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# A general equilibrium evolutionary model with generic utility functions and generic bell-shaped attractiveness maps, generating fashion cycle dynamics

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**Abstract**

We propose a discrete-time exchange economy evolutionary model, in which two groups of agents are possibly characterized by heterogeneous preference structures. With respect to the classical Walrasian framework, in our setting the definition of equilibrium, in addition to utility functions and endowments, depends also on population shares, which affect the market clearing conditions. We prove that, despite such difference with the standard framework, for every economy and for each population shares there exists at least one equilibrium and we show that, for all population shares, generically in the set of the economies, equilibria are finite and regular. We then introduce the dynamic law governing the evolution of the population shares, and we investigate the existence and the stability of the resulting stationary equilibria. More precisely, we assume that the reproduction level of a group is related to its attractiveness degree, which depends on the social visibility level, determined by the consumption choices of the agents in that group. The attractiveness of a group is described via a generic bell-shaped map, increasing for low visibility levels, but decreasing when the visibility of the group exceeds a given threshold value, due to a congestion effect. Thanks to the combined action of the price mechanism and of the share updating rule, the model may reproduce the recurrent dynamic behavior typical of the fashion cycle, presenting booms and busts in the agents' consumption choices, and in the groups' attractiveness and population shares. We illustrate the emergence of fashion cycle dynamics in the case of Stone-Geary utility functions, which generalize the Cobb-Douglas utility functions, and for different formulations of the attractiveness maps, already considered in the literature.

**Keywords:** general equilibrium; heterogeneous agents; evolution; bifurcation; fashion cycle.

**JEL classification:** C62, D11, D51, D91

## 1 Introduction

In the past decades, some dynamical models have been proposed to give a formal representation of the fashion cycle (see, for instance, Bianchi, 2002; Caulkins et al., 2007; Coelho and McClure, 1993; Corneo and Jeanne, 1999; Di Giovinazzo and Naimzada, 2015; Frijters, 1998; Gardini et al., 2018; Karni and Schmeidler, 1990; Matsuyama, 1991; Pesendorfer, 1995; Zhang, 2016, 2017), i.e., of the

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oscillatory behavior of the variable describing the consumed or purchased amount of a certain good, characterized by booms and busts.

Differently from the above mentioned works, following the line of research started in Chang and Stauber (2009) and further developed in Naimzada and Pireddu (2016, 2018a, 2018b, 2018c, 2018d, 2019b), we here present an evolutive general equilibrium model, in which we may investigate the combined effects of the price formation mechanism and of the population share updating mechanism, that describes the socio-economic interaction of two groups of agents exhibiting both bandwagon and snob behaviors. However, unlike the previous contributions, in which only Cobb-Douglas utility functions were considered, we now allow the agents' utility functions to vary in a suitable set of maps. This increased generality in the class of considered utility functions urges us to investigate not only the dynamic features of our model, but also to study the existence and the generic regularity of the market equilibria, which link the equilibrium price and the optimal consumption choices to population shares. Namely, due to the presence of the population shares inside the market clearing conditions, the validity of the standard results about exchange economies has to be carefully verified. We perform such task by adopting the extended approach illustrated in Villanacci et al. (2002). More precisely, we show that for every economy and for any population shares there exists at least a market equilibrium and that, for all population shares, generically in the set of the economies, equilibria are finite and regular. In particular, due to the need to deal with function spaces, we have to employ the infinite-dimensional version of the Implicit Function Theorem by Gloeckner (2006), used in the restricted participation frameworks in Carosi et al. (2009), Gori et al. (2013) and Hoelle et al. (2015), too. Furthermore, in order to avoid indeterminacy issues, we check that a unique equilibrium exists when dealing with utility functions that yield individual demand functions with the gross substitute property.

We then introduce the concept of market stationary equilibrium, that is a market equilibrium which does not vary with time and that solves in any period the dynamic equation describing the share updating rule. In regard to the latter, we stress that, unlike in Chang and Stauber (2009) and in Naimzada and Pireddu (2016, 2018a, 2018b, 2018c), the share updating rule is here based on sociological, rather than biological or merely economic, aspects. In more detail, like in Naimzada and Pireddu (2018d, 2019b), in the present work we deal with an evolutionary mechanism, described via a discrete exponential replicator rule (see e.g. Nachbar, 1990; Sandholm, 2010; Taylor and Jonker, 1978), in which the reproduction level of a group is based on a comparison between the attractiveness degree of the two groups of agents. The attractiveness of a group depends on its social visibility level, which is determined as a linear combination of the amount of the two goods consumed in equilibrium by the agents in that group, multiplied by commodity-specific visibility factors. That is why we restrict our attention to utility functions which imply the existence of a unique equilibrium for every economy and for any population shares. The attractiveness of a group is initially increasing with its visibility level, and this describes the bandwagon regime, characterized by an imitative behavior. However, when the visibility of a group exceeds a given threshold value, a congestion effect is produced. The attractiveness of that group then becomes a decreasing function of its visibility level, and this describes the snob regime, characterized by a predominating wish for distinction. Namely, according to Vigneron and Johnson (1999), on the basis of the empirical literature, too, in the context of luxury and prestige-seeking consumption agents may oscillate between snob and bandwagon behaviors. Such two opposite forces, imitation and distinction, drive the fashion cycle (cf. Simmel, 1957), which for us, thanks also to the price formation mechanism, emerges at the aggregate level as a continuous oscillation of the attractiveness and of the shares of the two groups, while on the individual level it is characterized by oscillatory consumption choices, presenting booms and busts, over the two goods for the agents belonging to the two groups.

While in Naimzada and Pireddu (2018d) just one particular formulation for the attractiveness was considered, like in Naimzada and Pireddu (2019b) we here deal with generic bell-shaped attractiveness functions for the two groups of agents. However, as explained above, with respect to Naimzada and Pireddu (2019b), also utility functions are now free to vary. In order to illustrate the general

results we obtain on existence and local stability of the stationary equilibria, as well as to investigate the emergence of fashion cycle dynamics, we add a simulative section where we consider Stone-Geary utility functions (see Geary, 1950 and Stone, 1954), which generalize the Cobb-Douglas utility functions, and we employ the formulations of the attractiveness maps dealt with in Naimzada and Pireddu (2018d, 2019b). Performing a bifurcation analysis on varying the parameter representing a suitable sensitivity measure, we find that the nontrivial equilibrium, characterized by the coexistence between the agents of the two groups, loses stability via a subcritical or a supercritical flip bifurcation, followed by a cascade of period-doubling bifurcations leading to chaos. Like in Naimzada and Pireddu (2018d, 2019b), also in this work we detect interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors, and oscillatory behaviors both for the population shares and the consumed quantities are allowed. In particular, the dynamic coexistence between groups and the oscillatory nature of the consumption activities display the recurrent dynamic behavior typical of the fashion cycle, with booms and busts. Due to its importance in view of understanding the functioning of the model dynamics, we also provide an interpretation of the main scenarios we found in our bifurcation analysis from a sociological and economic viewpoint, in terms of visibility and attractiveness.
We stress that, since in the simulative part we will consider Stone-Geary utility functions, which generalize the Cobb-Douglas utility functions, and we will deal with the formulations of the attractiveness maps introduced in Naimzada and Pireddu (2018d, 2019b), as a byproduct of our bifurcation analysis we complement the investigation performed in those works, where the bifurcation parameter was the heterogeneity degree between agents, using now as bifurcation parameter a suitable sensitivity measure. Despite some minor dissimilarities, we can deduce that, even with this choice for the bifurcation parameter, in the case of Cobb-Douglas utility functions we obtain scenarios which are dynamically analogous to those observed in Naimzada and Pireddu (2018d, 2019b) and that display the same kinds of multistability phenomena. This suggests that the findings in the previous papers are robust with respect to the employed bifurcation parameter. However, when the bifurcation parameter is the above mentioned sensitivity measure, dealing with Stone-Geary utility functions rather than with Cobb-Douglas utility functions allows to observe a greater variety of dynamical outcomes, obtaining frameworks which bear resemblance to, but that do not coincide with, those detected in Naimzada and Pireddu (2018d, 2019b).
The remainder of the paper is organized as follows. In Section 2 we present and study our model. In particular in Subsection 2.1 we analyze the existence, generic regularity and uniqueness of Walrasian equilibria with population shares under suitable conditions on the economies, while in Subsection 2.2 we present the dynamic equation describing the share updating rule and, after introducing the concept of stationary equilibria, we investigate their existence and local stability. In Section 3 we show the emergence of fashion cycle dynamics for our system when considering Stone-Geary utility functions and different formulations for the attractiveness, giving a sociological and economic interpretation of the main scenarios we find. In Section 4 we briefly discuss our results and describe some variants and extensions of the model. In the Appendix we perform a more detailed bifurcation analysis related to Section 3.

## 2 The model

### 2.1 The Walrasian equilibria with shares

Let us consider an exchange economy with a continuum of agents, which may be of type $\alpha$ or of type $\beta$.[1] There are two consumption goods, $x$ and $y$, and agents' preferences, as in most literature on smooth economies (see e.g. Villanacci et al., 2002), are described by the class of utility functions

[1] We keep the notation used in Chang and Stauber (2009) and in Naimzada and Pireddu (2016, 2018a, 2018b, 2018c, 2018d, 2019b), where $i \in \{\alpha, \beta\}$ was the weight assigned to good $x$ in the Cobb-Douglas utility functions by agents in group $i$, with $0 < \alpha < \beta < 1$.

introduced in the following definition.

**Definition 1.** *For $i \in \{\alpha, \beta\}$, we define $\mathcal{U}_i$ as the set of utility functions $u_i : (0, +\infty)^2 \to \mathbb{R}$ such that*

$(A1)$ $u_i \in \mathcal{C}^2((0, +\infty)^2)$;

$(A2)$ *$u_i$ is differentiably strictly increasing, i.e., $Du_i(x, y) >> 0$, $\forall (x, y) \in (0, +\infty)^2$;*

$(A3)$ *$u_i$ is differentiably strictly quasiconcave, i.e., $Du_i(x, y)v = 0$ implies $vD^2u_i(x, y)v < 0$, $\forall (x, y) \in (0, +\infty)^2$, $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$;*

$(A4)$ *$\forall (\overline{x}, \overline{y}) \in (0, +\infty)^2$, $\{(x, y) \in (0, +\infty)^2 : u_i(x, y) \geq u_i(\overline{x}, \overline{y})\}$ is closed in the topology of $\mathbb{R}^2$.*

Assumption $(A1)$ allows to perform computations and to employ tools from Calculus such as the implicit function theorem. Assumption $(A2)$ says that households always prefer a bundle with slightly more of anything, no matter what they are consuming. Assumption $(A3)$ says that households prefer bundles in which commodities are fairly distributed and ensures the existence of a unique solution to the household maximization problem. Assumption $(A4)$ implies that indifference curves of utility functions do not touch the axes and thus that the solution is interior. By the implicit function theorem, it is possible to show that such solution depends in a smooth manner on endowments and on prices (see Theorem 8.3.397 in Villanacci et al., 2002). Actually, employing an extension of the implicit function theorem valid for maps from topological vector spaces to Banach spaces by Gloeckner (2006) (see Theorem 2 below), it can be easily shown that the solution to the household maximization problem depends in a smooth manner also on utility functions.
We notice that, by Definition 1, we have $\mathcal{U}_\alpha = \mathcal{U}_\beta$. Hence, we will denote the set of utility functions simply by $\mathcal{U}$.
In our model we assume that time is discrete, i.e., that $t \in \mathbb{N}$. The quantity of good $x$ $(y)$ consumed by an agent of type $i \in \{\alpha, \beta\}$ at time $t$ is denoted by $x_{i,t}$ $(y_{i,t})$. Both kinds of agents have the same positive endowments of the two goods, denoted respectively by $w_x$ and $w_y$. We define the set of economies as $\mathcal{E} = \mathcal{U}^2 \times (0, +\infty)^2$, containing the elements $E = (u_\alpha, u_\beta, w_x, w_y)$.[2] In view of the result on generic regularity of market equilibria (cf. Proposition 2), we assume that $\mathcal{E} \subseteq \mathscr{E}$ is endowed with the topology induced by the Hausdorff topological vector space

$$\mathscr{E} = \left[\mathcal{C}^2((0, +\infty)^2)\right]^2 \times \mathbb{R}^2, \tag{1}$$

where $\mathscr{E}$ is endowed with the product topology of the natural topologies on each of the spaces in the Cartesian product. In particular, on the $\mathcal{C}^2$ function space we consider the $\mathcal{C}^2$ compact-open topology.
We denote by $p_{x,t} > 0$ and $p_{y,t} > 0$ the prices at time $t$ for goods $x$ and $y$, respectively. The size of the population of kind $\alpha$ $(\beta)$ at time $t$ is denoted by $A_t$ $(B_t)$. The normalized variable $a_t = A_t/(A_t + B_t) \in [0, 1]$ represents the population fraction composed by the agents of type $\alpha$ and $b_t = 1 - a_t = B_t/(A_t + B_t) \in [0, 1]$ represents the population fraction composed by the agents of type $\beta$.
We are now ready to provide the definition of market equilibrium.

**Definition 2.** *Given the economy $E \in \mathcal{E}$ and the population share $a_t \in [0, 1]$, a market equilibrium at time $t$ is a vector $(p^*_{x,t}, p^*_{y,t}, x^*_{i,t}, y^*_{i,t})$, with $i \in \{\alpha, \beta\}$, such that:*

- *every kind of agent $i$ chooses a utility-maximizing consumption bundle $(x^*_{i,t}, y^*_{i,t})$, given $(p^*_{x,t}, p^*_{y,t})$, i.e., the agents of group $i \in \{\alpha, \beta\}$ at time $t$ solve*

[2] We notice that if $u_\alpha = u_\beta$ then agents are homogeneous since endowments of both goods coincide between groups. We analyzed the case of heterogeneous endowments in the continuous-time setting in Naimzada and Pireddu (2018c). However, in order not to overburden notation and not to excessively complicate the analysis, we will here focus on the case with $w_{x,\alpha} = w_{x,\beta} = w_x$ and $w_{y,\alpha} = w_{y,\beta} = w_y$, implicitly assuming that $u_\alpha \neq u_\beta$, like it happens in the example considered in Section 3.

$$\max_{(x_{i,t},y_{i,t})\in(0,+\infty)^2} u_i(x_{i,t},y_{i,t}) \quad s.t.$$
$$p_{x,t}\,x_{i,t}+p_{y,t}\,y_{i,t}\leq p_{x,t}\,w_x+p_{y,t}\,w_y \tag{2}$$

– *the markets for the two goods clear, i.e., at time $t$ for good $j\in\{x,y\}$ it holds that*

$$a_t\,j_{\alpha,t}+(1-a_t)\,j_{\beta,t}=a_t\,w_j+(1-a_t)w_j=w_j. \tag{3}$$

We notice that, since utility functions are differentiably strictly increasing, problem (2) may be rewritten as

$$\max_{(x_{i,t},y_{i,t})\in(0,+\infty)^2} u_i(x_{i,t},y_{i,t}) \quad \text{s.t.}$$
$$p_{x,t}\,x_{i,t}+p_{y,t}\,y_{i,t}=p_{x,t}\,w_x+p_{y,t}\,w_y \tag{4}$$

and from the budget constraint we obtain $x_{i,t}=w_x+p_tw_y-p_t\,y_{i,t}$, where we set $p_t=p_{y,t}/p_{x,t}$. Hence, (4) simply becomes

$$\max_{y_{i,t}\in(0,+\infty)} u_i(w_x+p_tw_y-p_t\,y_{i,t},y_{i,t}).$$

From here, thanks to the fact that $u_i$ is differentiably strictly quasiconcave, there exists a unique optimal consumption choice for good $y$, depending on $p_t$, that we call $y^*_{i,t}(p_t)$. Hence, the optimal consumption choice for agent $i$ of good $x$, depending on $p_t$, that we call $x^*_{i,t}(p_t)$, is given by $x^*_{i,t}(p_t)=w_x+p_t\,w_y-p_t\,y^*_{i,t}(p_t)$. The equilibrium price $p^*_t$ can then be determined by using one of the two market clearing conditions in (3), since by Walras' law the other market clearing condition is redundant. In this manner $p^*_t$ will be influenced by the population share $a_t$, so that we can write $p^*_t(a_t)$. Inserting $p^*_t(a_t)$ into $x^*_{i,t}(p_t)$ and $y^*_{i,t}(p_t)$, we find the equilibrium consumption choices $x^*_{i,t}$ and $y^*_{i,t}$ for agent $i$, which will depend on $a_t$, as well. Indeed, using the extended approach based on first order conditions and market clearing conditions to characterize equilibria (cf. Paragraph 8.4 in Villanacci et al., 2002), it is possible to prove that, in any time period, for all $E\in\mathcal{E}$ and $a_t\in(0,1)$, there exists at least a market equilibrium[3] (see Proposition 1) and that, for all population shares, generically in the set of economies, market equilibria are finite and regular, i.e., they depend in a smooth manner on economies and population shares (cf. Proposition 2 where, like in Carosi et al., 2009, Gori et al., 2013, and Hoelle et al., 2015, we complement the finite-dimensional analysis performed in Villanacci et al. 2002 using the infinite-dimensional version of the Implicit Function Theorem by Gloeckner, 2006). Moreover, in order to avoid indeterminacy issues, we check in Proposition 3 that a unique equilibrium exists when dealing with utility functions that yield individual demand functions with the gross substitute property. We recall that a characterization of such class of utility functions has been provided in Fisher (1972). The need to restrict our attention to utility functions which imply the existence of a unique equilibrium for every economy and for all population shares comes from the fact that, as we shall see in Subsection 2.2, in our model the attractiveness of a group depends on its social visibility level, which is obtained as a linear combination of the amount of the two goods consumed in equilibrium by the agents in that group. Hence, the existence of a unique equilibrium prevents indeterminacy issues, in which different attractiveness levels correspond to the same economy.

We stress that, although by now no dynamic aspects have been introduced in the model, and thus we are just considering a variation of the classical exchange economy setting with two consumers, in which we take into account population shares in the market clearing conditions, we need to check that all the steps in the original proofs of existence, generic regularity and uniqueness of equilibria still

[3] We remark that the argument above suggests that market equilibria exist for any economy even when $a=0$ and $a=1$, although such extreme cases are not encompassed in Proposition 1 due to the need to deal with open sets because of the differential topology kind of proof. We also notice that considering an open interval of the form $(-\varepsilon,1+\varepsilon)$, with $\varepsilon>0$ arbitrarily small, would not solve the issue, as some steps in the proof of Proposition 1 would not work anymore.

hold true in our framework. In particular, such verification cannot be performed on the Edgeworth box because the two groups of agents in general do not have the same numerosity. We also remark that, in order to show the existence of equilibria, we could use continuity arguments applied to the excess demand function (cf. pages 584-585 in Mas-Colell et al., 1995). However, for the homogeneity's sake with the proof of generic regularity of equilibria, we prefer to employ the extended approach to show that equilibria exist, too. The corresponding results read as follows:

**Proposition 1.** *For every economy $E \in \mathcal{E}$, for every $t \in \mathbb{N}$ and for every population share $a_t \in (0,1)$, there exists at least a market equilibrium at time $t$.*

**Proposition 2.** *For every $t \in \mathbb{N}$ and for every population share $a_t \in (0,1)$, there exists an open and full measure*[4] *subset $\mathcal{D}(a_t)$ of $\mathcal{E}$ such that, for any $E \in \mathcal{D}(a_t)$, there is a (positive) finite number of associated market equilibria which locally smoothly depend on the elements $(E, a)$ of $\mathcal{E} \times (0,1)$.*

In relation to Proposition 2, we assume that $\mathcal{E} \times (0,1)$ is endowed with the topology induced by the Hausdorff topological vector space $\mathscr{E} \times \mathbb{R}$, where $\mathscr{E}$ has been introduced in (1) and $\mathscr{E} \times \mathbb{R}$ is endowed with the product topology.
In the proofs of Propositions 1 and 2 we are going to use the two following results (cf. Theorem 7.5.368 in Villanacci et al., 2002 for the former, and for the latter see Theorem 2.3 in Gloeckner, 2006, of which Theorem 2 is a simplified version).

**Theorem 1.** *Let $M$, $N$ be $\mathcal{C}^2$ boundaryless manifolds of the same dimension, $y \in N$ and $\Phi, \Gamma : M \to N$ be continuous functions. Assume that $\Gamma$ is $\mathcal{C}^1$ in an open neighborhood $U$ of $\Gamma^{-1}(y)$, $y$ is a regular value for $\Gamma$ restricted to $U$, the cardinality of $\Gamma^{-1}(y)$ is finite and odd, and there exists a continuous homotopy $\Psi : M \times [0,1] \to N$ from $\Phi$ to $\Gamma$ such that $\Psi^{-1}(y)$ is compact. Then $\Phi^{-1}(y) \neq \emptyset$.*

**Theorem 2.** *Let us consider $f : O \times B \to \mathbb{R}^n$, where $O$ is an open subset of $\mathbb{R}^n$ and $B$ is an open subset of a topological Hausdorff vector space $\mathscr{B}$. Assume that $f \in \mathcal{C}^1(O \times B, \mathbb{R}^n)$ and let $(x_0, b_0) \in O \times B$ be such that $f(x_0, b_0) = 0$ and $D_x f(x_0, b_0)$ is invertible. Then there exist $O(x_0) \subseteq O$ open neighborhood of $x_0$, $B(b_0) \subseteq B$ open neighborhood of $b_0$ and $\varphi : B(b_0) \to O(x_0)$ such that*

1. *$\varphi \in \mathcal{C}^1(B(b_0), O(x_0))$,*
2. *$\varphi(b_0) = x_0$,*
3. *$\{(x,b) \in O(x_0) \times B(b_0) : f(x,b) = 0\} = \{(x,b) \in O(x_0) \times B(b_0) : x = \varphi(b)\}$.*

In regard to Theorem 2, we report some definitions related to its statement. Given a topological Hausdorff vector space $\mathscr{S}$ containing an open set $S$ and a function $g : S \to \mathbb{R}^n$, we say that $g \in \mathcal{C}^0(S, \mathbb{R}^n)$ if $g$ is continuous, while we say that $g \in \mathcal{C}^1(S, \mathbb{R}^n)$ if it is continuous, the limit

$$dg(s, \sigma) = \lim_{\varepsilon \to 0} \frac{g(s + \varepsilon\sigma) - g(s)}{\varepsilon}, \tag{5}$$

exists for all $s \in S$, $\sigma \in \mathscr{S}$, and the function $dg : S \times \mathscr{S} \to \mathbb{R}^n$ is continuous. For further mathematical details, see Hoelle et al. (2015). In particular, as explained therein, the need to use the Gloeckner implicit function theorem comes from the fact that the $\mathcal{C}^2$ compact-open topology, the set of twice continuously differentiable utility functions is commonly endowed with, is not generated by a norm (see page 35 in Hirsch, 1976) and thus the standard implicit function theorem cannot be applied in

[4] Actually, along the proof we show the validity of a stronger result, i.e., that in any time period and for every population share, for each choice of the utility functions in $\mathcal{U}$, there exists an open and full measure subset of the space of endowments where the generic regularity property holds. In particular, the fact that the smooth dependence of equilibria holds for all share values, for almost all endowment combinations, is crucial, as we will characterize market stationary equilibria in terms of population shares.

our framework. On the other hand, since $\mathscr{E}$ in (1) is a topological Hausdorff vector space, Theorem 2 can be used instead.

*Proof of Proposition 1:* Since the arguments we shall employ are independent of the considered time period, in order not to overburden notation, we will omit the subscript $t$, as well as the stars, which will refer to the Pareto Optimal allocation only.
We define $\theta = \Big(p_x, p_y, (x_i, y_i)_{i\in\{\alpha,\beta\}}\Big) \in \Theta = (0,+\infty)^6$ and, according to Definition 2, we say that $\theta \in \Theta$ is a market equilibrium given $E \in \mathcal{E}$ and $a \in (0,1)$ if, for every $i \in \{\alpha, \beta\}$, $(x_i, y_i)$ solves problem (4) at $(p_x, p_y, E)$ and $(x_i, y_i)_{i\in\{\alpha,\beta\}}$ satisfies market clearing conditions (3) at $(E, a)$.
We denote by $\Theta(E, a)$ the set of market equilibria for $E$ and $a$, and we introduce

$$\Theta_{\mathrm{n}}(E,a) = \{\theta \in \Theta(E,a) : p_x = 1\},$$

that is, the set of normalized[5] market equilibria for $E$ and $a$. We will show that $\Theta_{\mathrm{n}}(E, a) \neq \emptyset$, for every $(E, a) \in \mathcal{E} \times (0, 1)$.
As one Walras' law holds true in our model, just one market clearing condition in Definition 2 is significant. To fix ideas, we will focus on the equation for commodity $y$, i.e., on

$$a\, y_\alpha + (1-a)\, y_\beta = w_y.$$

The extended system for our economy with two groups of agents, whose numerosity may not coincide, reads as

$$\begin{cases} \frac{\partial u_\alpha}{\partial x_\alpha}(x_\alpha, y_\alpha) - \lambda_\alpha p_x = 0 \\ \frac{\partial u_\alpha}{\partial y_\alpha}(x_\alpha, y_\alpha) - \lambda_\alpha p_y = 0 \\ -p_x\, x_\alpha - p_y\, y_\alpha + p_x\, w_x + p_y\, w_y = 0 \\ \frac{\partial u_\beta}{\partial x_\beta}(x_\beta, y_\beta) - \lambda_\beta p_x = 0 \\ \frac{\partial u_\beta}{\partial y_\beta}(x_\beta, y_\beta) - \lambda_\beta p_y = 0 \\ -p_x\, x_\beta - p_y\, y_\beta + p_x\, w_x + p_y\, w_y = 0 \\ a\, y_\alpha + (1-a)\, y_\beta - w_y = 0 \\ p_x - 1 = 0 \end{cases} \tag{6}$$

Since we are going to study market equilibria in terms of first-order conditions associated with households' maximization problems and (significant) market clearing conditions, we define

$$\xi = \Big(p_x, p_y, (x_i, y_i, \lambda_i)_{i\in\{\alpha,\beta\}}\Big) \in (0,+\infty)^8,$$

and the function

$$\mathcal{F} : (0,+\infty)^8 \times \mathcal{E} \times (0,1) \to \mathbb{R}^8, \quad \mathcal{F}(\xi, E, a) = \text{lhs of (6)}. \tag{7}$$

Given $(E, a) \in \mathcal{E} \times (0, 1)$, it is immediate to prove that if $\theta = \Big(p_x, p_y, (x_i, y_i)_{i\in\{\alpha,\beta\}}\Big) \in \Theta_{\mathrm{n}}(E, a)$, then there exists the vector $(\lambda_\alpha, \lambda_\beta) \in (0,+\infty)^2$ such that $\xi = \Big(p_x, p_y, (x_i, y_i, \lambda_i)_{i\in\{\alpha,\beta\}}\Big) \in (0,+\infty)^8$ solves the system $\mathcal{F}(\xi, E, a) = 0$. Vice versa, if $\xi = \Big(p_x, p_y, (x_i, y_i, \lambda_i)_{i\in\{\alpha,\beta\}}\Big) \in (0,+\infty)^8$ solves $\mathcal{F}(\xi, E, a) = 0$, then $\Big(p_x, p_y, (x_i, y_i)_{i\in\{\alpha,\beta\}}\Big) \in \Theta_{\mathrm{n}}(E)$.
Let us then show that for all $(E, a) \in \mathcal{E} \times (0, 1)$ there exists $\xi \in (0,+\infty)^8$ which solves system

[5] We stress that, differently from what done in Villanacci et al. (2002), we normalize the price of the first commodity, rather than of the last one. Of course, this change does not affect the validity of the results.

$\mathcal{F}(\xi, E, a) = 0$. Fixing $(E, a) \in \mathcal{E} \times (0,1)$, we define

$$F : (0, +\infty)^8 \to \mathbb{R}^8, \quad F(\xi) = \mathcal{F}(\xi, E, a).$$

Let us also introduce the homotopy

$$H : (0, +\infty)^8 \times [0,1] \to \mathbb{R}^8, \quad H(\xi, \tau) = \text{lhs of (8)},$$

with

$$\begin{cases} \frac{\partial u_\alpha}{\partial x_\alpha}(x_\alpha, y_\alpha) - \lambda_\alpha p_x = 0 \\ \frac{\partial u_\alpha}{\partial y_\alpha}(x_\alpha, y_\alpha) - \lambda_\alpha p_y = 0 \\ -p_x\, x_\alpha - p_y\, y_\alpha + p_x\left((1-\tau)w_x + \tau x_\alpha^*\right) + p_y\left((1-\tau)\, w_y + \tau y_\alpha^*\right) = 0 \\ \frac{\partial u_\beta}{\partial x_\beta}(x_\beta, y_\beta) - \lambda_\beta p_x = 0 \\ \frac{\partial u_\beta}{\partial y_\beta}(x_\beta, y_\beta) - \lambda_\beta p_y = 0 \\ -p_x\, x_\beta - p_y\, y_\beta + p_x\left((1-\tau)w_x + \tau x_\beta^*\right) + p_y\left((1-\tau)\, w_y + \tau y_\beta^*\right) = 0 \\ a\, y_\alpha + (1-a)\, y_\beta - \Big((1-\tau)w_y + \tau(a y_\alpha^* + (1-a) y_\beta^*)\Big) = 0 \\ p_x - 1 = 0, \end{cases} \tag{8}$$

where $(x_i^*, y_i^*)_{i \in \{\alpha,\beta\}} \in (0, +\infty)^4$ is a Pareto Optimal allocation[6], whose existence can be proven as in Section 8.5 in Villanacci et al. (2002). In particular, denoting by $r$ the vector of the total resources associated with $E$, we have $r = (w_x, w_y) \in (0, +\infty)^2$ and, denoting by $\underline{U}^r$ the set of utility level vectors attainable with resources $r$ in correspondence to $E \in \mathcal{E}$ and $a \in (0,1)$, we have

$$\underline{U}^r = \left\{ (\underline{u}_\alpha, \underline{u}_\beta) \in \mathbb{R}^2 : \exists (x_\alpha, x_\beta, y_\alpha, y_\beta) \in (0, +\infty)^4 \text{ s.t. } \begin{array}{l} a\, j_\alpha + (1-a)\, j_\beta = w_j, \text{ for } j \in \{x, y\}, \\ \text{and } u_i(x_i, y_i) - \underline{u}_i = 0, \text{ for } i \in \{\alpha, \beta\} \end{array} \right\}.$$

We notice that $H(\xi, 0) = F(\xi)$, $\forall \xi \in (0, +\infty)^8$. Setting

$$G : (0, +\infty)^8 \to \mathbb{R}^8, \quad G(\xi) = H(\xi, 1),$$

it holds that $F$, $H$ and $G$ are continuous functions. If we prove that, for some $\widehat{\xi} \in (0, +\infty)^8$,

$$G^{-1}(0) = \{\widehat{\xi}\} \text{ and } G \text{ is } \mathcal{C}^1 \text{ in an open neighborhood of } \widehat{\xi}, \tag{9}$$

$$D_\xi G(\widehat{\xi}) \text{ is not singular,} \tag{10}$$

$$H^{-1}(0) \text{ is compact,} \tag{11}$$

then Theorem 1 can be applied with the identifications $M = (0, +\infty)^8$, $N = \mathbb{R}^8$, $y = 0 \in \mathbb{R}^8$, $\Phi = F$, $\Gamma = G$, $\Psi = H$, to get $F^{-1}(0) \neq \emptyset$, so that a market equilibrium exists in correspondence to the fixed $(E, a) \in \mathcal{E} \times (0,1)$.
The proof of conditions (9), (10) and (11) follows by standard arguments and it is omitted. □

*Proof of Proposition 2:* Since the arguments we shall employ are independent of the considered time period, like in the proof of Proposition 1, in order not to overburden notation, we will omit the subscript $t$.
Recalling the definition of the map $\mathcal{F}$ in (7), it is evident that $\mathcal{F}$ is continuous. In order to apply Theorem 2 with the identifications $f = \mathcal{F}$, $\mathbb{R}^n = \mathbb{R}^8$, $O = (0, +\infty)^8$, $B = \mathcal{E} \times (0,1)$, $\mathscr{B} = \mathscr{E} \times \mathbb{R}$, with $\mathscr{E}$ as in (1), we have to check that $\mathcal{F} \in \mathcal{C}^1((0, +\infty)^8 \times \mathcal{E} \times (0,1), \mathbb{R}^8)$ according to the definition in (5),

[6]See Naimzada and Pireddu (2019a) for the corresponding definition and for the proof, in our framework, of the first fundamental theorem of welfare economics, according to which every (stationary) equilibrium allocation is Pareto optimal.

i.e., that

$$d\mathcal{F}: \big((0,+\infty)^8 \times \mathcal{E} \times (0,1)\big) \times \big(\mathbb{R}^8 \times \mathscr{E} \times \mathbb{R}\big) \to \mathbb{R}^8$$

is well defined and continuous. Considering any $(\xi, E, a) \in (0,+\infty)^8 \times \mathcal{E} \times (0,1)$ and $(\nu, \eta, \kappa) \in \mathbb{R}^8 \times \mathscr{E} \times \mathbb{R}$, it holds indeed that the limit

$$d\mathcal{F}(\xi, E, a) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}(\xi + \varepsilon\nu, E + \varepsilon\eta, a + \varepsilon\kappa) - \mathcal{F}(\xi, E, a)}{\varepsilon}$$

exists, and it is easy to check that the map $d\mathcal{F}$ is continuous.
Next, we want to show that for every $a \in (0,1)$ the set

$$\mathcal{D}(a) = \{E \in \mathcal{E} : \mathcal{F}(\xi, E, a) = 0 \Rightarrow \det D_\xi \mathcal{F}(\xi, E, a) \neq 0\}$$

is an open and full measure subset of $\mathcal{E}$. Namely, applying Theorem 2, we obtain the smooth dependence of the market equilibria associated to all population shares $a \in (0,1)$ and to any economy $E \in \mathcal{D}(a)$ on the elements $(E, a) \in \mathcal{E} \times (0,1)$.
Let us then sketch the proof that, for every $a \in (0,1)$, $\mathcal{D}(a)$ is an open and full measure subset of $\mathcal{E}$. The openness of $\mathcal{D}(a)$ follows by the closeness of the complement set $\mathcal{E} \setminus \mathcal{D}(a)$, due to the continuity of the involved functions. In order to show that $\mathcal{D}(a)$ is a full measure subset of $\mathcal{E}$, it suffices to show that for every $(u_\alpha, u_\beta, a) \in \mathcal{U}^2 \times (0,1)$ the set

$$\mathcal{D}(u_\alpha, u_\beta, a) = \big\{(w_x, w_y) \in (0,+\infty)^2 : \mathcal{F}(\xi, u_\alpha, u_\beta, w_x, w_y, a) = 0 \Rightarrow \det D_\xi \mathcal{F}(\xi, E, a) \neq 0\big\}$$

is a full measure subset of $(0,+\infty)^2$. Since it can be proven that 0 is a regular value for the map

$$\widetilde{\mathcal{F}}: (0,+\infty)^8 \times (0,+\infty)^2 \to \mathbb{R}^8, \quad \widetilde{\mathcal{F}}(\xi, w_x, w_y) = \mathcal{F}(\xi, u_\alpha, u_\beta, w_x, w_y, a),$$

then, by a transversality result (cf. Theorem 6.3.294 in Villanacci et al., 2002), there exists a full measure subset $\widetilde{\mathcal{D}}(u_\alpha, u_\beta, a)$ of $(0,+\infty)^2$ such that, for all $(w_x, w_y) \in \widetilde{\mathcal{D}}(u_\alpha, u_\beta, a)$, 0 is a regular value for the map $\widetilde{\mathcal{F}}(\,\cdot\,, w_x, w_y)$. As $\widetilde{\mathcal{D}}(u_\alpha, u_\beta, a) \subseteq \mathcal{D}(u_\alpha, u_\beta, a)$, it follows that $\mathcal{D}(u_\alpha, u_\beta, a)$ is a full measure subset of $(0,+\infty)^2$. Consequently, for every $a \in (0,1)$, $\mathcal{D}(a)$ is a full measure subset of $\mathcal{E}$.
The finiteness of the number of market equilibria associated to an element $(a, E)$, with $a \in (0,1)$ and $E \in \mathcal{D}(a)$, comes now from the fact that, as it is easy to check, the projection

$$\pi : \mathcal{F}^{-1}(0) \to \mathcal{E} \times (0,1), \quad \pi(\xi, E, a) = (E, a),$$

is proper, i.e., the inverse image through $\pi$ of a compact subset of $\mathcal{E} \times (0,1)$ is compact, too.
The proof is complete. □

In view of the introduction in Subsection 2.2 of the concepts of visibility and attractiveness of a group (cf. (13) and Definition 4), based on the equilibrium consumption choices, in order to avoid indeterminacy issues, we check in the next result that a unique equilibrium exists, even in the case of groups of agents with a different numerosity, when dealing with utility functions that yield individual demand functions with the gross substitute property. Omitting the (now irrelevant) dependence on time, we recall that, in the case of an exchange economy with two goods, the Walrasian demand function for a certain consumer of good $j \in \{x, y\}$, that we denote by $j^*(p_x, p_y)$, satisfies the gross substitute property if whenever $\widetilde{p}_j = p_j$ and $\widetilde{p}_k > p_k$, for $k \in \{x, y\}$, $k \neq j$, it holds that $j^*(\widetilde{p}_x, \widetilde{p}_y) > j^*(p_x, p_y)$.

**Proposition 3.** *Assume that, given $E \in \mathcal{E}$, the individual demand functions, for agents $\alpha$ and $\beta$, of goods $x$ and $y$ satisfy the gross substitute property. Then, for all $E \in \mathcal{E}$, $t \in \mathbb{N}$ and $a_t \in (0,1)$, there exists a unique market equilibrium at time $t$.*

*Proof.* By Theorem 1 we know that, for any $E \in \mathcal{E}$, $t \in \mathbb{N}$ and $a_t \in (0,1)$, there exists at least a

market equilibrium at time $t$. Let us then check that the equilibrium is unique, omitting as usual the dependence on $t$, in order not to overburden notation.
Recalling Definition 2 and the formulation of the market clearing conditions with shares in (3), we have to prove that if the gross substitute property holds for all the individual demand functions there exists one solution $(\widehat{p}_x, \widehat{p}_y) \in (0, +\infty)^2$ to the system

$$a\, x^*_\alpha(p_x, p_y) + (1-a)\, x^*_\beta(p_x, p_y) - w_x = 0 = a\, y^*_\alpha(p_x, p_y) + (1-a)\, y^*_\beta(p_x, p_y) - w_y. \tag{12}$$

Since we can normalize one price, to show the uniqueness of the solution, we may focus on the price vectors $(1, p) \in (0, +\infty)^2$, with $p = p_y/p_x$, Assuming that $a\, x^*_\alpha(1, \widehat{p}) + (1-a)\, x^*_\beta(1, \widehat{p}) - w_x = 0 = a\, y^*_\alpha(1, \widehat{p}) + (1-a)\, y^*_\beta(1, \widehat{p}) - w_y$, let us prove that no $(1, \widetilde{p})$, with $\widetilde{p} \neq \widehat{p}$, may solve (12). Namely, if $\widetilde{p} < \widehat{p}$, by the gross substitute property it would hold that $x^*_i(1, \widetilde{p}) < x^*_i(1, \widehat{p})$, $i \in \{\alpha, \beta\}$, and thus, since $a \in (0, 1)$, we would have

$$a\, x^*_\alpha(1, \widetilde{p}) + (1-a)\, x^*_\beta(1, \widetilde{p}) - w_x < a\, x^*_\alpha(1, \widehat{p}) + (1-a)\, x^*_\beta(1, \widehat{p}) - w_x = 0,$$

so that $(1, \widetilde{p})$ would not be a solution to (12). Similarly, if $\widetilde{p} > \widehat{p}$, by the gross substitute property it would hold that $x^*_i(1, \widetilde{p}) > x^*_i(1, \widehat{p})$, $i \in \{\alpha, \beta\}$, and thus, since $a \in (0, 1)$, it would follow that

$$a\, x^*_\alpha(1, \widetilde{p}) + (1-a)\, x^*_\beta(1, \widetilde{p}) - w_x > a\, x^*_\alpha(1, \widehat{p}) + (1-a)\, x^*_\beta(1, \widehat{p}) - w_x = 0,$$

so that $(1, \widetilde{p})$ would not solve (12).
This completes the proof. □

## 2.2 The share updating rule and the stationary equilibria

In Subsection 2.1 we introduced the notion of market equilibrium in Definition 2, which encompasses both the agents' choice of utility-maximizing consumption bundles and the market clearing conditions. We stress that such concept differs from the dynamic notion of equilibrium, i.e., of market stationary equilibrium. Indeed, by the latter we mean a market equilibrium in which population shares, and consequently prices and optimal consumption choices, are constant over time. In particular, shares are constant because in any period $t$ they solve the dynamic equation governing the share updating mechanism (see (15)). If shares are constant, the equilibrium price determined through the market clearing condition is constant, too, and consequently also the equilibrium consumption choices are constant.
Accordingly, we introduce the following definition of market stationary equilibrium:

**Definition 3.** *Given the economy $E \in \mathcal{E}$, the vector $(a^*, p^*, x^*_i, y^*_i)$, $i \in \{\alpha, \beta\}$, is a market stationary equilibrium if $a^* \in [0, 1]$ is constant and if, given $a^*$, $(p^*, x^*_i, y^*_i)$, $i \in \{\alpha, \beta\}$, is a market equilibrium in every time $t$.*

We remark that, in order not to overburden notation and terminology, although $a^*$ is not part of the market equilibrium vector introduced in Definition 2, we call the objects described in Definition 3 (market stationary) equilibria, and we use the symbol * even for the shares. We also notice that in Definition 2 there are time subscripts, missing in Definition 3, as the latter describes a stationary, time-unvarying, situation.
For the sake of brevity, when each economy admits a unique market equilibrium for all population shares, we shall identify market stationary equilibria just with the population share $a^*$, since it determines all other equilibrium components. Namely, according to what explained in Subsection 2.2, when dealing e.g. with utility functions that yield individual demand functions with the gross substitute property, it holds that, for every economy, $a^*$ determines a unique equilibrium price $p^*$, which in turns determines a unique equilibrium allocation $(x^*_i, y^*_i)_{i\in\{\alpha,\beta\}}$. In fact, in what follows we will

focus on the subset $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ in which utility functions yield individual demand functions with the gross substitute property, so that we will identify market stationary equilibria for the economies in $\widetilde{\mathcal{E}}$ with the corresponding population share.
The market stationary equilibria will be called trivial if they are not characterized by the coexistence between the two groups of agents, and nontrivial otherwise.

Let us now illustrate the population share evolutionary mechanism, based on a sociological payoff. In a social interaction framework, consumption choices produce visibility. Since the various goods induce different visibility levels, we introduce the positive parameters $v_x$ and $v_y$ describing the degree of visibility that each agent derives from the consumption of a unit of commodity $x$ and of commodity $y$, respectively. Given such assumptions, we define the social visibility level $V_{i,t}$ of an agent of type $i \in \{\alpha, \beta\}$ at time $t$ as a linear combination of the units $x_{i,t}$ and $y_{i,t}$ of goods $x$ and $y$ he consumes, weighted respectively with the positive parameters $v_x$ and $v_y$, i.e., $V_{i,t} = v_x x_{i,t} + v_y y_{i,t}$. In particular, at the market equilibrium, which is unique under the maintained assumption that our utility functions yield individual demand functions with the gross substitute property, it holds that

$$V^*_{i,t} = v_x x^*_{i,t} + v_y y^*_{i,t}, \quad \text{for } i \in \{\alpha, \beta\}\,. \tag{13}$$

For ease of notation, since we will consider visibility values just in correspondence to the market equilibrium, we will denote $V^*_{i,t}$ simply by $V_{i,t}$. The same remark applies to the attractiveness functions we shall introduce in Definition 4 below, which we will denote by $\mathscr{A}_{i,t}$, rather than by $\mathscr{A}^*_{i,t}$.
We stress that, although we assume that the parameters $v_x$ and $v_y$ cannot vanish, the case of polarized values for $v_x$ and $v_y$, in which both of them are positive but one is much larger than the other, is allowed and well approximates those frameworks in which visibility and attractiveness are produced by the consumption of a single good.
Agents' consumption choices, deriving by the underlying preference structures, give rise to different attractiveness degrees, when the preference structures of the two groups do not coincide. Indeed, in a social interaction setting, the attractiveness of a preference structure depends on its visibility in a non-monotone manner: the social attractiveness of a preference structure is increasing in its visibility as long as the latter is not excessive, and then such dependence becomes decreasing.
Introducing the attractiveness $\mathscr{A}_{i,t}$ of group $i$ as a function of $V_{i,t}$, $i \in \{\alpha, \beta\}$, suitable hypotheses on such map are that it is bell-shaped, increasing with the visibility level up to a certain threshold value $\overline{V}$, above which it becomes decreasing in a symmetric manner with respect to $\overline{V}$, due to a congestion effect. Hence, $\mathscr{A}_{i,t}$ will be for us any differentiable and strictly decreasing function of $d(\overline{V}, V_{i,t}) = |\overline{V} - V_{i,t}|$, where $d$ denotes the Euclidean distance. In particular, in order to avoid differentiability issues, we will consider functions depending on $d^2(\overline{V}, V_{i,t}) = (\overline{V} - V_{i,t})^2$. For simplicity, in what follows we will sometimes denote $d^2(\overline{V}, V_{i,t})$ just by $d_i^2$. We stress that it would be possible to view the attractiveness as an index, normalizing such variable and assuming that $\mathscr{A}_{i,t}$ varies in the interval $(-1, 1)$, so that negative values of the attractiveness could be interpreted as a repulsion, positive values of $\mathscr{A}_{i,t}$ would describe an attraction, and a null attractiveness would represent indifference toward a certain lifestyle. However, since such normalization would complicate the expression of attractiveness, we do not impose any restriction on the values that $\mathscr{A}_{i,t}$ may assume.

**Definition 4.** *Given a map $f : [0, +\infty) \to (-\infty, f(0)]$, differentiable and strictly decreasing, and $\sigma > 0$, $\overline{V} > 0$, recalling the definition of $V_{i,t}$ in* (13)*, for $i \in \{\alpha, \beta\}$, any function*

$$\mathscr{A}_{i,t} : (0, +\infty)^3 \to (-\infty, \mathscr{A}_{i,t}(\overline{V})], \quad \mathscr{A}_{i,t}(V_{i,t}; \sigma, \overline{V}) = f(\sigma\, d^2(\overline{V}, V_{i,t})), \tag{14}$$

*will be called attractiveness of group $i$ at time $t$. The set of the admissible attractiveness functions of group $i$ at time $t$ will be denoted by $\mathcal{A}_{i,t}$.*

We notice that, by definition, we have $\mathcal{A}_{\alpha,t} = \mathcal{A}_{\beta,t}$, for every $t$, and that, for $i \in \{\alpha, \beta\}$, it holds that $\mathcal{A}_{\alpha,t'} = \mathcal{A}_{\alpha,t''}$, for all $t'$, $t'' \in \mathbb{N}$. Hence, we will denote the set of the admissible attractiveness functions simply by $\mathcal{A}$. In particular, in order to simplify computations and, most importantly, in view of making the share updating rule depending just on a comparison between the distance of the visibility levels for the two groups from the threshold value $\overline{V}$, we will assume that the same map $f$ in (14) describes the attractiveness of both groups $\alpha$ and $\beta$. Nonetheless, we will specify after the main results we obtain what it would be possible to infer when allowing for different attractiveness formulations between groups.

As concerns the parameters, we remark that, by (14), the attractiveness depends in an explicit manner only on $\sigma$. The dependence of $\mathscr{A}_{i,t}$ on the other model parameters is indirect, and occurs due to the presence of $V_{i,t}$. The parameter $\sigma$ in (14) describes the sensitivity of the attractiveness $\mathscr{A}_{i,t}$, for $i \in \{\alpha, \beta\}$, with respect to the distance between the visibility level $V_{i,t}$ and the threshold level $\overline{V}$. In particular, a visibility level coinciding with $\overline{V}$ allows maximizing the attractiveness degree. We stress that when $\sigma \to 0$ agents are insensitive even to a large distance between $V_{i,t}$ and $\overline{V}$, and $\mathscr{A}_{i,t}$ tends towards its maximum value $\mathscr{A}_{i,t}(\overline{V})$. When instead $\sigma \to +\infty$, even a small, but still positive, distance between $V_{i,t}$ and $\overline{V}$ leads to a very low attractiveness level because of an excessive sensitivity to such distance.

The attractiveness functions satisfying the assumptions in Definition 4 are suitable to describe, in relation to group $i \in \{\alpha, \beta\}$, both the bandwagon behavior, which occurs as long as $V_{i,t} < \overline{V}$, and the snob behavior, which occurs when $V_{i,t} > \overline{V}$. Indeed, according to Simmel (1957), two contrasting tendencies operate in determining people behavior towards fashion. On the one hand, to follow fashion makes people feel accepted and socially integrated, answering to their innate tendency for conformity. This generates imitation of others, i.e., the so called bandwagon behavior. Such phenomenon is reproduced by $\mathscr{A}_{i,t}$ as long as $V_{i,t}$ is smaller than $\overline{V}$, when increasing values for $V_{i,t}$ imply higher and higher attractiveness degrees $\mathscr{A}_{i,t}$. On the other hand, when the visibility of a group becomes excessive, a congestion effect arises and the people wish for distinction predominates. That phenomenon is known as snob behavior and it is reproduced by $\mathscr{A}_{i,t}$ when $V_{i,t}$ is larger than $\overline{V}$, because in this regime an increase in the visibility level $V_{i,t}$ leads to a decrease in $\mathscr{A}_{i,t}$. Namely, according to Vigneron and Johnson (1999), on the basis of the empirical literature, too, in the context of luxury and prestige-seeking consumption agents may oscillate between snob and bandwagon behaviors. Those two opposite forces, imitation and distinction, drive the fashion cycle, which for us emerges at the aggregate level as a continuous oscillation of the attractiveness and of the shares of the two groups, while on the individual level it is characterized by oscillatory consumption choices, presenting booms and busts, over the two goods for the agents belonging to the two groups. We shall find evidence of such phenomenon in Section 3 (cf. Figures 1 and 2 therein, and also the Appendix), when considering Stone-Geary utility functions and for different formulations of the attractiveness maps, already dealt with in Naimzada and Pireddu (2018d, 2019b).

The share of agents which choose to belong to a given group in the next period depends on the present attractiveness levels of the two groups. More precisely, following Taylor and Jonker (1978), Nachbar (1990) and Sandholm (2010), we do consider a discrete exponential replicator mechanism to formalize the population share updating rule, so that the evolution of the fraction $a_t$ of agents of type $\alpha$ is described by the discrete choice model

$$\begin{aligned} a_{t+1} &= \frac{a_t \exp(\mu \mathscr{A}_{\alpha,t})}{a_t \exp(\mu \mathscr{A}_{\alpha,t}) + (1-a_t) \exp\big(\mu \mathscr{A}_{\beta,t}\big)} \\ &= \frac{a_t}{a_t + (1-a_t) \exp\Big(\mu\big(\mathscr{A}_{\beta,t} - \mathscr{A}_{\alpha,t}\big)\Big)}, \end{aligned} \tag{15}$$

where $\mu$ is a positive parameter measuring the sensitivity of the share formation mechanism to the difference in the groups attractiveness levels. When we will need to underline the dependence, for $i \in \{\alpha, \beta\}$, of $\mathscr{A}_{i,t}$ and of $V_{i,t}$ on $a_t$, we will also write $\mathscr{A}_{i,t}(a_t)$ and $V_{i,t}(a_t)$, respectively, or, at the market stationary equilibria, simply $\mathscr{A}_i(a)$ and $V_i(a)$ (see for instance the map $g$ in (17), whose fixed

points are the system market stationary equilibria).[7] In particular, recalling the definition of visibility in (13) and that, by Proposition 2 and Footnote 4, the equilibrium consumption choices depend smoothly on $a_t \in (0,1)$, we have that also $V_{i,t}$, $i \in \{\alpha, \beta\}$, depend smoothly on $a_t \in (0,1)$, for every $t$, and that, in correspondence to a market stationary equilibrium, $V_i(a)$, $i \in \{\alpha, \beta\}$, depend in a smooth way on $a \in (0,1)$.
By (14) and (15), we obtain

$$a_{t+1} = g(a_t), \tag{16}$$

where the one-dimensional map $g : [0,1] \to \mathbb{R}$ is defined as

$$g(a) = \frac{a}{a + (1-a)\exp\left(\mu\left(f(\sigma d^2(\overline{V}, V_\beta(a))) - f(\sigma d^2(\overline{V}, V_\alpha(a)))\right)\right)}. \tag{17}$$

We stress that if $\mu \to 0$, independently of $V_\alpha$ and $V_\beta$, then $a_{t+1} = a_t$, for all $t$, and thus there is no evolution, the population shares remain unchanged with respect to the initial ones, as agents are insensitive to the attractiveness degrees of the two groups; when instead $\mu \to +\infty$, agents are extremely sensitive to the social attractiveness of the groups and they instantaneously move towards the "best" one, i.e., the one which gave a visibility level closer to $\overline{V}$.

Let us start our analysis by deriving the expressions of the market stationary equilibria for (16).

**Proposition 4.** *Given the economy $E \in \widetilde{\mathcal{E}}$ and the attractiveness $\mathscr{A}_{i,t}(V_{i,t}; \sigma, \overline{V}) = f(\sigma\, d^2(\overline{V}, V_{i,t})) \in \mathcal{A}$, for $i \in \{\alpha, \beta\}$, equation (16) admits as market stationary equilibria, in addition to the trivial $a = 0$, $a = 1$, also all solutions in $(0,1)$ to the equation $V_\alpha(a) = V_\beta(a)$, if any, as well as all solutions in $(0,1)$ to the equation $(V_\alpha(a) + V_\beta(a))/2 = \overline{V}$, if any.*

*Proof.* The conclusion immediately follows by observing that the solutions to the fixed-point equation $g(a) = a$, with $g$ as in (17), are given by $a = 0$, $a = 1$, as well as by all solutions to the equation $V_\alpha(a) = V_\beta(a)$, if any, and by all the solutions to the equation $(V_\alpha(a) + V_\beta(a))/2 = \overline{V}$, if any. Namely, the nontrivial equilibria for equation (16) are found as solutions to the equation $\mathscr{A}_\alpha = f(\sigma\, d^2(\overline{V}, V_\alpha)) = f(\sigma\, d^2(\overline{V}, V_\beta)) = \mathscr{A}_\beta$. Since $f$ is strictly decreasing, all solutions have to satisfy $d(\overline{V}, V_\alpha) = d(\overline{V}, V_\beta)$, i.e., $V_\alpha = V_\beta$ or $(V_\alpha + V_\beta)/2 = \overline{V}$, as desired. This concludes the proof. □

We stress that the condition $(V_\alpha + V_\beta)/2 = \overline{V}$ in Proposition 4 means that $\overline{V}$ is the midpoint between $V_\alpha$ and $V_\beta$. Hence, at the nontrivial market stationary equilibria, the population share of group $\alpha$ has to make $V_\alpha$ and $V_\beta$ coincide or $V_\alpha$ and $V_\beta$ have to lie at the same distance from $\overline{V}$, but on its opposite sides.
We also notice that assuming in Proposition 4 that $\mathscr{A}_\alpha = f_\alpha(\sigma\, d^2(\overline{V}, V_\alpha))$ and $\mathscr{A}_\beta = f_\beta(\sigma\, d^2(\overline{V}, V_\beta))$, for some $f_\alpha \neq f_\beta$, with $f_\alpha$, $f_\beta$ differentiable and strictly decreasing maps, we then find that, in addition to the trivial $a = 0$, $a = 1$, the market stationary equilibria are the solutions in $(0,1)$ to the equation $f_\alpha(\sigma\, d^2(\overline{V}, V_\alpha)) = f_\beta(\sigma\, d^2(\overline{V}, V_\beta))$, if any. We finally stress that if both $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$ were described by the same map $f$, but if $\mathscr{A}_\alpha = f(\sigma_\alpha\, d^2(\overline{V}, V_\alpha))$ and $\mathscr{A}_\beta = f(\sigma_\beta\, d^2(\overline{V}, V_\beta))$, for some $\sigma_\alpha \neq \sigma_\beta$, then the nontrivial market stationary equilibria would be given by all solutions in $(0,1)$ to the equation $d(\overline{V}, V_\alpha) = d(\overline{V}, V_\beta)\sqrt{\frac{\sigma_\beta}{\sigma_\alpha}}$, i.e., to the equations $V_\alpha - \sqrt{\frac{\sigma_\beta}{\sigma_\alpha}}\, V_\beta = \overline{V}\left(1 - \sqrt{\frac{\sigma_\beta}{\sigma_\alpha}}\right)$ and $V_\alpha + \sqrt{\frac{\sigma_\beta}{\sigma_\alpha}}\, V_\beta = \overline{V}\left(1 + \sqrt{\frac{\sigma_\beta}{\sigma_\alpha}}\right)$. Namely, setting $\sigma_\alpha = \sigma_\beta$ in such expressions, we find again the results obtained in Proposition 4.

In the next result we analytically investigate the local stability of the market stationary equilibria for the map $g$ in (17).

[7] We remark that $V_i$, and consequently $\mathscr{A}_i$, also depend on the economy $\widetilde{\mathcal{E}}$, which together with $a$ determines the market equilibrium. However, in order not to overburden notation, we will not make such dependence explicit.

**Proposition 5.** *The equilibrium $a = 0$ is locally asymptotically stable for the map $g$ in* (17) *if* $(V_\beta(0) - \overline{V})^2 < (V_\alpha(0) - \overline{V})^2$.
*The equilibrium $a = 1$ is locally asymptotically stable for map $g$ if* $(V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2$.
*When $\mathscr{A}_i$, $i \in \{\alpha, \beta\}$, do not depend on $\mu$, calling $\hat{a}$ any solution to the equation $V_\alpha(a) = V_\beta(a)$, if it exists in $(0,1)$, the equilibrium $a = \hat{a}$ is locally asymptotically stable for map $g$ if* $(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a})) < 0$ *and*

$$\mu < \hat{\mu} = \frac{1}{\sigma\hat{a}(1-\hat{a})\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\hat{a}}(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a}))}. \tag{18}$$

*In particular, a flip bifurcation occurs at $a = \hat{a}$ if $\mu = \hat{\mu}$.*
*If* $(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a})) > 0$, *then $\hat{a} \in (0,1)$ is unstable.*
*When $\mathscr{A}_i$, $i \in \{\alpha, \beta\}$, do not depend on $\mu$, calling $\tilde{a}$ any solution to the equation $(V_\alpha(a) + V_\beta(a))/2 = \overline{V}$, if it exists in $(0,1)$, the equilibrium $a = \tilde{a}$ is locally asymptotically stable for map $g$ if* $(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a})) < 0$ *and*

$$\mu < \tilde{\mu} = \frac{1}{\sigma\tilde{a}(1-\tilde{a})\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\tilde{a}}(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a}))}. \tag{19}$$

*In particular, a flip bifurcation occurs at $a = \tilde{a}$ if $\mu = \tilde{\mu}$.*
*If* $(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a})) > 0$, *then $\tilde{a} \in (0,1)$ is unstable.*

*Proof.* Since $g'(0) = 1/(\exp(\mu(\mathscr{A}_\beta(0) - \mathscr{A}_\alpha(0))))$, the condition $g'(0) > -1$ is always fulfilled, while $g'(0) < 1$ is fulfilled for $\mathscr{A}_\beta(0) > \mathscr{A}_\alpha(0)$. Independently of $f$, exploiting the strictly decreasing behavior of $\mathscr{A}_i$ with respect to $d^2(\overline{V}, V_i)$, we then find $(V_\alpha(0) - \overline{V})^2 > (V_\beta(0) - \overline{V})^2$, as desired.
As concerns the local stability of $a = 1$ for map $g$, we find $g'(1) = \exp(\mu(\mathscr{A}_\beta(1) - \mathscr{A}_\alpha(1)))$. Hence, the condition $g'(1) > -1$ is always fulfilled, while $g'(1) < 1$ is fulfilled for $\mathscr{A}_\beta(1) < \mathscr{A}_\alpha(1)$. Independently of $f$, exploiting again the strictly decreasing behavior of $\mathscr{A}_i$ with respect to $d^2(\overline{V}, V_i)$, we find $(V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2$.
In regard to $a = \hat{a}$, recalling that $V_\alpha(\hat{a}) = V_\beta(\hat{a})$, we find that $g'(\hat{a}) = 1 - 2\mu\sigma\hat{a}(1-\hat{a})\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\hat{a}}(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a}))$. Since $\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\hat{a}}$ is negative, then $a = \hat{a} \in (0,1)$ is unstable for map $g$ for all positive values of $\mu$ if $(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a})) > 0$. If instead $(V_\beta(\hat{a}) - \overline{V})(V'_\beta(\hat{a}) - V'_\alpha(\hat{a})) < 0$, then $g'(\hat{a}) < 1$ is always fulfilled, while, if $\mathscr{A}_\beta$ does not depend on $\mu$, $g'(\hat{a}) > -1$ is fulfilled for $\mu < \hat{\mu}$, with $\hat{\mu}$ as in (18), obtaining the desired conclusion about the local stability of $a = \hat{a} \in (0,1)$ for map $g$, too. The condition for the flip bifurcation follows by setting $g'(\hat{a}) = -1$.
Finally, in regard to $a = \tilde{a}$, recalling that $V_\beta(\tilde{a}) - \overline{V} = \overline{V} - V_\alpha(\tilde{a})$, we find that $g'(\tilde{a}) = 1 - 2\mu\sigma\tilde{a}(1-\tilde{a})\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\tilde{a}}(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a}))$. Since $\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d^2_\beta)}|_{a=\tilde{a}}$ is negative, then $a = \tilde{a} \in (0,1)$ is unstable for map $g$ for all positive values of $\mu$ if $(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a})) > 0$. If instead $(\overline{V} - V_\alpha(\tilde{a}))(V'_\alpha(\tilde{a}) + V'_\beta(\tilde{a})) < 0$, then $g'(\tilde{a}) < 1$ is always fulfilled, while, if $\mathscr{A}_\beta$ does not depend on $\mu$, $g'(\tilde{a}) > -1$ is fulfilled for $\mu < \tilde{\mu}$, with $\tilde{\mu}$ as in (19). The condition for the flip bifurcation follows by setting $g'(\tilde{a}) = -1$. This concludes the proof. □

We notice that if we assumed that $\mathscr{A}_\alpha = f_\alpha(\sigma\, d^2(\overline{V}, V_\alpha))$ and $\mathscr{A}_\beta = f_\beta(\sigma\, d^2(\overline{V}, V_\beta))$, for some $f_\alpha \neq f_\beta$, with $f_\alpha$, $f_\beta$ differentiable and strictly decreasing maps, we would find that $a = 0$ is locally asymptotically stable for the map $g$ in (17) if $\mathscr{A}_\beta(0) > \mathscr{A}_\alpha(0)$ and that $a = 1$ is locally asymptotically stable if $\mathscr{A}_\beta(1) < \mathscr{A}_\alpha(1)$. However, in such more general setting, those conditions on attractiveness could not be translated into conditions on visibility and thus they would remain more vague. Moreover, if $f_\alpha$, $f_\beta$ do not depend on $\mu$, calling $\bar{a}$ any solution to the equation $f_\alpha(\sigma\, d^2(\overline{V}, V_\alpha)) = f_\beta(\sigma\, d^2(\overline{V}, V_\beta))$, if it exists in $(0,1)$, we find that $a = \bar{a}$ is locally asymptotically stable for the map $g$ in (17) if

$\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=\bar{a}} \left(V_\beta(\bar{a}) - \overline{V}\right) V_\beta'(\bar{a}) - \frac{\partial \mathscr{A}_\alpha(a)}{\partial(\sigma d_\alpha^2)}|_{a=\bar{a}} \left(V_\alpha(\bar{a}) - \overline{V}\right) V_\alpha'(\bar{a}) > 0$ and

$$\mu < \bar{\mu} = \frac{1}{\sigma\bar{a}(1-\bar{a})\left(\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=\bar{a}} \left(V_\beta(\bar{a}) - \overline{V}\right) V_\beta'(\bar{a}) - \frac{\partial \mathscr{A}_\alpha(a)}{\partial(\sigma d_\alpha^2)}|_{a=\bar{a}} \left(V_\alpha(\bar{a}) - \overline{V}\right) V_\alpha'(\bar{a})\right)}.$$

In particular, a flip bifurcation occurs at $a = \bar{a}$ if $\mu = \bar{\mu}$.
If $\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=\bar{a}} \left(V_\beta(\bar{a}) - \overline{V}\right) V_\beta'(\bar{a}) - \frac{\partial \mathscr{A}_\alpha(a)}{\partial(\sigma d_\alpha^2)}|_{a=\bar{a}} \left(V_\alpha(\bar{a}) - \overline{V}\right) V_\alpha'(\bar{a}) < 0$, then $a = \bar{a}$ is unstable for all positive values of $\mu$.
We also stress that it is not possible to derive general conditions similar to (18) and (19) when investigating the local stability of the nontrivial equilibria with respect to $\sigma$. Namely, even if $V_i$, $i \in \{\alpha, \beta\}$, do not depend on $\sigma$, such parameter is usually present in the expression of $\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=\hat{a}}$ and of $\frac{\partial \mathscr{A}_\beta(a)}{\partial(\sigma d_\beta^2)}|_{a=\tilde{a}}$, as it is easy to check, for instance, with the formulations for attractiveness in (23) and in (24). In fact, with the Gaussian formulation in (24), for suitable parameter configurations, there may not exist threshold stability values analogous to those in (18) and (19) with respect to $\sigma$, since e.g. $a = \hat{a} \in (0,1)$ may be stable for any $\sigma > 0$.[8]
In agreement with Proposition 5, in Section 3 we will consider $\mu$ as bifurcation parameter, so that the theoretical investigation performed so far will serve as a guideline for the qualitative bifurcation analysis we shall conduct below.

# 3 Bifurcation analysis and possible scenarios in the case of Stone-Geary utility functions

In the present section we perform a qualitative bifurcation analysis, investigating the stability gain/loss of stationary equilibria and the emergence/disappearance of periodic and chaotic attractors on varying the sensitivity parameter $\mu$. Namely, assuming a constitutive heterogeneity between groups in terms of the structure of preferences, our aim is that of investigating the model asymptotic heterogeneity, i.e., we discuss the possible dynamics arising when choosing initial conditions characterized by the coexistence between heterogeneous agents in view of understanding whether the initial heterogeneity eventually disappears and, in case it persists, if it is stationary or oscillatory in nature, of periodic or chaotic kind. In order to make the heterogeneity between groups explicit in our setting, we need to specify an analytical formulation for the utility functions.
Recalling that in Chang and Stauber (2009) and in Naimzada and Pireddu (2018d, 2019b) agents' preferences were described by Cobb-Douglas utility functions, we will now deal with Stone-Geary utility functions[9], which generalize the previously considered maps, still yielding, for suitable parameter values, to individual demand functions with the gross substitute property.[10] In this manner, according to Proposition 3, for all endowments and population shares there exists a unique equilibrium as described in (21) and thus no indeterminacy issues on visibility and attractiveness arise.
The formulation of the Stone-Geary utility functions over the two consumption goods $x$ and $y$ is given by

$$U_i(x,y) = (x + c_i)^i (y + d_i)^{1-i}, \text{ for } i \in \{\alpha, \beta\}, \text{ with } 0 < \beta < \alpha < 1. \tag{20}$$

[8]This happens for instance with the parameter configuration considered in Scenario A of Section 3, when fixing $\mu = 2$ and letting $\sigma > 0$ free to vary.

[9]The Stone-Geary utility functions were derived by Geary (1950) in a comment on an earlier work, while Stone (1954) estimated the Linear Expenditure System, arising from the utility functions in (20).

[10]Although such feature is mentioned in Fisher (1972) for Stone-Geary utility functions when the coefficients $c_i$, $d_i$ are non-negative, for $i \in \{\alpha, \beta\}$, a direct proof using the expression for the individual demand functions in (21) shows that the gross substitute property holds also in the case of negative coefficients, as long as their value is not excessive in absolute value. See the discussion after (21) for more details.

Such maps are certainly well defined when $c_i \geq 0$ and $d_i \geq 0$, case considered in Fisher (1972), and indeed for $c_i = d_i = 0$ we obtain the Cobb-Douglas utility functions. However, as we shall see below, in order to ensure that the equilibrium consumption levels in (21) are positive, $c_i$ and $d_i$ can not assume excessively large positive values. As concerns the negativity of $c_i$ and $d_i$, albeit not affecting the positivity of the equilibrium consumption levels in (21), it may prevent the individual demand functions from having the gross substitute property. Both features, i.e., the positivity of the equilibrium consumption levels and individual demand functions with the gross substitute property, are ensured for values of $c_i$ and $d_i$ not too large in absolute value (cf. (22)). We stress that allowing for negative values of $c_i$ and $d_i$ is important from an interpretative viewpoint. Namely, Stone-Geary functions are often used to model problems involving subsistence levels of consumption. In these cases, a certain minimal level of some good has to be consumed, irrespective of its price or of the consumer's income. We also notice that the Stone-Geary utility functions satisfy Assumptions $(A1)-(A4)$ in Definition 1.

The next analysis is performed in terms of the relative price $p_t = p_{y,t}/p_{x,t}$, where $p_{x,t} > 0$ and $p_{y,t} > 0$ are the prices at time $t$ for goods $x$ and $y$, respectively. Solving the consumer maximization problem in (4) and using one market clearing condition in (3), with the Stone-Geary utility functions as optimal equilibrium price we find

$$p_t^* = \frac{a_t(1-\alpha)(w_x+c_\alpha)+(1-a_t)(1-\beta)(w_x+c_\beta)}{a_t(\alpha w_y+d_\alpha)+(1-a_t)(\beta w_y+d_\beta)}$$

and the consumption equilibrium quantities of the two goods for an agent of type $i \in \{\alpha, \beta\}$ are given by

$$\begin{aligned}
x^*_{\alpha,t} &= \alpha w_x - (1-\alpha)c_\alpha + p_t^*(\alpha w_y + d_\alpha) = \\
&= \tfrac{a_t w_x(\alpha w_y+d_\alpha)+(1-a_t)\big[(1-\beta)d_\alpha w_x+(1-\beta)c_\beta d_\alpha-c_\alpha\beta w_y-c_\alpha d_\beta+w_x\alpha d_\beta+\alpha c_\alpha\beta w_y+\alpha c_\alpha d_\beta+\alpha w_x w_y+\alpha w_y c_\beta-\alpha\beta c_\beta w_y\big]}{a_t(\alpha w_y+d_\alpha)+(1-a_t)(\beta w_y+d_\beta)}, \\
x^*_{\beta,t} &= \alpha w_x - (1-\alpha)c_\beta + p_t^*(\alpha w_y + d_\beta) = \\
&= \tfrac{a_t\big[(1-\alpha)d_\beta w_x+(1-\alpha)d_\beta c_\alpha-\alpha c_\beta w_y-d_\alpha c_\beta+\beta d_\alpha w_x+\alpha\beta c_\beta w_y+\beta d_\alpha c_\beta+\beta w_x w_y+\beta c_\alpha w_y-\alpha\beta c_\alpha w_y\big]+(1-a_t)w_x(\beta w_y+d_\beta)}{a_t(\alpha w_y+d_\alpha)+(1-a_t)(\beta w_y+d_\beta)}, \\
y^*_{\alpha,t} &= (1-\alpha)\left(\tfrac{w_x+c_\alpha}{p_t^*}+w_y\right) - d_\alpha = \\
&= \tfrac{a_t(1-\alpha)w_y(w_x+c_\alpha)+(1-a_t)\big[(1-\alpha)(w_x d_\beta+\beta c_\alpha w_y+c_\alpha d_\beta+w_x w_y+c_\beta w_y-\beta c_\beta w_y)-(1-\beta)d_\alpha(w_x+c_\beta)\big]}{a_t(1-\alpha)(w_x+c_\alpha)+(1-a_t)(1-\beta)(w_x+c_\beta)}, \\
y^*_{\beta,t} &= (1-\alpha)\left(\tfrac{w_x+c_\beta}{p_t^*}+w_y\right) - d_\beta = \\
&= \tfrac{a_t\big[(1-\beta)(w_x d_\alpha+\alpha c_\beta w_y+d_\alpha c_\beta+w_x w_y+c_\alpha w_y-\alpha c_\alpha w_y)-(1-\alpha)d_\beta(w_x+c_\alpha)\big]+(1-a_t)(1-\beta)(w_x+c_\beta)}{a_t(1-\alpha)(w_x+c_\alpha)+(1-a_t)(1-\beta)(w_x+c_\beta)}.
\end{aligned} \tag{21}$$

Recalling (13), in such context the social visibility level $V_{i,t}$ of an agent of type $i \in \{\alpha, \beta\}$ at time $t$ is given by $V_{i,t} = v_x x^*_{i,t} + v_y y^*_{i,t}$, with the expressions for $x^*_{i,t}$ and $y^*_{i,t}$ just derived and for values of the visibility weight parameters $v_x$ and $v_y$ to be fixed below.

From (21) we immediately find that a sufficient condition for the positivity of $x^*_{i,t}$, $i \in \{\alpha, \beta\}$, is $c_i < \frac{\alpha w_x}{1-\alpha}$, while a sufficient condition for the positivity of $y^*_{i,t}$, $i \in \{\alpha, \beta\}$, is $d_i < (1-\alpha)w_y$. On the other hand, recalling that $p_t = p_{y,t}/p_{x,t}$, from (21) we obtain $\alpha w_y + d_i > 0$ as sufficient condition for $x^*_{i,t}$, $i \in \{\alpha, \beta\}$, to display the gross substitute property, and $w_x + c_i > 0$ as sufficient condition for $y^*_{i,t}$, $i \in \{\alpha, \beta\}$, to display the gross substitute property. Hence, the positivity of the equilibrium consumption levels and individual demand functions with the gross substitute property are simultaneously ensured for

$$-w_x < c_i < \frac{\alpha w_x}{1-\alpha}, \quad -\alpha w_y < d_i < (1-\alpha)w_y, \quad i \in \{\alpha, \beta\}. \tag{22}$$

As regards the attractiveness functions, we will deal with the bell-shaped formulations considered in Naimzada and Pireddu (2018d, 2019b), i.e., respectively,

$$\mathscr{A}'_{i,t}(V_{i,t};\sigma,\overline{V}) = \frac{1}{1+\sigma(\overline{V}-V_{i,t})^2}, \tag{23}$$

the Gaussian map

$$\mathscr{A}''_{i,t}(V_{i,t};\sigma,\overline{V}) = \exp(-\sigma(\overline{V}-V_{i,t})^2), \tag{24}$$

and the parabolic function

$$\mathscr{A}'''_{i,t}(V_{i,t};\sigma,\overline{V}) = 1-\sigma(\overline{V}-V_{i,t})^2. \tag{25}$$

This choice comes from a simple observation. Since the Stone-Geary utility functions are a generalization of the Cobb-Douglas utility functions, dealt with in Naimzada and Pireddu (2018d, 2019b), as a byproduct of the bifurcation analysis we are going to perform in the Appendix using the sensitivity measure $\mu$ as bifurcation parameter, when considering in (20) $c_i = d_i = 0$, $i \in \{\alpha, \beta\}$, we complement the investigation performed in the previous works, where the bifurcation parameter was the heterogeneity degree between groups of agents $\Delta = \alpha - \beta$.

We stress that the attractiveness functions in (23)–(25) satisfy the conditions in Definition 4. Namely, they are differentiable, bell-shaped, increasing with the visibility level $V_{i,t}$, $i \in \{\alpha, \beta\}$, up to the threshold value $\overline{V}$, after which they become decreasing in $V_{i,t}$ in a symmetric manner with respect to $\overline{V}$, due to their dependence on $d^2(\overline{V}, V_{i,t})$. We recall that $\sigma$ is a positive parameter describing the sensitivity of the attractiveness with respect to the distance between the visibility level $V_{i,t}$ and the threshold visibility level $\overline{V}$, while $\mu$, that we shall employ as bifurcation parameter, measures the sensitivity of the share formation mechanism to the difference in the preference structures attractiveness levels (see also the discussion after (17)).

In view of the qualitative bifurcation analysis we will perform in the Appendix, we explain which are the stationary equilibria of our system and we illustrate some of their features.

As concerns the trivial equilibria, according to Proposition 5, $a = 0$ is locally asymptotically stable for the map $g$ in (17) if $(V_\beta(0) - \overline{V})^2 < (V_\alpha(0) - \overline{V})^2$ and $a = 1$ is locally asymptotically stable for map $g$ if $(V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2$. Hence, the stability of the trivial equilibria is influenced neither by the value of $\mu$, nor by the formulation of the map describing attractiveness. In particular, as we shall see in the Appendix, both $a = 0$ and $a = 1$ are locally asymptotically stable for every positive value of $\mu$ in all the considered configurations.

In regard to the nontrivial stationary equilibria of (16), according to Proposition 4, they are given by the solutions belonging to the interval $(0, 1)$ to $V_\alpha(a) = V_\beta(a)$, as well as by the solutions in $(0, 1)$ to the equation $(V_\alpha(a) + V_\beta(a))/2 = \overline{V}$. Hence, their number and expression are independent of the chosen formulation for attractiveness and of the value of $\mu$. On the other hand, recalling (18) and (19), the stability of the nontrivial stationary equilibria is influenced by the value of $\mu$ and by the formulation of the map describing attractiveness. Indeed, the latter affects the stability threshold value of nontrivial equilibria when considering $\mu$ as bifurcation parameter. Moreover, in analogy with the findings in Naimzada and Pireddu (2018d, 2019b), we expect that varying $\mu$ the global dynamics may differ according to the formulation for attractiveness, e.g. due to the occurrence of subcritical or supercritical flip bifurcations at the nontrivial equilibria.

The parameter configuration we will deal with is given by $v_x = 0.9$, $w_x = 0.2$, $v_y = 0.15$, $w_y = 2$, $\beta = 0.1$, $\alpha = 0.9$, $\sigma = 8$, $\overline{V} = 0.8$, while we let $\mu$ free to vary.[11] We stress that such parameter values coincide with those considered in Naimzada and Pireddu (2018d, 2019b), even if, as explained above, in those works we used $\Delta = \alpha - \beta$ as bifurcation parameter, while setting $\mu = 6.5$. As concerns the

[11] We here consider polarized values for $v_x$ and $v_y$, as both of them are positive but one is much larger than the other. Such case approximates those frameworks in which visibility and attractiveness are produced by the consumption of a single good. However, our results hold true also for more balanced values of $v_x$ and $v_y$.

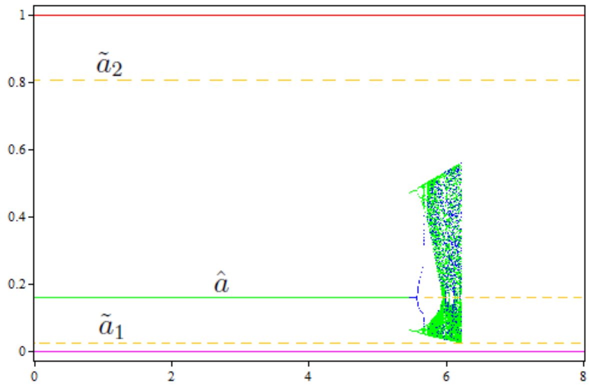



Figure 1: The bifurcation diagram of $g$ for $c_\alpha = -0.19$, $c_\beta = -0.14$, $d_\alpha = -0.3$, $d_\beta = -0.16$, $\mu \in (0, 8)$ and the attractiveness $\mathscr{A}'$ in (23). We denote in magenta $a = 0$, in red $a = 1$, in blue (green) the points generated by the initial condition $a_0 = 0.3437$ ($a_0 = 0.8$) and in orange $a = \hat{a}$ when it is no more stable, as well as the unstable equilibria $a = \tilde{a}_1$ and $a = \tilde{a}_2$. Solid (dashed) lines refer to stable (unstable) equilibria and cycles.

new parameters $c_i$ and $d_i$ characterizing the Stone-Geary utility functions, from (22) we obtain the following bounds:[12]

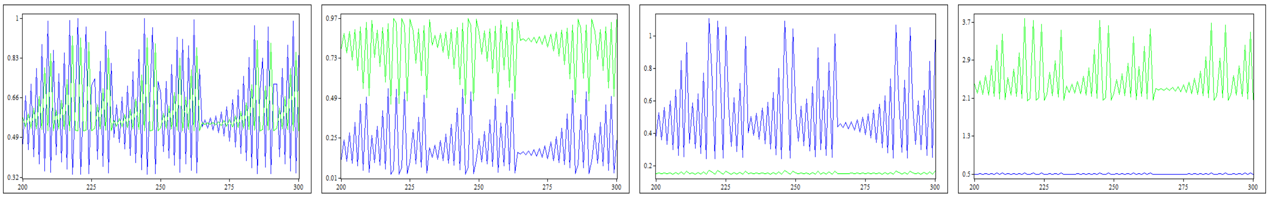
$$-0.2 < c_i < 1.8, \quad -1.8 < d_i < 0.2, \quad i \in \{\alpha, \beta\}.$$

In the Appendix we will consider three different frameworks, in correspondence to each of the attractiveness formulations in (23)–(25).
The first framework, we will call Scenario A, is characterized by $c_i = d_i = 0$, for $i \in \{\alpha, \beta\}$. In this case Stone-Geary utility functions reduce to Cobb-Douglas utility functions and we will treat such scenario as a benchmark context, which allows for a comparison between the bifurcation analysis performed in Naimzada and Pireddu (2018d, 2019b) in terms of $\Delta = \alpha - \beta$ and here in terms of $\mu$. The second framework, we will call Scenario B, is characterized by positive values for all parameters in the Stone-Geary utility functions. In particular, we will consider $c_\alpha = 0.3$, $c_\beta = 0.12$, $d_\alpha = 0.1$, $d_\beta = 0.15$. The third framework, we will call Scenario C, is characterized by negative values for the parameters in the Stone-Geary utility functions. In particular, we will deal with $c_\alpha = -0.19$, $c_\beta = -0.14$, $d_\alpha = -0.3$, $d_\beta = -0.16$.
For the brevity's sake, we here discuss just the latter framework, which is both the best grounded from an interpretative viewpoint, as well as the most interesting in terms of the arising dynamics. We shall now introduce only those aspects needed to illustrate the emergence of fashion cycle dynamics in our model for the attractiveness formulation in (23). The analysis of Scenarios A and B, as well as the remaining details regarding Scenario C, can be found in the Appendix.
In Figure 1 we report the bifurcation diagram of $g$ for $\mu \in (0, 8)$. In addition to the trivial equilibria, which are locally asymptotically stable for all positive values of $\mu$, we notice that for small values of $\mu$ just $a = \hat{a}$ is stable. When $\mu$ increases, an external attractor, born via a fold bifurcation of map $g$ as a period-two cycle and then undergoing a sequence of period-doubling bifurcations leading to chaos, coexists first with $a = \hat{a}$ and next with the period-two cycle following the supercritical flip bifurcation occurring at $a = \hat{a}$ for $\mu = \hat{\mu} = 5.549$. For still larger values of $\mu$, the latter period-two cycle falls within the basin of attraction of the external attractor, which for increasing values of parameter $\mu$ from being a two-piece chaotic attractor becomes a one-piece chaotic attractor, that disappears for $\mu = 6.228$ due to a contact bifurcation with $a = \tilde{a}_1$. After such bifurcation, the only attractors are

[12] We remark that the symmetry between the bounds of $c_i$ and $d_i$ is caused by the fact that, for the parameter configuration we deal with, it holds that $w_x = (1 - \alpha)w_y$. Of course, such peculiarity does not affect the outcomes. Indeed, in Scenarios B and C the parameters $c_i$ and $d_i$ will not bear any symmetry.

given by $a = 0$ and $a = 1$. We stress that, due to the absorption of the internal period-two cycle into the external attractor, after which $a = \hat{a}$ is repelling, suddenly oscillations in the agents' consumption choices and in the population shares become strong. We recall that this kind of framework is a mixture between the outcomes found in Naimzada and Pireddu (2018d, 2019b). Namely, like in Figure 5 in Naimzada and Pireddu (2019b), in Figure 1 the nontrivial equilibrium $a = \hat{a}$ undergoes a supercritical flip bifurcation. However, the period-two cycle following it, rather than undergoing a classical cascade of flip bifurcations to chaos, is soon absorbed by the external attractor, and this suddenly leads to wide oscillations, similar to those which follow the subcritical flip bifurcation in Figure 5 in Naimzada and Pireddu (2018d) and in Figure 1 in Naimzada and Pireddu (2019b). As in such settings, we here find interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors, suitable to represent the variety of historical experiences across different countries in relation to the approach they adopt towards consumption choices and fashion.
Summarizing, Figure 1 highlights that in our model fashion cycle dynamics may be generated by intermediate values of the sensitivity parameter $\mu$. Indeed, when $\mu$ is too small, the only attractors are steady states and this excludes the possibility of any sort of non-convergent dynamics. When $\mu$ increases, the nontrivial equilibrium $a = \hat{a}$ loses stability and the period-two cycle following it coexists with an external attractor which, contrarily to the internal period-two cycle, persists, giving rise to periodic and chaotic dynamics. However, when the sensitivity measure becomes excessive, no oscillatory behaviors occur anymore, as the main chaotic attractor disappears due to a contact bifurcation with one of the nontrivial equilibria, and thus the system, according to the chosen initial condition, asymptotically converges toward one or the other of the trivial steady states, characterized by the presence of a unique group of agents. In fact, even for lower values of the sensitivity parameter $\mu$, trajectories will tend toward one of the three or four coexisting attractors according to the chosen initial datum. In particular, when the initial conditions, which represent a summary of the past history, are excessively close to the extreme values $a = 0$ and $a = 1$, trajectories are attracted by one of the two trivial equilibria, and a preference structure totally prevails over the other. However, when the initial datum is not too close to $a = 0$ and $a = 1$, for intermediate values of $\mu$, our dynamical system may lead to outcomes characterized by the coexistence among heterogeneous agents. Due to the combined effect of the price formation mechanism, of the social interaction mechanism, according to which consumption choices produce visibility, and of the evolutionary mechanism, based on the relative attractiveness of the different lifestyles, the group coexistence may be stationary or oscillatory in nature, both periodic, in a neighborhood of the flip bifurcation, and erratic, due to the presence of the chaotic attractor. In either case, the dynamic coexistence among heterogeneous agents displays the recurrent behavior typical of the fashion cycle, characterized by booms and busts. We report, for the same parameter configuration considered in Figure 1 with $\mu = 6.1$ and for the periods $t \in [200, 300]$, the time series for all the relevant variables in Figure 2. Namely, the fashion cycle for us emerges at the aggregate level as a continual oscillation of the attractiveness and of the shares of the two groups, while on the individual level it is characterized by oscillatory consumption choices over the two goods for the agents of the two groups.

We stress that, although the proof of chaos performed in Naimzada and Pireddu (2018d) using the method of the turbulent maps in Block and Coppel (1992), working with homoclinic orbits, can be directly transposed to the present setting, the social and economic interpretation of the main scenarios provided in Naimzada and Pireddu (2018d, 2019b) cannot be precisely repeated here. For such reason, we conclude the present section by briefly reporting the interpretation of the most important phenomena we found, due also to the role it plays in understanding the functioning of the model dynamics. The main scenarios in Figure 1 are: the convergence towards a trivial steady state, so that the agents' original heterogeneity contained in the initial condition does not persist; the convergence towards a nontrivial steady state, that is a static form of heterogeneity; the convergence towards a period-two cycle, which represents the simplest case of oscillatory behavior, that encompasses agents' heterogeneity and in which population shares, prices and consumption bundles vary over time. We



(a) (b) (c) (d)

Figure 2: The time series corresponding to the periods $t \in [200, 300]$ for $\mathscr{A}'_{\alpha,t}$ (in blue) and $\mathscr{A}'_{\beta,t}$ (in green) in (a), for $a_t$ (in blue) and $b_t = 1 - a_t$ (in green) in (b), for $x^*_{\alpha,t}$ (in blue) and $x^*_{\beta,t}$ (in green) in (c), for $y^*_{\alpha,t}$ (in blue) and $y^*_{\beta,t}$ (in green) in (d), for the same parameter configuration considered in Figure 1, with $\mu = 6.1$ and $a_0 = 0.4$.

notice that the other possible scenarios may be explained combining the arguments we shall employ below.

As concerns the convergence towards a trivial steady state, let us consider for instance $\mu = 2$ and $a_0 = 0.85$ and let us clarify why the system converges towards $a = 1$. Since $V_\alpha < \overline{V} < V_\beta$, with $(V_\alpha - \overline{V})^2 < (V_\beta - \overline{V})^2$, we expect an increase in the share of the agents of group $\alpha$ and, consequently, a raise in the aggregate demand of commodity $x$, agents of group $\alpha$ have a stronger preference for. Indeed, we do observe a decrease in the relative price $p$. The optimal consumption quantities for the two goods are determined and, since the value of $x^*_\alpha$ decreases more than $x^*_\beta$ and $v_x$ is much larger than $v_y$, the value of $V_\alpha$ raises less than the value of $V_\beta$, so that the ordering among $V_\alpha$, $\overline{V}$ and $V_\beta$ is maintained but the distance $(V_\beta - \overline{V})^2$ increases, while the distance $(V_\alpha - \overline{V})^2$ decreases. The repetition of such process eventually leads to the extinction of the agents of group $\beta$. The explanation of the scenario which leads to the convergence towards $a = 0$ is omitted, as it is completely symmetric.

In regard to the convergence towards a nontrivial steady state, let us consider for instance $\mu = 2$ and $a_0 = 0.7$ and let us clarify why the system tends to $a = \hat{a} = 0.160$. Since $V_\alpha < \overline{V} < V_\beta$, with $(V_\alpha - \overline{V})^2 > (V_\beta - \overline{V})^2$, we expect a decrease in the share of the agents of group $\alpha$ and, consequently, a fall in the aggregate demand of commodity $x$. Indeed, we do observe a raise in the relative price $p$. The optimal consumption quantities for the two goods are determined and, since the value of $x^*_\alpha$ increases more than $x^*_\beta$ and $v_x$ is much larger than $v_y$, the value of $V_\alpha$ increases while the value of $V_\beta$ decreases, so that the new ordering among $V_\alpha$, $\overline{V}$ and $V_\beta$ is given by $V_\alpha < V_\beta < \overline{V}$. Since $V_\beta$ is closer than $V_\alpha$ to $\overline{V}$, the share of agents of type $\alpha$ falls. The repetition of such process leads to visibility values which satisfy $V_\alpha < V_\beta < \overline{V}$, with $V_\alpha$ that increases and $V_\beta$ that decreases, so that the share of agents of type $\alpha$ is progressively reduced, as long as it holds that $V_\alpha = V_\beta = 0.48$ in correspondence to the nontrivial steady state $a = \hat{a} = 0.160$.

Let us finally provide the explanation of a framework which leads to the convergence towards a period-two cycle. Considering e.g. $\mu = 5.5$ and starting from $a_0 = 0.06$, we have that the agents of type $\alpha$ are few and thus we expect that the aggregate demand for commodity $x$ is low. Indeed, we do observe a high value for the relative price $p$. The optimal consumption quantities for the two goods are determined and, since $p$ is high, the value of $x^*_\alpha$ is high, too. Due to the fact that $v_x$ is much larger than $v_y$, we find that $V_\beta < V_\alpha < \widehat{V}$. Since the distance between $V_\alpha$ and $\widehat{V}$ is smaller than the distance between $V_\beta$ and $\widehat{V}$, the share of agents of type $\alpha$ raises and exceeds $a = 0.47$. We then expect that the aggregate demand for commodity $x$ is higher than before. Indeed, we observe a lower value for the relative price $p$. The optimal consumption quantities for the two goods are determined and, since $p$ is lower now, the value of $x^*_\alpha$ decreases. As $v_x$ is still much larger than $v_y$, we find that $V_\alpha < V_\beta < \widehat{V}$. Since $V_\beta$ is closer to $\widehat{V}$ than $V_\alpha$, the share of agents of type $\alpha$ decreases and is again near $a = 0.06$, giving rise to the period-two cycle, whose values are $a = 0.062$ and $a = 0.476$.

## 4 Conclusions

In the present work we proposed a discrete-time exchange economy evolutionary model with two groups of agents, in which the reproduction level of a group is related to its attractiveness degree, that depends on the social visibility level, determined by the consumption choices of the agents in that group. Like in Naimzada and Pireddu (2019b) we dealt with generic bell-shaped attractiveness functions for the two groups of agents, increasing for low visibility levels and decreasing when the visibility of the group exceeds a given threshold value, due to a congestion effect. However, differently from that paper, rather than considering just Cobb-Douglas utility functions, we let utility functions free to vary in a suitable set of maps. This increased generality in the class of considered utility functions urged us to investigate not only the dynamic features of our model, but also to study the existence and the generic regularity of the market equilibria, which link the equilibrium price and the optimal consumption choices to population shares. In order to illustrate the general results we obtained on equilibria, as well as to investigate the emergence of fashion cycle dynamics, we added a simulative section where we considered Stone-Geary utility functions, which generalize the Cobb-Douglas utility functions, and we employed the formulations of the attractiveness maps dealt with in Naimzada and Pireddu (2018d, 2019b). Like in those settings, also in the present framework we observed interesting multistability phenomena, involving equilibria, as well as periodic or chaotic attractors. In particular, thanks to the combined action of the price formation mechanism and of the share updating rule, with the alternation between the bandwagon and snob regimes, the dynamic coexistence between groups and the oscillatory nature of the consumption activities over the two goods for the agents belonging to the two groups display the recurrent dynamic behavior typical of the fashion cycle (see Simmel, 1957), characterized by booms and busts.

As concerns future study directions, our setting could be employed to represent the fashion cycle in frameworks with capital accumulation, such as the OLG model by Diamond (1965).

A further extension of our setting would consist in assuming two different time scales for consumption choices and for the evolutionary mechanism. Indeed, since the consumption activities require less time than the updating process of the population shares, it would be suitable to assume a continuous-time formulation for the consumption choices and a discrete-time formulation for the share updating rule. This would lead to the study of a hybrid dynamical system, like those analyzed e.g. in Cavalli and Naimzada (2016) and in Cavalli et al. (2018).

Alternatively, we could assume that the utility of some consumers depends also on the attractiveness level of their group, in order to investigate whether the more "rational" agents, which are aware of the functioning of the share updating mechanism, perform better from an evolutionary viewpoint.

Finally, modifying the argument of agents' utility functions so as to enter a setting with strategic interaction, like those considered in Heifetz et al. (2007a, 2007b), and displaying strategic complementarity and/or substitutability, we could investigate the possible dynamical effects produced by two groups heterogeneous in the structure of preferences in those contexts, too.

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# Appendix

## Bifurcation analysis

For the sake of completeness, in the present Appendix we analyze the dynamics arising in Scenarios A–C.
For the reader's convenience, we recall that Scenario A is characterized by setting $c_i = d_i = 0$ in (20), for $i \in \{\alpha, \beta\}$, so that Stone-Geary utility functions reduce to Cobb-Douglas utility functions, Scenario B is characterized by $c_\alpha = 0.3$, $c_\beta = 0.12$, $d_\alpha = 0.1$, $d_\beta = 0.15$, while Scenario C is characterized by $c_\alpha = -0.19$, $c_\beta = -0.14$, $d_\alpha = -0.3$, $d_\beta = -0.16$.
Let us start focusing on Scenario A. With such choice for the parameters, recalling (13) and (21), we have

$$\begin{aligned} &x^*_{\alpha,t} = \tfrac{3.6}{16\,a_t+2}, \quad x^*_{\beta,t} = \tfrac{0.4}{16\,a_t+2}, \quad y^*_{\alpha,t} = \tfrac{4}{-16\,a_t+18}, \quad y^*_{\beta,t} = \tfrac{36}{-16\,a_t+18}, \\ &V_{\alpha,t} = 0.9\,x^*_{\alpha,t} + 0.15\,y^*_{\alpha,t}, \quad V_{\beta,t} = 0.9\,x^*_{\beta,t} + 0.15\,y^*_{\beta,t}. \end{aligned} \tag{A1}$$

According to Proposition 5, for every positive value of $\mu$ and for all formulations of the map describing attractiveness, $a = 0$ is locally asymptotically stable for the map $g$ in (17) because $0.102 = (V_\beta(0) - \overline{V})^2 < (V_\alpha(0) - \overline{V})^2 = 0.728$, and $a = 1$ is locally asymptotically stable for the map $g$ because $0.102 = (V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2 = 3.686$. As concerns the nontrivial equilibria, the equation $V_\alpha(a) = V_\beta(a)$ admits as unique solution $\hat{a} = 0.344$, which according to (18) is stable for $\mu < \hat{\mu}' = 4.372$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \hat{\mu}'' = 2.997$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \hat{\mu}''' = 1.321$ for attractiveness $\mathscr{A}'''$ in (25), while the equation $(V_\alpha(a) + V_\beta(a))/2 = \overline{V}$ admits $\tilde{a}_1 = 0.055$ and $\tilde{a}_2 = 0.851$ as solutions. Based on (19), $a = \tilde{a}_1$ is stable for $\mu < \tilde{\mu}'_1 = -4.182$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_1 = -2.843$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_1 = -1.010$ for attractiveness $\mathscr{A}'''$ in (25), while $a = \tilde{a}_2$ is stable for $\mu < \tilde{\mu}'_2 = -3.217$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_2 = -2.392$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_2 = -0.455$ for attractiveness $\mathscr{A}'''$ in (25). Hence, for the considered parameter configuration, $a = \tilde{a}_1$ and $a = \tilde{a}_2$ are always unstable[13], as confirmed by the bifurcation diagrams in Figure A1, where we denote in magenta $a = 0$, in red $a = 1$, in blue the points generated by the initial condition $a_0 = 0.3437$ and in green the points generated by the initial condition $a_0 = 0.8$, while we denote in orange the stationary equilibrium $a = \hat{a}$ when it is no more stable, as well as the unstable equilibria $a = \tilde{a}_1$, $a = \tilde{a}_2$ and the unstable period-two cycle colliding with $a = \hat{a}$ when the flip bifurcation is subcritical. The same color legend applies also to Figures A2 and A3, which correspond to Scenarios B and C, respectively.[14] We stress that in Scenario C no subcritical flip bifurcations arise and that also the flip bifurcations detected in Figure A1 (c) and in Figure A2 (c) are supercritical.

[13] We stress that, although for the parameter configurations we deal with in Scenarios A–C and for the attractiveness formulations in (23)–(25) $a = \tilde{a}_1$ and $a = \tilde{a}_2$ are always unstable and just play the role of separating the basins of attraction of the various coexisting attractors, this is not true in general. Taking, for instance, $c_i = d_i = 0$, $i \in \{\alpha, \beta\}$, and $\beta = 0.1$, $\alpha = 0.9$, $\mu = 6.5$, $\sigma = 8$, $v_x = 0.8$, $w_x = 0.1$, $v_y = 0.5$, $w_y = 2$, $\overline{V} = 0.85$, the only nontrivial equilibria in the interval $(0, 1)$ are $a = \tilde{a}_1$ and $a = \tilde{a}_2$, of which the former is locally stable, as well as $a = 1$. Considering instead $c_i = d_i = 0$, $i \in \{\alpha, \beta\}$, and $\beta = 0.1$, $\alpha = 0.9$, $\mu = 6.5$, $\sigma = 8$, $v_x = 0.2$, $w_x = 0.3$, $v_y = 0.25$, $w_y = 1.4$, $\overline{V} = 0.36$, we find that also $a = \hat{a}$ belongs to $(0, 1)$, and the locally stable equilibria are given by $a = 0$, $a = \tilde{a}_1$ and $a = 1$. Hence, both with $a = \hat{a}$ belonging to $(0, 1)$ or not, we may have multistability phenomena involving $a = \tilde{a}_1$ and $a = \tilde{a}_2$, too. Indeed, as shown in Proposition 5, $a = \tilde{a}_1$ and $a = \tilde{a}_2$ lose stability via a flip bifurcation occurring for $\mu = \tilde{\mu}$ in (19) when increasing $\mu$.

[14] We remark that in Figure A3 (a) we should represent also the unstable period-two cycle which emerges, together with the stable period-two cycle, via a fold bifurcation of the map $g$ for $\mu = 5.430$. On the other hand, since the unstable period-two cycle does not collide with $a = \hat{a}$ when the (supercritical) flip bifurcation occurs, and it rather undergoes a sequence of period-doubling bifurcations, we chose not to draw it, in order not to overburden the diagram.

Moving now to Scenario B we have

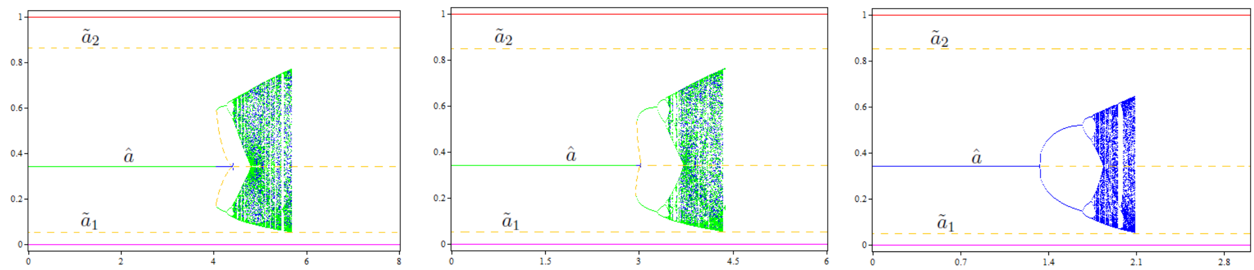
$$x^*_{\alpha,t} = \frac{-22\,a_t+60}{155\,a_t+35}, \quad x^*_{\beta,t} = \frac{-22\,a_t+7}{155\,a_t+35}, \quad y^*_{\alpha,t} = \frac{27\,a_t+23}{-119\,a_t+144}, \quad y^*_{\beta,t} = \frac{27\,a_t+288}{-119\,a_t+144},$$
$$V_{\alpha,t} = 0.9\,x^*_{\alpha,t} + 0.15\,y^*_{\alpha,t}, \quad V_{\beta,t} = 0.9\,x^*_{\beta,t} + 0.15\,y^*_{\beta,t}.$$

Also in this case, for every positive value of $\mu$ and for all formulations of the map describing attractiveness, $a = 0$ is locally asymptotically stable for the map $g$ in (17) because $0.102 = (V_\beta(0) - \overline{V})^2 < (V_\alpha(0) - \overline{V})^2 = 0.587$, and $a = 1$ is locally asymptotically stable for map $g$ because $0.102 = (V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2 = 1.037$. In regard to the nontrivial equilibria, the equation $V_\alpha(a) = V_\beta(a)$ admits as unique solution $\hat{a} = 0.463$, which according to (18) is stable for $\mu < \hat{\mu}' = 4.172$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \hat{\mu}'' = 2.860$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \hat{\mu}''' = 1.261$ for attractiveness $\mathscr{A}'''$ in (25), while the equation $(V_\alpha(a)+V_\beta(a))/2 = \overline{V}$ admits $\tilde{a}_1 = 0.071$ and $\tilde{a}_2 = 0.899$ as solutions. According to (19), $a = \tilde{a}_1$ is stable for $\mu < \tilde{\mu}'_1 = -4.83$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_1 = -1.204$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_1 = -1.085$ for attractiveness $\mathscr{A}'''$ in (25), while $a = \tilde{a}_2$ is stable for $\mu < \tilde{\mu}'_2 = -3.945$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_2 = -0.971$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_2 = -0.756$ for attractiveness $\mathscr{A}'''$ in (25). Hence, for the considered parameter configuration, $a = \tilde{a}_1$ and $a = \tilde{a}_2$ are always unstable, as confirmed by the bifurcation diagrams in Figure A2.
Similarly, in Scenario C we have

$$x^*_{\alpha,t} = \frac{211\,a_t+89}{1460\,a_t+40}, \quad x^*_{\beta,t} = \frac{211\,a_t+8}{1460\,a_t+40}, \quad y^*_{\alpha,t} = \frac{-25\,a_t+27}{-53\,a_t+54}, \quad y^*_{\beta,t} = \frac{-25\,a_t+108}{-53\,a_t+54},$$
$$V_{\alpha,t} = 0.9\,x^*_{\alpha,t} + 0.15\,y^*_{\alpha,t}, \quad V_{\beta,t} = 0.9\,x^*_{\beta,t} + 0.15\,y^*_{\beta,t}.$$

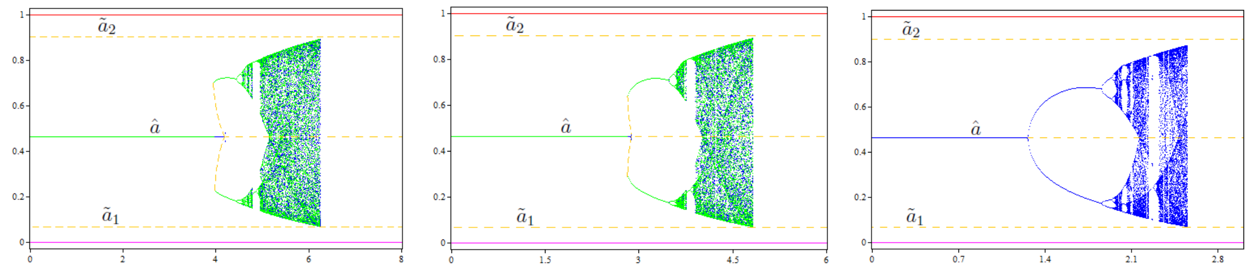
For every positive value of $\mu$ and for all formulations of the map describing attractiveness, we have again that $a = 0$ is locally asymptotically stable for map $g$ in (17) because $0.102 = (V_\beta(0) - \overline{V})^2 < (V_\alpha(0) - \overline{V})^2 = 1.630$, and $a = 1$ is locally asymptotically stable for map $g$ because $0.102 = (V_\alpha(1) - \overline{V})^2 < (V_\beta(1) - \overline{V})^2 = 138.661$. As concerns the nontrivial equilibria, the equation $V_\alpha(a) = V_\beta(a)$ admits as unique solution $\hat{a} = 0.160$, which according to (18) is stable for $\mu < \hat{\mu}' = 5.549$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \hat{\mu}'' = 3.804$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \hat{\mu}''' = 1.677$ for attractiveness $\mathscr{A}'''$ in (25), while the equation $(V_\alpha(a)+V_\beta(a))/2 = \overline{V}$ admits $\tilde{a}_1 = 0.028$ and $\tilde{a}_2 = 0.809$ as solutions. According to (19), $a = \tilde{a}_1$ is stable for $\mu < \tilde{\mu}'_1 = -2.946$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_1 = -2.006$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_1 = -0.801$ for attractiveness $\mathscr{A}'''$ in (25), while $a = \tilde{a}_2$ is stable for $\mu < \tilde{\mu}'_2 = -2.889$ for attractiveness $\mathscr{A}'$ in (23), for $\mu < \tilde{\mu}''_2 = -2.490$ for attractiveness $\mathscr{A}''$ in (24) and for $\mu < \tilde{\mu}'''_2 = -0.293$ for attractiveness $\mathscr{A}'''$ in (25). Hence, for the considered parameter configuration, $a = \tilde{a}_1$ and $a = \tilde{a}_2$ are always unstable, as confirmed by the bifurcation diagrams in Figure A3.
Comparing Figures A1–A3, we observe that, for the attractiveness formulation in (25), the flip bifurcation occurring at $a = \hat{a}$ for $\mu = \hat{\mu}$ is supercritical in Scenarios A–C, while, for the attractiveness formulations in (23) and (24), in Scenarios A and B that flip bifurcation is subcritical, and it is supercritical in Scenario C.[15] Nonetheless, while in Figure A3 (b) the flip bifurcation is followed by a cascade of period-doubling bifurcations leading to chaos and the unique multistability phenomenon concerns the central attractor and the trivial equilibria, in Figure A3 (a) the internal period-two cycle, born for $\mu = \hat{\mu}$ via the flip bifurcation, bears an increase in its values and soon falls within the basin of attraction of the external attractor. Like in Figure A1 (a, b) and in Figure A2 (a, b), the external attractor in Figure A3 (a) was born via a fold bifurcation of the map $g$ as a period-two cycle, then undergoing a sequence of period-doubling bifurcations, leading first to a chaotic attractor in two pieces,

[15]We stress that the occurrence of a subcritical or of a supercritical flip bifurcation could be proven as done in Proposition III.2 in Naimzada and Pireddu (2018d) and in Proposition 3.1 in Naimzada and Pireddu (2019b), checking that the conditions in Wiggins (2003), page 516, as well as those in Theorem 8.2 in Sharkovsky et al. (1997) are fulfilled. We omit the corresponding proofs for brevity's sake.




(a) (b) (c)

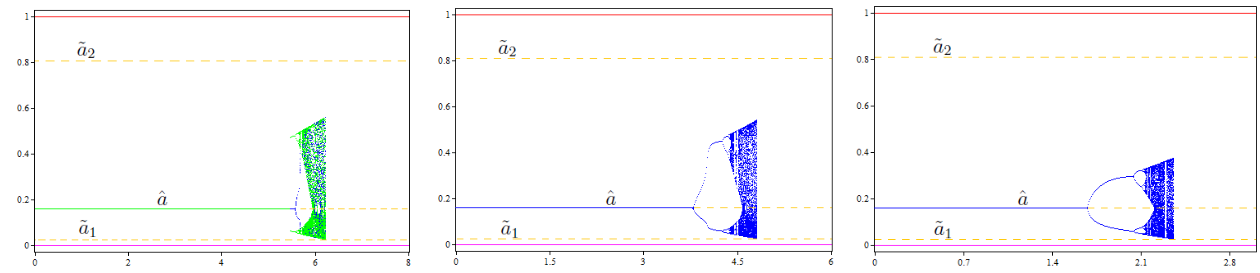
Figure A1: For $c_i = d_i = 0$, $i \in \{\alpha, \beta\}$, we report the bifurcation diagram of $g$ in (a) with attractiveness $\mathscr{A}'$ for $\mu \in (0, 8)$, in (b) with attractiveness $\mathscr{A}''$ for $\mu \in (0, 6)$, and in (c) with attractiveness $\mathscr{A}'''$ for $\mu \in (0, 3)$. Solid (dashed) lines refer to stable (unstable) equilibria and cycles.




(a) (b) (c)

Figure A2: For $c_\alpha = 0.3$, $c_\beta = 0.1$, $d_\alpha = 0.1$, $d_\beta = 0.15$, we report the bifurcation diagram of $g$ in (a) with attractiveness $\mathscr{A}'$ for $\mu \in (0, 8)$, in (b) with attractiveness $\mathscr{A}''$ for $\mu \in (0, 6)$, and in (c) with attractiveness $\mathscr{A}'''$ for $\mu \in (0, 3)$. Solid (dashed) lines refer to stable (unstable) equilibria and cycles.

that for increasing values of the parameter $\mu$ join into a one-piece chaotic attractor, which disappears due to a contact bifurcation with $a = \tilde{a}_1$. We remark that also the chaotic attractors in Figure A1 (c), in Figure A2 (c) and in Figure A3 (b, c) disappear due to a contact bifurcation with $a = \tilde{a}_1$, and that the same phenomenon occurred in Naimzada and Pireddu (2018d, 2019b), as well.
Due to the absorption of the internal period-two cycle into the external attractor in Figure A3 (a) or due to the subcritical flip bifurcations in Figure A1 (a, b) and in Figure A2 (a, b), after which $a = \hat{a}$ is repelling and the only stable set is the external attractor previously coexisting with $a = \hat{a}$, suddenly oscillations in the agents' consumption choices and in the population shares become strong, while the flip bifurcations in Figure A1 (c), in Figure A2 (c) and in Figure A3 (b, c) are initially followed by small oscillations, which gradually grow for increasing values of $\mu$, until we observe large oscillations, similar to those occurring soon after the flip bifurcation for $\mu = \hat{\mu}$ in all other cases. We notice that, although the global dynamics may not coincide for different parameter configurations and attractiveness formulations, in all scenarios we find interesting multistability phenomena, involving trivial and nontrivial equilibria, as well as periodic or chaotic attractors. In particular, the scenarios with the highest degree of complexity from a multistability viewpoint are those in Figure A1 (a) and in Figure A3 (a), because in the other cases at most we observe the coexistence between $a = \hat{a}$, the external period-two cycle and the trivial equilibria, which are always stable.
As already stressed, since for $c_i = d_i = 0$, $i \in \{\alpha, \beta\}$, Stone-Geary utility functions in (20) reduce to Cobb-Douglas utility functions, Scenario A coincides with the framework analyzed in Naimzada and Pireddu (2018d, 2019b), where indeed Cobb-Douglas utility functions were considered, in Naimzada and Pireddu (2018d) for the attractiveness formulation in (23) and in Naimzada and Pireddu (2019b)




(a) (b) (c)

Figure A3: For $c_\alpha = -0.19$, $c_\beta = -0.14$, $d_\alpha = -0.3$, $d_\beta = -0.16$, we report the bifurcation diagram of $g$ in (a) with attractiveness $\mathscr{A}'$ for $\mu \in (0, 8)$, in (b) with attractiveness $\mathscr{A}''$ for $\mu \in (0, 6)$, and in (c) with attractiveness $\mathscr{A}'''$ for $\mu \in (0, 3)$. Solid (dashed) lines refer to stable (unstable) equilibria and cycles.

for the attractiveness formulations in (24) and in (25). Comparing the bifurcation analysis performed in those works in terms of $\Delta = \alpha - \beta$ and here in terms of $\mu$, we notice that, while the value of the nontrivial stationary equilibria does not depend on $\mu$, their value is not constant with respect to $\Delta$. In fact, when moving $\Delta$, it holds that $a = \hat{a}$, $a = \tilde{a}_1$ and $a = \tilde{a}_2$ may become negative or exceed 1, and transcritical bifurcations occur when a trivial and a nontrivial equilibria merge, leading to an exchange in stability, so that the previously stable equilibrium loses its stability in favor of the unstable one, which becomes stable. Hence, with respect to $\Delta$, for instance, it does not hold that $a = 0$ and $a = 1$ are always stable and that $a = \tilde{a}_1$ and $a = \tilde{a}_2$ are always unstable. Except for such difference, we find that the same kind of flip bifurcations and of global dynamics occur when considering either $\Delta$ or $\mu$ as bifurcation parameters. Moreover, when letting $\mu$ vary, the situation does not change from a dynamical viewpoint if instead of dealing with Cobb-Douglas utility functions we consider Stone-Geary utility functions with positive values for the parameters $c_i$ and $d_i$, $i \in \{\alpha, \beta\}$, in (20). Indeed, we observe a strong analogy between Figures A1 and A2 for the attractiveness formulations in (23)–(25). However, when taking into account negative values for $c_i$ and $d_i$ in (20), while the flip bifurcation for the attractiveness formulation in (25) is still supercritical, the flip bifurcations occurring, for increasing values of $\mu$, for the attractiveness formulations in (23) and (24) from subcritical become supercritical. As observed above, this may allow either for a gradual rise in the width of the oscillations of the population shares when, like in Figure A3 (b, c), the flip bifurcation is followed by a standard cascade of period-doubling bifurcations leading to chaos, or for sudden strong oscillations when, like in Figure A3 (a), the period-two cycle following the flip bifurcation is captured by the external attractor.

Thus, we can conclude that, while the simulative results in Naimzada and Pireddu (2018d, 2019b) seem to be robust with respect to the employed bifurcation parameter, considering utility functions that do not coincide with the Cobb-Douglas formulation, for instance taking negative values of $c_i$ and $d_i$ in (20), leads to a greater variety of dynamical outcomes, allowing for frameworks, like that in Figure A3 (a), which bear resemblance to, but that do not coincide with, those detected when letting $\mu$ vary just in correspondence to null or positive values of $c_i$ and $d_i$.