \ (x, y) \in \bigcap_{i \in N} R_i \right\}$. To construct E, we may use a naive algorithm that mechanically goes through all agent orderings to determine the pairs of alternatives over which there is unanimous agreement. Thus, the construction of G can be completed in a maximum of $\binom{m}{2} \cdot n \binom{m}{2} = O(m^4 n)$ operations.[8] With the graph G defined, we apply to it a **topological ordering** algorithm (see pages 77-79 in Ahuja et al. [1]) in which ties are broken lexicographically by consulting R^0 and picking the alternative that is best-ranked by it. The output of this algorithm will be $f(R_N)$.

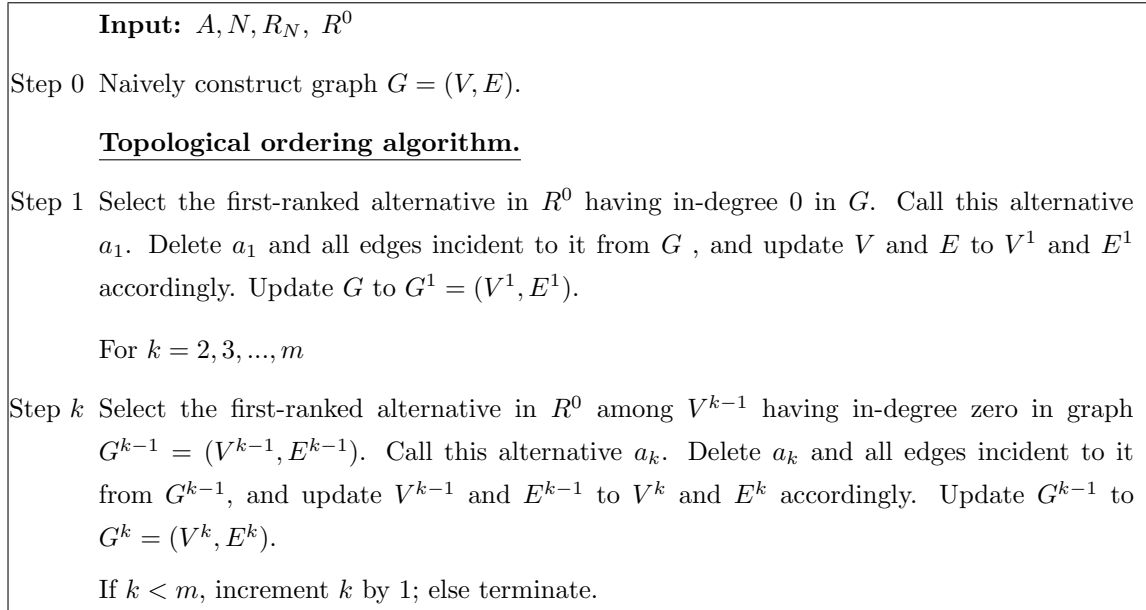Informally, the full algorithm works as follows:

Input: $A, N, R_N, \ R^0$

Step 0 Naively construct graph $G = (V, E)$.

Topological ordering algorithm.

Step 1 Select the first-ranked alternative in R^0 having in-degree 0 in G. Call this alternative a_1. Delete a_1 and all edges incident to it from G, and update V and E to V^1 and E^1 accordingly. Update G to $G^1 = (V^1, E^1)$.

For $k = 2, 3, ..., m$

Step k Select the first-ranked alternative in R^0 among V^{k-1} having in-degree zero in graph $G^{k-1} = (V^{k-1}, E^{k-1})$. Call this alternative a_k. Delete a_k and all edges incident to it from G^{k-1}, and update V^{k-1} and E^{k-1} to V^k and E^k accordingly. Update G^{k-1} to $G^k = (V^k, E^k)$.

If $k < m$, increment k by 1; else terminate.

Figure 4: Computational implementation of lexicographic status-quo rule with reference ordering R^0.

The algorithm in Figure 4 terminates after Step m with ordering $a_1 a_2 ... a_m$ as output. The tie-breaking rule it employs ensures that $a_1 a_2 ... a_m$ is the strongly efficient ordering that is first-ranked by $\succeq_{R^0}$. Hence, $f(R_N) = a_1 a_2 ... a_m$.

The running time of the topological ordering algorithm is $O(|E|) = O(m^2)$ (Ahuja

---

[8]Here we are assuming that agent orderings are stored as $m \times m$ 0-1 arrays.

et al. [1]). Considering that Step 0 can be naively completed in $O(m^4 n)$ operations, the total complexity of the algorithm in Figure 4 is $O(m^4 \cdot n + m^2) = O(m^4 \cdot n)$. Evidently, the most computationally burdensome part of the process is the construction of graph G. As there probably exist more efficient ways of determining the partial order $\bigcap_{i \in N} R_i$, lexicographic status-quo rules are likely to be implementable at lower computational cost.

Example 4. Suppose $A = \{a, b, c, d, e\}$ and let f denote the lexicographic status-quo rule with reference ordering $R^0 = abcde$. Let $N = \{1, 2, 3\}$ and consider the profile satisfying

$$R_1 = edbca, \quad R_2 = decba, \quad R_3 = ecdab.$$

The above preferences imply $\bigcap_{i \in N} R_i = \{(c, a), (d, a), (d, b), (e, a), (e, b), (e, c)\}.$[9] The corresponding graph G is given by Figure 5.
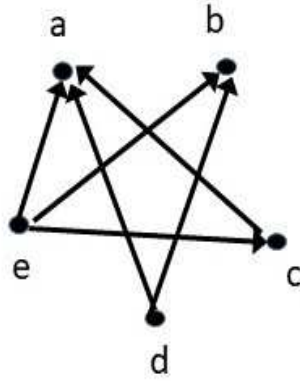

Figure 5: Graph $G = (V, E)$ for profile (R_1, R_2, R_3)

Figure 6 illustrates the topological ordering algorithm applied to profile (R_1, R_2, R_3). At every step $k = 1, 2, .., 5$ the selected alternative a_k is depicted in a red box. We obtain $f(R_1.R_2, R_3) = debca$.

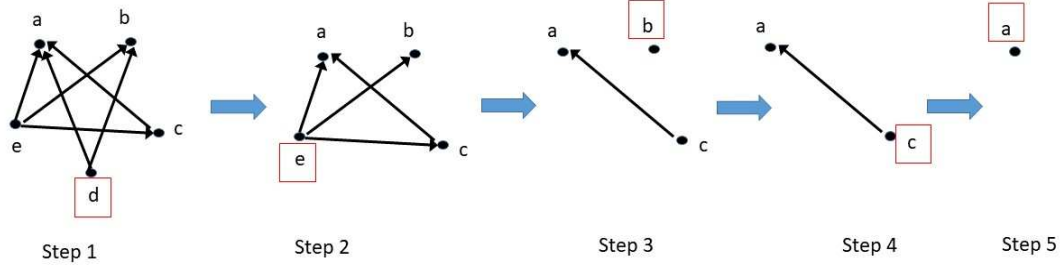[9]For simplicity we again omit pairs of identical alternatives.

Figure 6: Application of algorithm of Figure 4 to profile (R_1, R_2, R_3). Recall $R^0 = abcde$.

Now, suppose ordering R_1 is replaced by $R_1' = edcba$, i.e., agent 1 switches the order of adjacent alternatives b and c. These preferences imply $R_1' \cap R_2 \cap R_3 = \{(c, a), (d, a), (d, b), (e, a), (e, b), (e, c), (c, b)\}$. Hence, compared to profile (R_1, R_2, R_3), there exists unanimous agreement on an additional pair of alternatives, namely (c, b). Figure 7 displays the application of the algorithm to this profile, yielding $f(R_1', R_2, R_3) = decab$.

Furthermore, note that we obtain the exact same pattern of unanimous agreement, and thus also outcome, if agent 1 were to exit the population. That is, $f(R_1', R_2, R_3) = f(R_2, R_3) = decab$.
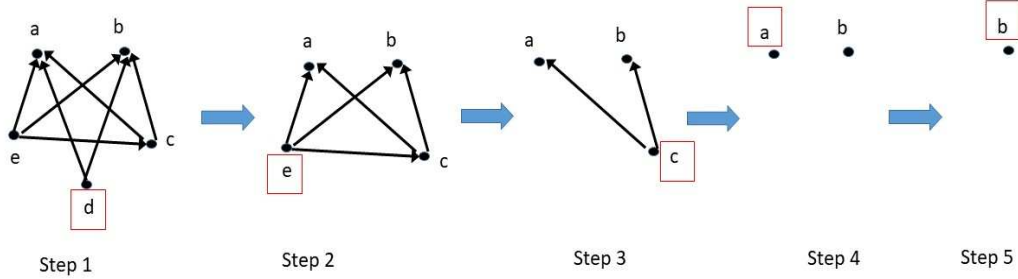

Figure 7: Application of algorithm of Figure 4 to profile (R_1', R_2, R_3) or (R_2, R_3). Recall $R^0 = abcde$.

Finally, consider profile R_N and suppose ordering R_3 is replaced by $R_3'' = cedab$. These preferences imply $R_1 \cap R_2 \cap R_3'' = \{(c, a), (d, a), (d, b), (e, a), (e, b)\}$. Hence, compared to profile (R_1, R_2, R_3), there is no longer unanimous agreement on pair (e, c). Figure 8 displays the application of the algorithm to this profile, yielding $f(R_1, R_2, R_3'') = cdeab$.
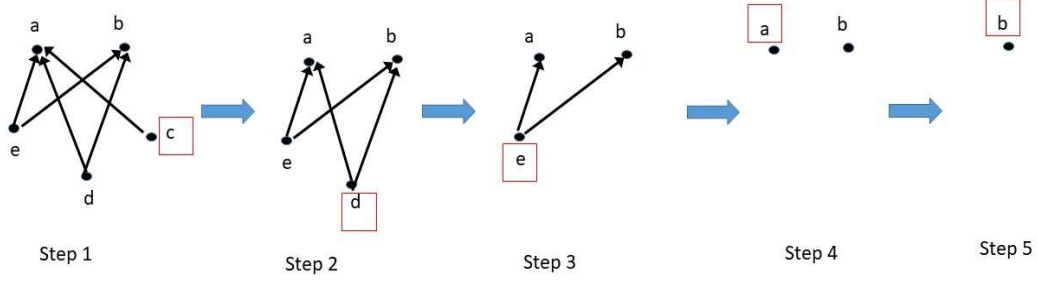
Figure 8: Application of algorithm of Figure 4 to profile (R_1, R_2, R_3''). Recall $R^0 = abcde$.

Let us compare the above outcomes to those obtained with the equivalent lattice status-quo rule. Let g denote the lattice status-quo rule with reference ordering $R^0 = abcde$. Applying Harless's improvement algorithm [10] it is easy to see that

$$g(R_N) = abcde, \quad g(R_1', R_2, R_3) = cdeab, \quad g(R_1, R_2, R_3'') = abcde.$$

Table 2 summarizes the results of rules f and g to enable easier comparison.

profile	unanimous agreement	f	g
(R_1, R_2, R_3)	$\big\{$(c,a),(d,a),(d,b), (e,a),(e,b), (e,c)$\big\}$	$debca$	$abcde$
(R_1', R_2, R_3) or (R_2, R_3)	$\big\{$(c,a),(d,a),(d,b), (e,a),(e,b), (e,c),(c,b)$\big\}$	$decab$	$cdeab$
(R_1, R_2, R_3'')	$\big\{$(c,a),(d,a),(d,b), (e,a),(e,b)$\big\}$	$cdeab$	$abcde$

Table 2: Example 4. Comparing the lexicographic status-quo rule f to its lattice counterpart g. Recall that $R_1 = edbca$, $R_2 = decba$, $R_3 = ecdab$, $R_1' = edcba$, $R_3'' = cedab$.

Profile (R_1, R_2, R_3) provides a vivid example of how lattice status-quo rules may fail strong efficiency. The ordering $g(R_1, R_2, R_3) = abcde$ violates all six unanimous binary comparisons present in the electorate.

Conversely, suppose agent 1 changes her preferences to R_1' or exits the aggregation procedure. This results in $f(R_1', R_2, R_3) = f(R_2, R_3) = decab$ and $g(R_1', R_2, R_3) = g(R_2, R_3) = cdeab$. The violation of strong efficiency of rule g has been attenuated to just pair (e, c). Meanwhile, consistent with g's solidarity properties, agents 2 and 3 unambiguously prefer the new ordering to $g(R_1, R_2, R_3)$. The same is not true of f, since $(b, a) \in R_2$ and $(c, b) \in R_2$ so that $f(R_1', R_2, R_3)$ and $f(R_2, R_3)$ are not

comparable to $f(R_1, R_2, R_3)$ for agent 2. Hence, adjacent welfare dominance, as well as population monotonicity, fail rule f at this profile.

Finally, suppose agent 3 changes her preferences to R_3''. This results in $f(R_1, R_2, R_3'') = cdeab$ and $g(R_1, R_2, R_3'') = abcde$. Once again, rule g fails strong efficiency in a significant way. On the other hand, $(e, c) \in R_2$ and $(c, b) \in R_2$ so that $f(R_1, R_2, R_3'')$ is not comparable to $f(R_1, R_2, R_3)$ for agent 2. Thus, f fails the welfare dominance requirement (note that this is not the case for g). ∎

3.4 Weak orders

An interesting extension of the model involves the introduction of indifferences to the reference partial orders as well as to the outcome space, analogous to Harless [10]. This would necessitate changes to both the properties that rules aspire to satisfy and the rules themselves. The concept of betweeness would need to be redefined so that for all $R, R', R'' \in \bar{\mathcal{R}}$, where $\bar{\mathcal{R}}$ is the set of weak orderings on A, $R \in [R', R'']$ is equivalent to $R' \cap R'' \subseteq R \subseteq R' \cup R''$. This would in turn affect the definitions of strategy-proofness and weak solidarity. Furthermore, strong efficiency might need to be adapted by supplementing the requirement that $f(R_N) \supseteq \bigcap_{i=1}^{n} R_i$ with a property stipulating that for any pair of alternatives (a, b) with $a \neq b$, $(a, b) \in \bigcap_{i=1}^{n} R_i \implies (b, a) \notin f(R_N)$. Finally, the introduction of indifferences would necessitate the definition of fixed-order status-quo rules on the basis of conclusive partial orders on $\bar{\mathcal{R}}$.

Given the above, it is unclear whether the proof of Proposition 4 can be adapted to this more general framework and, more generally, whether the corresponding desirable properties of fixed order status-quo rules continue to hold. Elucidating this point would enable a fuller comparison of fixed-order and lattice status-quo rules.

4 A characterization of fixed order status-quo rules

We now offer an axiomatic characterization of fixed order status-quo rules. Our analysis is reminiscent of Theorem 1 in Bossert and Sprumont where monotonic majority alteration rules are shown to uniquely satisfy majority-basedness, unanimity, and strategy-proofness.[10]

We proceed by introducing a property that is similar, at least in a technical sense, to the majority-basedness requirement of Bossert and Sprumont. Given $N \subset \mathcal{N}, R_N \in$

---

[10]Majority basedness stipulates that profiles with the same majority relation be assigned the same outcomes, whereas unanimity is a minimal standard of efficiency implying that if all agents have the same orderings, then the rule must also pick this ordering.

$\mathcal{R}^N$, define the *unanimity relation* $U(R_N)$ on A by

$$U(R_N) \equiv \bigcap_{i \in N} R_i = \{(a,b) \in A \times A : \ |i \in N : aR_ib| = n\}. \qquad (4)$$

For any profile R_N, the relation $U(R_N)$ is a partial order on A. By definition, all strongly efficient orderings are supersets of $U(R_N)$; furthermore, for any pair of alternatives $(a,b) \in A \times A$ inclusion in $U(R_N)$ obviously implies inclusion in profile R_N's majority relation.

The unanimity relation plays a central role in the following axiom.

Unanimity-basedness. For all $N \subset \mathcal{N}, R_N, \tilde{R}_N \in \mathcal{R}^N$,

$$U(R_N) = U(\tilde{R}_N) \Rightarrow f(R_N) = f(\tilde{R}_N). \qquad (5)$$

Unanimity-basedness requires that a rule be invariant to changes in agent preferences that do not alter the underlying unanimity relation of the electorate. As such, it implies that the rule can be recast as a function from $\mathcal{U}$ to $\mathcal{R}$, where $\mathcal{U}$ is the set of partial orders on A. It is straightforward to check that unanimity-basedness implies Bossert and Storcken's [6] *extrema independence* axiom, and thus also precludes dictatoriality. Variants of unanimity-basedness has also been analyzed in other models of social choice (see, e.g., Gordon [11, 13]).

Among the strategy-proof rules examined by Bossert and Sprumont, lattice status-quo rules are the only ones that are unanimity-based (that monotonic majority alteration rules and Condorcet-Kemeny rules are not unanimity-based is obvious).

Proposition 5 *All lattice status-quo rules are unanimity-based.*

We proceed with the characterization. The proof frequently invokes the following Lemma.

Lemma 1 *Suppose rule f satisfies unanimity-basedness and strategy-proofness. For all $N \subset \mathcal{N}, R_N^1, R_N^2 \in \mathcal{R}^N$ we have*

$$U(R_N^1) \subseteq U(R_N^2) \subseteq f(R_N^1) \ \Rightarrow f(R_N^1) = f(R_N^2).$$

Lemma 1 is reminiscent of *Arrow's choice axiom* (Arrow [2]). It also has an exact analog (Lemma 3) in the cycle-location model studied by Gordon [12]. It implies that

if an ordering R is selected from a set of strongly efficient orderings, then R must *also* be picked when considering any subset of this set for which it does not violate strong efficiency. For example, if $A = \{a, b, c\}$ and rule f picks ordering $R = abc$ when there is no unanimous agreement among agents with respect to any pair of alternatives, then it should also pick R when unanimous agreement among agents is confined to everyone ranking a over b.

Recall that $\mathcal{U}$ denotes the set of partial orders on A. By unanimity-basedness, we can re-write Lemma 1 to suppress its dependence on profiles: For all $U^1, U^2 \in \mathcal{U}$,

$$U^1 \subseteq U^2 \subseteq f(U^1) \ \Rightarrow \ f(U^1) = f(U^2). \tag{6}$$

We are now ready to state this section's main result.

Theorem 2 *A rule satisfies strong efficiency, unanimity-basedness, and strategy-proofness if and only if it is a fixed order status-quo rule.*

Remark 1. The above characterization is tight since (a) Condorcet-Kemeny rules satisfy strong efficiency and strategy-proofness but not unanimity-basedness, (b) lattice status-quo rules satisfy unanimity-basedness and strategy-proofness but not strong efficiency, and (c) we can construct rules satisfying unanimity-basedness and strong efficiency that violate strategy-proofness. The latter can be done by taking a function $f : \mathcal{R}^N \mapsto \mathcal{R}$ that respects unanimity-basedness (meaning that it can be recast as $f : \mathcal{U} \mapsto \mathcal{R}$) and strong efficiency, but violates Lemma 1 and thus also strategy-proofness. For example, suppose $A = \{a, b, c\}$, f is unanimity-based and strongly efficient and satisfies $f(\emptyset) = abc$ and $f(\{(a, b)\}) = cab$.[11] Assume, further, that profile R_N is such that $U(R_N) = \{(a, b)\}$ so that $f(R_N) = cab$. Suppose now that agent i has preferences $R_i = abc$. If this agent misreports his preferences submitting $R'_i = bac$, we have $U(R'_i, R_{N \setminus i}) = \emptyset$, so that $f(R'_i, R_{N \setminus i}) = abc$ and strategy-proofness is violated.

Remark 2. Given Theorem 1 and the dubious normative content of unanimity-basedness, it is natural to wonder whether unanimity-basedness can be substituted with weak welfare dominance and weak population monotonocity in the characterization of fixed order status-quo rules. Unfortunately, the following argument shows why this is not possible. Consider $R^0 \in \mathcal{R}$ and its exact opposite $-R^0$. Suppose f is a fixed order status-quo rule with a conclusive partial order $\succeq$ that ranks R^0 and $-R^0$ in first and

---

[11]Once again, to avoid clutter, we omit pairs of identical alternatives in the definition of U.

second place, respectively. That is, $\succeq$ is conclusive and also satisfies $R^0 \succeq -R^0 \succeq R$ for all $R \neq R^0, -R^0$. Consider the following rule g:

$$
g(R_N) = \begin{cases} R^0, & \text{if } R^0 \in R_N \text{ and } U(R_N) = \emptyset \\ -R^0, & \text{if } R^0 \notin R_N \text{ and } U(R_N) = \emptyset \\ f(R_N), & \text{otherwise} \end{cases}
$$

Rule g coincides with f for all profiles in which there exists at least one pair of (non-identical) alternatives over whose relative order all agents agree on. By contrast, when $U(R_N) = \emptyset$, rule g does the following: (i) if there exists an agent holding preferences R^0, the rule picks this ordering; (ii) if not, then it picks the exact opposite of R^0, namely $-R^0$. Evidently, g violates unanimity-basedness.

Using Proposition 4 and the structure of rule g, it is straightforward to prove that g satisfies strong efficiency, group strategy-proofness, group weak welfare dominance and group weak population monotonicity.[12] (Note how, consistent to Theorem 1, rule g stipulates R^0 as its reference ordering.)

Remark 3. As mentioned earlier, Theorem 2 is reminiscent of Bossert and Sprumont's characterization of monotonic majority alteration rules with majority-basedness, unanimity, and strategy-proofness. The common thread between the two results is that both majority- and unanimity-basedness are very strong properties that seem to be necessary for these sorts of characterizations to go through. This is because the efficiency and strategy-proofness axioms are too weak to sufficiently restrict the range of possible rules.

Remark 4. While admittedly a very strong property, unanimity-basedness is not without practical relevance. Criminal law jury trials in the United States require unanimous agreement to issue a verdict. If such consensus is not reached then a mistrial is declared, with the status-quo outcome being an acquittal on the related charges. In the political sphere, a number of prominent organizations are known to use unanimity-based methods in their decision-making processes. For example, the five permanent members of the United Nations Security Council have veto power in the approval of all "substantive" (as opposed to "draft") resolutions. These countries can also use their veto power in the selection of the United Nations' Secretary General. Another example of unanimity-basedness can be observed in the European Council's deliberative procedures. There, unanimous agreement is needed for decisions to be reached in many important areas including EU membership, taxation, foreign policy, common security, justice and home

---

[12]Details available upon request.

affairs.

5 Concluding thoughts

This paper has dealt with preference aggregation in the classical Arrovian framework. Within this setting, emphasis was placed on rules that satisfy notions of solidarity. We formulated novel relaxations of existing solidarity properties and showed how they imply, in conjunction with strategy-proofness, the emergence of reference outcomes holding privileged status. We proceeded to introduce a new class of rules, fixed order status-quo rules, satisfying these solidarity requirements as well as desirable criteria of efficiency and strategy-proofness. An appealing subfamily based on lexicographic orders was highlighted for its intuitive interpretation and computational tractability. Finally, fixed order status-quo rules were characterized by strong efficiency, strategy-proofness, and an additional axiom of a more technical nature, unanimity-basedness.

Status-quo rules, lattice as well as fixed order, are appropriate in settings characterized by high volatility, with agents changing their preferences often and entering/exiting the aggregation procedure at will. In such instances it is important to guarantee a measure of fairness that precludes dramatic, and diametrically opposed, reversals of fortune for agents who are not responsible for these changes. This requirement may be interpreted as a weak form of envy-freeness. Moreover, in view of their joint dependence on exogenous orders, status-quo rules are applicable in environments defined by strong pre-existing views on the desirability of different outcomes whose recommendations should be modified only if met with strenuous opposition.

The present work suggests natural directions for future research. Given Remark 2, we know that weak solidarity axioms together with strong efficiency and strategy-proofness are not sufficient to characterize fixed order status-quo rules. It would be interesting to determine what additional axioms may be brought to bear to achieve such a characterization. Furthermore, it would be worthwhile to delve deeper into lexicographic status-quo rules and better understand why they might be preferable to other fixed order status-quo rules. Finally, a promising extension of the proposed rules involves the introduction of indifferences to the reference partial orders as well as to the outcome space.

Appendix

Proposition 1. It is sufficient to prove the impossibility for the $m = 3$ case. This is because problems with three alternatives embed into larger ones by considering profiles in which (i) there exist three alternatives that appear among the top three positions in all agent orderings and (ii) all agents order the remaining alternatives in the same way.

Suppose $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$ and consider the following profile $R_1 = R_2 = abc$, $R_3 = cba$. Note that the majority relation of this profile equals the ordering abc. As a result, all Condorcet-Kemeny and monotonic majority alteration rules will select ordering abc. Now suppose agent 1 changes his preferences to cba and note that the majority relation of this new profile equals the ordering cba. Hence, once again, all Condorcet-Kemeny and monotonic majority alteration rules will select ordering cba, resulting in an unambiguously better (resp., worse) outcome for agent 3 (resp., 2). This violates weak welfare dominance.

Consider now a $\succeq$-Condorcet-Kemeny rule such that ordering $R = abc$ is ranked first by $\succeq$. Construct a profile with three agents such that $R_1 = R_2 = -abc = cba$ and $R_3 = abc$. When applied to profile (R_1, R_2, R_3), the $\succeq$-Condorcet-Kemeny rule will pick ordering cba. Now, suppose agent 1 departs. Then, in the reduced profile (R_2, R_3) the rule will pick abc, an outcome that is unambiguously worse for agent 2 and better for agent 3, thus violating weak population monotonicity. Similarly, consider a monotonic majority alteration rule such that when the majority relation equals $A \times A$, the rule picks ordering abc. When applied to profile (R_1, R_2, R_3) this monotonic majority alteration rule will pick ordering cba. Now, suppose agent 1 departs. Then, in the reduced profile (R_2, R_3) the rule will pick abc, an outcome that is unambiguously worse for agent 2 and better for agent 3, thus violating weak population monotonicity. Repeating the above argument for all $R \in \mathcal{R}$, establishes that all Condorcet-Kemeny and monotonic majority alteration rules fail weak population monotonicity. ■

Theorem 1. Let $N \subset \mathcal{N}$ and assume $|N| \geq m!$. A profile is *diversified* if for all $R \in \mathcal{R}$ there exists $i \in N$ such that $R_i = R$. Suppose R_N is diversified and let $f(R_N) = R^0$. By construction, there exist $i, j \in N$ such that $R_i = R^0$ and $R_j = -R^0$, where $-R$ denotes the exact opposite of R. By weak welfare dominance, $f(R_i, R_j, R'_k, R_{N \setminus \{i,j,k\}}) = R^0$ for all $R'_k \in \mathcal{R}$. Similarly, weak welfare dominance applied to agent $l \neq i, j, k$ implies $f(R_i, R_j, R'_l, R'_k, R_{N \setminus \{i,j,k,l\}}) = R^0$ for all $R'_k, R'_l \in \mathcal{R}$. Repeating this argument for all other agents except i and j obtains $f(R_i, R_j, R'_{N \setminus \{i,j\}}) = R^0$ for all $R'_{N \setminus \{i,j\}} \in \mathcal{R}^{N \setminus \{i,j\}}$. Strategy-proofness applied to

deviations by agent j implies that $f(R^0, R'_{N\setminus i}) = R^0$ for all $R'_{N\setminus i} \in \mathcal{R}^{N\setminus i}$.

We show that the above result holds irrespective of the identity of the agent holding preferences R^0. Consider the diversified profile R^p_N that is a relabeling of the orderings in R_N. That is, for all $x \in N$ there exists a unique $y \in N$ such that $R_x = R^p_y$. Suppose $f(R^p_N) = R^p$. By construction, there exist $k, l \in N$ such that $R^p_k = R^p$ and $R^p_l = -R^p$. We distinguish between two cases according to whether $k = i$, or $k \neq i$. If $k = i$, then $R_i = R^p_k$ and thus $R^0 = R^p$ and we are done. Conversely, if $k \neq i$, using weak welfare dominance and strategy-proofness as before implies $f(R^p, R'_{N\setminus k}) = R^p$ for all $R'_{N\setminus k} \in \mathcal{R}^{N\setminus k}$. In particular, setting $R'_i = R^0$ yields $f(R^p, R^0, R'_{N\setminus\{k,i\}}) = R^p$ for all $R'_{N\setminus\{k,i\}} \in \mathcal{R}^{N\setminus\{k,i\}}$. Given the previously established equality $f(R^0, R'_{N\setminus i}) = R^0$ for all $R'_{N\setminus i} \in \mathcal{R}^{N\setminus i}$, we conclude $R^p = R^0$. Thus, the equality $f(R^0, R'_{N\setminus i}) = R^0$ for all $R'_{N\setminus i} \in \mathcal{R}^{N\setminus i}$ holds for all $i \in N$.

We now extend the above result to all populations in $\mathcal{N}$. Consider the already-examined population N and a profile $\hat{R}_N$ in which there exist $i, j \in N$ such that $\hat{R}_i = R^0$ and $\hat{R}_j = -R^0$. By the previous argument, $f(\hat{R}_N) = R^0$. Let $N' \subset \mathcal{N}$ and distinguish between two cases:

(a) $N \cap N' = \emptyset$. Consider a profile $R'_{N'}$ in which there exist $i', j' \in N'$ such that $R'_{i'} = R^0$ and $R'_{j'} = -R^0$. Let $k' \in N'$ and introduce the profile $(\hat{R}_N, R'_{k'})$. Weak population monotonicity implies $f(\hat{R}_N) = f(\hat{R}_N, R'_{k'}) = R^0$. Repeating this argument iteratively for all agents in N', yields $f(\hat{R}_N, R'_{N'}) = f(\hat{R}_N) = R^0$.

Let $j \in N$. Recalling that $R'_{i'} = R^0$ and $R'_{j'} = -R^0$ and using the fact that $f(\hat{R}_N, R'_{N'}) = R^0$, weak population monotonicity implies $f(\hat{R}_{N\setminus j}, R'_{N'}) = R^0$. Repeating this argument iteratively for all agents in N, yields $f(R'_{N'}) = R^0 = f(\hat{R}_N, R'_{N'})$. Imposing strategy-proofness to deviations by agent j' and noting how the preferences of agents in $N' \setminus \{i', j'\}$ were arbitrary, we conclude $f(R^0, R_{N'\setminus i'}) = R^0$ for all $R_{N'\setminus i'} \in \mathcal{R}^{N'\setminus i'}$. As before, we can prove that this result holds for all $i' \in N'$.

(b) $N \cap N' \neq \emptyset$. Consider once again a profile $R'_{N'}$ in which there exist $i', j' \in N'$ such that $R'_{i'} = R^0$ and $R'_{j'} = -R^0$. Take $N^0 \subset \mathcal{N}$ such that $N^0 \cap N = N^0 \cap N' = \emptyset$. Consider a profile $R^0_{N^0}$ in which there exist $i^0, j^0 \in N^0$ such that $R^0_{i^0} = R^0$ and $R^0_{j^0} = -R^0$. By part (a), $f(R^0_{N^0}) = R^0$ and $f(R'_{N'}, R^0_{N^0}) = R^0$. By repeatedly removing agents belonging to N^0 from the population $N' \cup N^0$ and applying weak population monotonicity, we obtain $f(R'_{N'}) = R^0 = f(R^0_{N^0}, R'_{N'})$. Imposing strategy-proofness to deviations by agent j' and noting how in profile $R'_{N'}$ the

preferences of agents in $N' \setminus \{i', j'\}$ were arbitrary, we conclude $f(R^0, R_{N' \setminus i'}) = R^0$ for all $R_{N' \setminus i'} \in \mathcal{R}^{N' \setminus i'}$. As before, we can prove that this result holds for all $i' \in N'$.

We conclude that for all $N \subset \mathcal{N}$ and $R_N \in \mathcal{R}^N$, if there exists $i \in N$ with $R_i = R^0$ then $f(R_N) = R^0$. ∎

Proposition 2. Begin with part (i). Consider a partial order U satisfying the property in the statement of the Proposition. Let $N \subset \mathcal{N}$ and suppose R_N^u is a profile such that for all $R \in \mathcal{R}$ satisfying $R \supseteq U$, there exists at least one $i \in N$ such that $R_i^u = R$. Let $f(R_N^u) = R^u$. By construction, there exist $i, j \in N$ such that $R_i^u = R^u$ and $R_i^u \cap R_j^u = U$. Weak welfare dominance and strong efficiency imply $f(R_i^u, R_j^u, R_k', R_{N \setminus \{i,j,k\}}) = R^u$ for all $R_k' \in \mathcal{R}$ such that $\left(R_k' \cap \bigcap_{l \in N \setminus k} R_l^u \right) = U$. Repeating this argument iteratively for all $l \in N \setminus \{i, j\}$ we deduce that $f(R_i^u, R_j^u, R_{N \setminus \{i,j\}}') = R^u$ for all $R_{N \setminus \{i,j\}}' \in \mathcal{R}^{N \setminus \{i,j\}}$ such that $\left(\bigcap_{l \in N \setminus \{i,j\}} R_l' \cap R_i^u \cap R_j^u \right) = U$. Next, strategy-proofness applied to deviations by agent j and strong efficiency imply that $f(R^u, R_{N \setminus i}') = R^u$ for all $R_{N \setminus i}'$ such that $\left(\bigcap_{i \in N \setminus i} R_i' \cap R_i^u \right) = U$. As before, this equality holds for all $i \in N$.

Using similar arguments as in the proof of Theorem 1 we can extend the above result to an arbitrary $N \subset \mathcal{N}$. This concludes the proof of part (i).

Now consider part (ii). Consider a partial order U satisfying the property in the statement of the Proposition. Let $N \subset \mathcal{N}$ and suppose R_N^u is a profile such that for all $R \in \mathcal{R}$ satisfying $R \supseteq U$, there exists at least one $i \in N$ such that $R_i^u = R$. Let $f(R_N^u) = R^u$. By construction, there exists $i \in N$ such that $R_i^u = R^u$.

If $R^u \in \{R^1, R^2\}$, then there exists $j \in N$ such that $R_i^u \cap R_j^u = U$ and an identical proof to that of part (i) yields the desired result. Suppose instead $R_i^u = R^u \notin \{R^1, R^2\}$ and consider an agent $j \in N \setminus i$. Weak welfare dominance and strong efficiency imply that $f(R_i^u, R_j', R_{N \setminus \{i,j\}}^u) = R^u$ for all $R_j' \in \mathcal{R}$ such that $\left(R_j' \cap \bigcap_{l \in N \setminus j} R_l^u \right) = U$. If this were not the case, then either agents holding preferences R^1 would unambiguously gain and those holding preferences R^2 would unambiguously lose from j's preference change, or vice versa.[13] Repeating the same argument iteratively for all agents in $N \setminus i$ yields

---

[13]Given the structure of U, for a profile R_N to satisfy $\bigcap_{i \in N} R_i = U$ there must exist a group of agents holding preferences R^1 and another holding preferences R^2.

$f(R^u, R'_{N \setminus i}) = R^u$ for all $R'_{N \setminus i}$ such that $\left(\bigcap\limits_{i \in N \setminus i} R'_i \cap R^u_i \right) = U$. Note how we did not need to invoke strategy-proofness in this case. As before, this equality holds for all $i \in N$ and we can extend the result to all subsets of $\mathcal{N}$. ∎

Corollary 1. Suppose $m = 3$. Partial orders with three non-identical alternatives yield a unique strongly efficient ordering so the result trivially holds. Partial orders U consisting of two pairs of non-identical alternatives yield exactly two strongly efficient orderings that differ with respect to a single binary comparison. Thus part (i) of Proposition 2 implies the result. Partial orders U consisting of a single pair of non-identical alternatives are such that there exists a unique pair of orderings R^1, R^2 satisfying $R^1 \cap R^2 = U$ and a unique third ordering R^3 that is strictly between them. Thus part (ii) of Proposition 2 implies the result. ∎

Proposition 3. Group strategy-proofness, group population monotonicity, adjacent welfare dominance, and efficiency follow from Theorem 2 in Bossert and Sprumont [5] and Proposition 1 in Harless [10].

Let us prove group weak welfare dominance. Consider a lattice status-quo rule f with reference ordering R^0, and suppose, in contradiction, that there exists $N \subset \mathcal{N}$, a profile R_N, $S \subset N$, and a pair of agents $i, j \in N \setminus S$ such that $f(R_N) \in [R_i, f(R'_S, R_{N \setminus S})]$ and $f(R'_S, R_{N \setminus S}) \in [R_j, f(R_N)]$ and $f(R_N) \neq f(R'_S, R_{N \setminus S})$.

Since $f(R_N) \in [R_i, f(R'_S, R_{N \setminus S})]$ and $f(R_N) \neq f(R'_S, R_{N \setminus S})$ there must exist $(a, b) \in R_i \cap f(R_N)$ such that $(a, b) \notin f(R'_S, R_{N \setminus S})$. This pair must satisfy $(a, b) \notin R_j$; otherwise $f(R'_S, R_{N \setminus S}) \in [R_j, f(R_N)]$ would imply that $(a, b) \in f(R'_S, R_{N \setminus S})$, a contradiction.

Recall the sequential-improvement algorithm discussed by Harless [10] that is used to calculate the output of lattice status-quo rules. According to this algorithm, we start from R^0 and gradually move away from it by switching the ranks of pairs of alternatives (x, y) such that the following two conditions hold: (i) alternatives x and y are adjacent in the current ordering, and (ii) y is unanimously preferred to x by the agents. The algorithm stops when the current ordering is such that no pair of alternatives simultaneously satisfies conditions (i) and (ii). For further information and the formal details of the algorithm, consult Harless [10].

Now, distinguish between two cases:

(i) $(a, b) \in R^0$. Since $(a, b) \in R_i$ and $(a, b) \notin R_j$ and $i, j \in N \setminus S$, in profile $(R'_S, R_{N \setminus S})$ alternative b is not unanimously preferred to a. Thus, when applied

to $(R'_S, R_{N \setminus S})$, Harless' improvement algorithm will never replace the pair (a, b) with (b, a). Thus, $(b, a) \notin f(R'_S, R_{N \setminus S})$, a contradiction.

(ii) $(b, a) \in R^0$. Since $(a, b) \in R_i$ and $(a, b) \notin R_j$ and $i, j \in N \setminus S$, in profile R_N alternative a is not unanimously preferred to b. Thus, when applied to R_N, Harless' improvement algorithm will never replace the pair (b, a) with (a, b). Thus, $(a, b) \notin f(R_N)$, a contradiction.

$\blacksquare$

Proposition 4. That fixed order status-quo rules satisfy strong efficiency is obvious by construction. Suppose f is a fixed order status-quo rule associated with $\succeq$.

We begin with group strategy-proofness. Suppose that f fails group strategy-proofness so that there exist $N \subset \mathcal{N}, R_N \in \mathcal{R}^N, S \subset N$ and $R'_S \in \mathcal{R}^S$ such that $f(R'_S, R_{N \setminus S}) \in \bigcap_{i \in S}[R_i, f(R_N)]$ and $f(R'_S, R_{N \setminus S}) \neq f(R_N)$. By strong efficiency $\bigcap_{j \in N} R_j \subseteq f(R_N)$ and $\bigcap_{j \in N \setminus S} R_j \cap \bigcap_{j \in S} R'_j \subseteq f(R'_S, R_{N \setminus S})$.

Distinguish between three cases:

(i) $f(R'_S, R_{N \setminus S}) \succeq f(R_N)$. This implies $\bigcap_{j \in N} R_j \not\subseteq f(R'_S, R_{N \setminus S})$. By strong efficiency, there must exist a pair of alternatives $(a, b) \in \bigcap_{j \in S} R_j$ such that $(a, b) \in f(R_N)$ and $(a, b) \notin f(R'_S, R_{N \setminus S})$. This contradicts $f(R'_S, R_{N \setminus S}) \in \bigcap_{i \in S}[R_i, f(R_N)]$.

(ii) $f(R_N) \succeq f(R'_S, R_{N \setminus S})$. This implies $\bigcap_{j \in N \setminus S} R_j \cap \bigcap_{j \in S} R'_j \not\subseteq f(R_N)$. By strong efficiency, there must exist a pair of alternatives $(a, b) \in \bigcap_{j \in S} R'_j$ such that $(a, b) \notin \bigcap_{j \in S} R_j$, $(a, b) \notin f(R_N)$ and $(a, b) \in f(R'_S, R_{N \setminus S})$. Thus, $(b, a) \in R_i$ for some $i \in S$, $(b, a) \in f(R_N)$, and $(b, a) \notin f(R'_S, R_{N \setminus S})$. This contradicts $f(R'_S, R_{N \setminus S}) \in \bigcap_{i \in S}[R_i, f(R_N)]$.

(iii) $f(R_N)$ and $f(R'_S, R_{N \setminus S})$ are incomparable according to $\succeq$. Consider two sub-cases:

    (a) $f(R'_S, R_{N \setminus S}) \supseteq \bigcap_{j \in N} R_j$. Since $\succeq$ is conclusive, we must have $f(R_N) \succeq f(R'_S, R_{N \setminus S})$, a contradiction.

    (b) $f(R'_S, R_{N \setminus S}) \not\supseteq \bigcap_{j \in N} R_j$. There must exist $(a, b) \in \bigcap_{j \in N} R_j$ such that $(a, b) \notin f(R'_S, R_{N \setminus S})$. This contradicts $f(R'_S, R_{N \setminus S}) \in \bigcap_{i \in S}[R_i, f(R_N)]$.

We now address group weak population monotonicity. Suppose that f fails this property and there exist $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $S \subset N$ and $j, k \in N \setminus S$ such that $f(R_N) \in [R_j, f(R_{N \setminus S})]$ and $f(R_{N \setminus S}) \in [R_k, f(R_N)]$, with $f(R_N) \neq f(R_{N \setminus S})$. We distinguish between two cases:

(i) $f(R_N) \supseteq \bigcap_{i \in N \setminus S} R_i$. Strong efficiency implies $f(R_{N \setminus S}) \supseteq \bigcap_{i \in N \setminus S} R_i \supseteq \bigcap_{i \in N} R_i$. Since both $f(R_N)$ and $f(R_{N \setminus S})$ satisfy strong efficiency at both profiles R_N and $R_{N \setminus S}$, this implies $f(R_{N \setminus S}) \succeq f(R_N)$ and $f(R_N) \succeq f(R_{N \setminus S})$. This is a contradiction.

(ii) $f(R_N) \not\supseteq \bigcap_{i \in N \setminus S} R_i$. Then, $f(R_{N \setminus S}) \supseteq \bigcap_{i \in N \setminus S} R_i$ implies that there must exist $(a, b) \in f(R_{N \setminus S}) \cap \bigcap_{i \in N \setminus S} R_i$ such that $(a, b) \notin f(R_N)$. This contradicts $f(R_N) \in [R_j, f(R_{N \setminus S})]$.

Finally, we address group weak welfare dominance. Suppose that f fails this property and there exist $N \subset \mathcal{N}$, $R_N \in \mathcal{R}^N$, $S \subseteq N$, $R'_S \in \mathcal{R}^S$ and $i, j \in N \setminus S$ such that $f(R_N) \in [R_i, f(R'_S, R_{N \setminus S})]$ and $f(R'_S, R_{N \setminus S}) \in [R_j, f(R_N)]$ and $f(R_N) \neq f(R'_S, R_{N \setminus S})$. Trivially, we have: (a) $R_i \supseteq \left(\bigcap_{l \in S} R'_l \cap \bigcap_{l \in N \setminus S} R_l \right)$ and (b) $R_j \supseteq \bigcap_{l \in N} R_l$. This yields:

$$ f(R_N) \supseteq R_i \cap f(R'_S, R_{N \setminus S}) \overset{\text{(a) + str. eff.}}{\supseteq} \bigcap_{l \in S} R'_l \cap \bigcap_{l \in N \setminus S} R_l \tag{7} $$

$$ f(R'_S, R_{N \setminus S}) \supseteq R_j \cap f(R_N) \overset{\text{(b)+ str. eff.}}{\supseteq} \bigcap_{l \in N} R_l. \tag{8} $$

Eq. (7) implies that $f(R_N)$ satisfies strong efficiency at profile $(R'_S, R_{N \setminus S})$. Thus, $f(R'_S, R_{N \setminus S}) \succeq f(R_N)$. Equivalently, Eq. (8) implies $f(R_N) \succeq f(R'_S, R_{N \setminus S})$. This is a contradiction. ∎

Proposition 5. Consider $N \supset \mathcal{N}$, R_N and ordering R^0. Recall that the lattice status-quo rule with reference ordering R^0 assigns to each profile R_N the unique ordering R satisfying (i) $R \in [R^0, R_i]$ for all $i \in N$ and; (ii) $\{R' \in [R^0, R_i] \text{ for all } i \in N\} \Rightarrow R' \in [R^0, R]$ for all $R' \in \mathcal{R}$ such that $R' \neq R$. Note that the condition $R \in [R^0, R_i]$ for all $i \in N$ is equivalent to requiring $R \supseteq R^0 \cap \bigcup_{i \in N} R_i$.

Now, consider two profiles R_N and R'_N such that $U(R_N) = U(R'_N)$ and suppose there exists a pair of alternatives (a, b) such that $(a, b) \in \bigcup_{i \in N} R_i$ but $(a, b) \notin \bigcup_{i \in N} R'_i$.

This implies that $(b, a) \in U(R'_N)$ and $(b, a) \notin U(R_N)$, a contradiction. Thus, there can exist no such pair (a, b).

Hence, for all pairs of profiles R_N and R'_N we have $U(R_N) = U(R'_N) \Rightarrow \bigcup_{i \in N} R_i = \bigcup_{i \in N} R'_i$. Combining this fact with the definition of lattice status-quo rules mentioned above, we conclude that two profiles having the same unanimity relation will necessarily yield identical outcomes under any lattice status-quo rule. ∎

Lemma 1. Consider $N \subset \mathcal{N}$ and two profiles R^1_N, R^2_N satisfying $U(R^1_N) \subseteq U(R^2_N)$ and $f(R^1_N) \supseteq U(R^2_N)$. Let $\tilde{R}^1_N$ be a profile in which there exist two agents $i, j \in N$ with orderings $\tilde{R}^1_i = R^1_1$ and $\tilde{R}^1_j = R^1_2$ such that $U(\tilde{R}^1_N) = R^1_1 \cap R^1_2 = U(R^1_N)$.[14] This implies that $\tilde{R}^1_l \in [R^1_1, R^1_2]$ for all other agents $l \neq i, j$.

Similarly, consider a profile $\tilde{R}^2_N$ in which there exist $i, j \in N$ with orderings $\tilde{R}^2_i = R^2_1$ and $\tilde{R}^2_j = R^2_2$ such that $U(\tilde{R}^2_N) = R^2_1 \cap R^2_2 = U(R^2_N)$. This implies that $\tilde{R}^2_l \in [R^2_1, R^2_2]$ for all $l \neq i, j$. Furthermore, suppose there exists an agent $h \in N$ with preferences $\tilde{R}^2_h = f(R^1_N)$.

Finally, assume without loss of generality that $R^2_1 \in [R^1_1, R^1_2]$. Figure 9 provides a schematic view of the orderings $R^1_1, R^1_2, R^2_1, R^2_2$ and our hypothesis regarding $f(R^1_N)$.
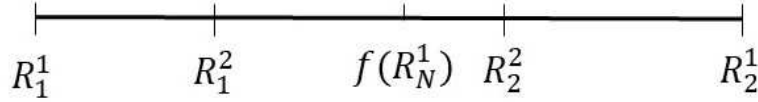

Figure 9: The orderings used in the proof of Lemma 1. An ordering lying between two others denotes betweeness in the Grandmont [14] sense.

Note that unanimity-basedness implies $f(R^1_N) = f(\tilde{R}^1_N)$ and $f(R^2_N) = f(\tilde{R}^2_N)$.

Let $\hat{R}_N$ be a profile in which there exist $i, j \in N$ such that $\hat{R}_i = R^1_1$ and $\hat{R}_j = R^2_2$ and such that $U(\hat{R}_N) = R^1_1 \cap R^2_2$.[15] This implies that $\hat{R}_l \in [R^1_1, R^2_2]$ for all $l \neq i, j$. Furthermore, suppose there exists $k \in N$ such that $\hat{R}_k = f(R^1_N)$.

First, we prove that $f(\hat{R}_N) = f(R^1_N)$. Suppose this is not true, so that $f(\hat{R}_N) \neq f(R^1_N) = f(\tilde{R}^1_N)$. Recall that, by assumption, $\hat{R}_k = f(R^1_N)$. Agent k can misreport

---

[14]This is a slight abuse of notation since R^1_1 and R^1_2 do not necessarily belong in profile R^1_N.

[15]If the reader finds this notation hard to follow, here is an illustrative example. Consider $(R^1_1, R^1_2) = (cdab, bacd)$ and $(R^2_1, R^2_2) = (cbda, cbad)$. This implies $[R^1_1, R^1_2] = [cdab, bacd]$, $[R^2_1, R^2_2] = [cbda, cbad]$, and $[R^1_1, R^2_2] = [cdab, cbad]$, so that (omitting identical pairs of alternatives) we obtain $U(\tilde{R}^1_N) = \{(c, d)\}$, $U(\tilde{R}^2_N) = \{(c, d), (c, b), (c, a), (b, d), (b, a)\}$, and $U(\hat{R}_N) = \{(c, d), (c, a), (c, b)\}$.

his preferences to $R'_k = R^1_2$ so that $U(R'_k, \hat{R}_{N \setminus k}) = U(\tilde{R}^1_N) = U(R^1_N)$. By unanimity-basedness, $f(R'_k, \hat{R}_{N \setminus k}) = f(R^1_N)$, violating strategy-proofness. Thus, $f(\hat{R}_N) = f(R^1_N)$.

Let us now prove that $f(\hat{R}_N) = f(R^2_N)$. Suppose this is not true, so that $f(\hat{R}_N) \neq f(R^2_N) = f(\tilde{R}^2_N)$. By assumption $\tilde{R}^2_h = f(R^1_N)$, and we have just proved that $f(\hat{R}_N) = f(R^1_N)$. Agent h can misreport his preferences to $R'_h = R^1_1$ so that $U(R'_h, \tilde{R}^2_{N \setminus h}) = U(\hat{R}_N)$. By unanimity-basedness, $f(R'_h, \tilde{R}^2_{N \setminus h}) = f(\hat{R}_N) = f(R^1_N)$, violating strategy-proofness. Thus, $f(\hat{R}_N) = f(\tilde{R}^2_N) = f(R^2_N)$.

We conclude that $f(R^1_N) = f(\hat{R}_N) = f(R^2_N)$. ∎

Theorem 2. Necessity follows from Proposition 4.

We turn to proving sufficiency. Recall that, by Lemma 1, given any two $U^1, U^2 \in \mathcal{U}$, the following holds:

$$U^1 \subseteq U^2 \subseteq f(U^1) \implies f(U^1) = f(U^2). \tag{9}$$

Rule f induces a binary relation $\succeq^f$ on $\mathcal{R}$ defined as follows

$$R^1 \succeq^f R^2 \Leftrightarrow \exists U \in \mathcal{U} \text{ such that } R^1, R^2 \supseteq U \text{ and } f(U) = R^1. \tag{10}$$

Let $\succeq^{f^*}$ denote the *transitive extension* of $\succeq^f$. That is,

1. $R^1 \succeq^f R^2 \Rightarrow R^1 \succeq^{f^*} R^2$, for all $R^1, R^2 \in \mathcal{R}$, and

2. $\{R^1 \succeq^f R^2 \text{ and } R^2 \succeq^f R^3\} \Rightarrow R^1 \succeq^{f^*} R^3$, for all distinct $R^1, R^2, R^3 \in \mathcal{R}$.

We will show that $\succeq^{f^*}$ is a conclusive partial order on $\mathcal{R}$.

First we show that $\succeq^{f^*}$ is reflexive. Given $R \in \mathcal{R}$, the partial order $U = R$ satisfies $R \supseteq U$; moreover, by strong efficiency $f(U) \supseteq R \Rightarrow f(U) = R$. Thus, the right-hand side of Eq. (10) implies $R \succeq^{f^*} R$.

Now we show that $\succeq^{f^*}$ is anti-symmetric. Suppose there exist R^1, R^2 such that $R^1 \succeq^{f^*} R^2$ and $R^2 \succeq^{f^*} R^1$. Hence there exist $U^1 \in \mathcal{U}$ such that $f(U^1) = R^1$ and $R^1, R^2 \supseteq U^1$, and $U^2 \in \mathcal{U}$ such that $f(U^2) = R^2$ and $R^1, R^2 \supseteq U^2$. Consider the partial order $R^1 \cap R^2$. Since $R^1 \supseteq U^1$ and $R^2 \supseteq U^1$, we have $U^1 \subseteq R^1 \cap R^2$. Similarly, $U^2 \subseteq R^1 \cap R^2$. Applying (9) to U^1 and $R^1 \cap R^2$, implies $R^1 = f(U^1) = f(R^1 \cap R^2)$. Doing the same for U^2 and $R^1 \cap R^2$ implies $R^2 = f(U^2) = f(R^1 \cap R^2)$. Hence, $R^1 = R^2$.

Finally we prove that $\succeq^{f^*}$ satisfies transitivity. For this purpose it is sufficient to show that for any three distinct orderings R^1, R^2, and R^3 we have $\{R^1 \succeq^f R^2 \text{ and } R^2 \succeq^f R^3\} \Rightarrow R^3 \not\succeq^f R^1$. Suppose otherwise. Thus, there exist $U^1, U^2, U^3 \in \mathcal{U}$ such that:

34

(a) $f(U^1) = R^1$, and $R^1, R^2 \supseteq U^1$.

(b) $f(U^2) = R^2$, and $R^2, R^3 \supseteq U^2$.

(c) $f(U^3) = R^3$, and $R^3, R^1 \supseteq U^3$.

By similar reasoning as when proving anti-symmetry, we must have $f(R^1 \cap R^2) = R^1$, $f(R^2 \cap R^3) = R^2$, and $f(R^3 \cap R^1) = R^3$. To avoid immediate contradictions via Eq. (10), this implies $R^3 \notin [R^1, R^2], R^1 \notin [R^2, R^3]$ and $R^2 \notin [R^1, R^3]$.

Consider the partial order $R^1 \cap R^2 \cap R^3$ on A. We distinguish between two cases:

(i) $f(R^1 \cap R^2 \cap R^3) = R^k \in \{R^1, R^2, R^3\}$. Applying Eq. (9) to partial orders $R^1 \cap R^2 \cap R^3$ and $R^{k-1} \cap R^k$ (defining $R^{k-1} \equiv R^3$ when $k = 1$), we obtain $f(R^1 \cap R^2 \cap R^3) = R^k = f(R^{k-1} \cap R^k) = R^{k-1}$, a contradiction.

(ii) $f(R^1 \cap R^2 \cap R^3) = R^l \notin \{R^1, R^2, R^3\}$. The definition of the betweeness relation implies the following:

$$\left\{ R \in \mathcal{R} : R \supseteq R^1 \cap R^2 \cap R^3 \right\} = \left\{ R \in \mathcal{R} : R \in [R^1, R^2] \cup [R^2, R^3] \cup [R^3, R^1] \right\}$$

Thus by strong efficiency $R^l = f(R^1 \cap R^2 \cap R^3) \in [R^{k-1}, R^k]$ for some $k = 1, 2, 3$ (again, defining $R^{k-1} \equiv R^3$ when $k = 1$), so that $f(R^1 \cap R^2 \cap R^3) \supseteq R^{k-1} \cap R^k$. Applying Eq. (9) to $R^1 \cap R^2 \cap R^3$ and $R^{k-1} \cap R^k$ we obtain $f(R^1 \cap R^2 \cap R^3) = R^l = f(R^{k-1} \cap R^k) = R^{k-1}$, a contradiction.

Thus, $\succeq^{f^*}$ is transitive, which implies that it is partial order on $\mathcal{R}$. Moreover, it is conclusive since for every partial order U on A, the ordering $f(U)$ is such that (i) $f(U) \supseteq U$ and (ii) $f(U) \succeq^{f^*} R$ for all $R \in \mathcal{R}$ satisfying $R \supseteq U$.

Let g be a fixed order status-quo rule associated with $\succeq^{f^*}$. Consider $N \subset \mathcal{N}$ and a profile R_N and let $f(R_N) = f(U(R_N)) = \tilde{R}$. By Eq. (10), $\succeq^{f^*}$ is such that $\tilde{R}$ ranked first among all orderings in the set $\{R \in \mathcal{R} : R \supseteq U(R_N)\}$. Hence, $g(R_N) = f(R_N)$. ∎

6 Acknowledgments

References

[1] Ahuja, R., Magnanti, T., and Orlin, J. (1993). *Network flows: theory, algorithms, and applications.* Prentice Hall, Upper Saddle River, NJ.

[2] Arrow, K. J. (1959). Rational choice functions and orderings. *Economica*, 26, 121-127.

[3] Arrow, K. J. (1963). *Social choice and individual values.* 2nd edition. John Wiley. New York.

[4] Athanasoglou, S. (2016). Strategyproof and efficient preference aggregation with Kemeny-based criteria. *Games and Economic Behavior*, 95, 156-167.

[5] Bossert, W., and Sprumont, Y. (2014). Strategy-proof preference aggregation: possibilities and characterizations. *Games and Economic Behavior*, 85, 109-126.

[6] Bossert, W., and Storcken, T. (1992). Strategy-proofness of social welfare functions: the use of the Kemeny distance between preference orderings. *Social Choice and Welfare*, 9(4), 345–360.

[7] Brandt, F., Conitzer, V., Endriss, U., Lang, J., and Procaccia, A. D. (Eds.). (2016). *Handbook of Computational Social Choice.* Cambridge University Press.

[8] Ching, S., and Thomson, W. (1992). Population-monotonic solutions in public good economies with single-peaked preferences. University of Rochester mimeo. Soc Choice Welfare (forthcoming).

[9] Harless, P. (2015). Reaching Consensus: Solidarity and strategic properties in binary social choice. *Social Choice and Welfare*, 45, 97-121.

[10] Harless, P. (2016). Solidarity in preference aggregation: Improving on a status quo. *Games and Economic Behavior*, 95, 73-87.

[11] Gordon, S. (2007). Public decisions: solidarity and the status quo. *Games and Economic Behavior*, 61(2), 225-241.

[12] Gordon, S. (2007). Solidarity in choosing locations on a cycle. *Social Choice and Welfare*, 29, 125-147.

[13] Gordon, S. (2015). Unanimity in attribute-based preference domains. *Social Choice and Welfare*, 44(1), 13-29.

[14] Grandmont, J. M. (1978). Intermediate preferences and the majority rule. *Econometrica*, 317-330.

[15] Guilbaud, G. T., and Rosenstiehl, P. (1963). Analyse algebrique d'un scrutin. *Math. Sci. Hum*, 4, 9-33.

[16] Kemeny, J.G. (1959). Mathematics without Numbers. *Daedalus*, 88, 575–591.

[17] Moulin, H. (1987). The pure compensation problem: egalitarianism versus laissez-fairism. *Quarterly Journal of Economics*, 102(4), 769-783.

[18] Nehring, K., and Puppe, C. (2002). Strategy-proof social choice on single-peaked domains: possibility, impossibility and the space between. Unpublished manuscript, Department of Economics, University of California at Davis.

[19] Thomson, W. (1983). The fair division of a fixed supply among a growing population. *Mathematics of Operations Research*, 8(3), 319-326.

[20] Thomson, W. (1983). Problems of fair division and the egalitarian solution. *Journal of Economic Theory*, 31, 211-226.

[21] Thomson, W. (1993). The replacement principle in public good economies with single-peaked preferences. *Economics Letters*, 42(1), 31-36.

[22] Young, H. P., and Levenglick, A. (1978). A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, 35(2), 285-300.