DEMS WORKING PAPER SERIES

Eductive stability may not imply evolutionary stability in the presence of information costs

Ahmad Naimzada and Marina Pireddu

No. 406 – May 2019

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

Eductive stability may not imply evolutionary stability in the presence of information costs

Ahmad Naimzada [a*], Marina Pireddu [b†]

[a] Dept. of Economics, Management and Statistics, University of Milano - Bicocca, U6 Building, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy.

[b] Dept. of Mathematics and its Applications, University of Milano - Bicocca, U5 Building, Via Cozzi 55, 20125 Milano, Italy.

May 2, 2019

Abstract

Starting from a Muthian cobweb model, we extend the profit-based evolutionary setting in Hommes and Wagener (2010) populated by pessimistic, optimistic and unbiased fundamentalists, by assuming that agents face heterogeneous information costs, inversely proportional to the entity of their bias. Hommes and Wagener (2010) proved that, when the unique steady state of their model is globally eductively stable in the sense of Guesnerie (2002), the equilibrium under evolutionary learning may be just locally, but not globally, stable, due to the presence of a period-two cycle. Thanks to the introduction of information costs, we find that the equilibrium, when globally eductively stable, may be not even locally stable under evolutionary learning. More precisely, we analyze our setting by measuring the influence of agents' heterogeneity through the parameter describing the degree of optimism and pessimism. According to the considered parameter configuration, the unique steady state, which coincides with the fundamental, may be (locally or globally) stable for every value of the bias, like in Hommes and Wagener (2010), or it may be stable

*Tel.: +39 0264485813. E-mail address: ahmad.naimzada@unimib.it

†Tel.: +39 0264485767. E-mail address: marina.pireddu@unimib.it

just for suitably small and for suitably large values of the bias. Hence, increasing beliefs' heterogeneity can be stabilizing when information costs are taken into account. We give an interpretation of such counterintuitive result in terms of profits, on which the share updating rule is based.

Keywords: Muthian cobweb model; heterogeneous agents; evolutionary learning; information costs; double stability threshold.

JEL classification: B52, C62, D84, E32

1 Introduction

Brock and Hommes (1997) present a Muthian cobweb type demand-supply model, where producers can choose between rational and naive expectations about prices, selecting the strategy on the basis of the recent profits the two forecasting rules allowed to realize. In particular, an information cost is associated to the use of the more sophisticated forecasting rule. Dealing with the same share updating mechanism adopted in Brock and Hommes (1997) for the case without memory, Hommes and Wagener (2010) consider a Muthian cobweb model framework in which producers can choose among three different forecasting rules: fundamentalists predict that prices will always be at their fundamental value, optimists predict that the price of the good will always be above the fundamental price, whereas pessimists always predict prices below the fundamental price. Despite the heterogeneity in the forecasting rules, in Hommes and Wagener (2010) all agents face a null information cost, regardless of their degree of rationality. Hommes and Wagener (2010) focus on the case in which the Muthian model is globally eductively stable in the sense of Guesnerie (2002), that is, on the case in which the model is stable under naive expectations, as the slopes of demand and supply satisfy the familiar "cobweb theorem" by Ezekiel (1938). They show that under evolutionary learning the steady state, which is always (locally or globally) stable, may coexist with a locally stable period-two cycle, along which prices fluctuate around the rational expectations price and most agents switch between optimistic and pessimistic strategies. This means that, although the model in Hommes and Wagener (2010) is globally eductively stable, the evolutionary system therein admits both the steady state and the period-two cycle as possible long-run outcomes, and thus, contrarily to the setting in

Brock and Hommes (1997), it may be not globally evolutionary stable.
Extending the model in Hommes and Wagener (2010) by assuming that agents face heterogeneous information costs, inversely proportional to the entity of their bias,[1] we find that the equilibrium, when globally eductively stable, may be not even locally stable under evolutionary learning. Hence, the introduction of differentiated information costs, in addition to making the characterization of agents' heterogeneity more complete than in Hommes and Wagener (2010), allows to obtain a stronger result, which gives a neater negative answer to the question *does eductive stability always imply evolutive stability?* addressed in that paper, and which was in turn inspired by the claim that *"reasonable" adaptive learning processes are asymptotically stable* in Guesnerie (2002).
In more detail, our setting is mainly analyzed by measuring the influence of agents' heterogeneity through the parameter describing the degree of optimism and pessimism, like done, in financial markets contexts, e.g. in De Grauwe and Rovira Kaltwasser (2012), in Naimzada and Pireddu (2015), and in Naimzada and Ricchiuti (2008, 2009). We find that the unique steady state, which coincides with the fundamental, may be stable either for all values of the bias or just for suitably small and for suitably large values of the bias. Thus, thanks to the presence of information costs, the steady state, according to the considered parameter configuration, may be either (locally or globally) stable or unstable when the Muthian model is globally eductively stable. In particular, the existence for us of a double stability threshold implies that the introduction of information costs may not only produce a destabilization of the system for intermediate values of the bias of optimistic and pessimistic agents, but that a sufficiently strong beliefs' heterogeneity may be stabilizing in our setting. Such counterintuitive result can be easily explained in terms of profits. Namely, when the bias is large enough, the fitness of optimists and pessimists is very low for prices in a neighborhood of the fundamental steady state and it becomes relatively more advantageous being unbiased, despite the higher information cost. The consequent increase

[1] Namely, according to Hommes (2013), page 150, *A fundamentalists strategy, however, requires structural knowledge of the economy and information about "economic fundamentals", and therefore we assume positive information-gathering costs for fundamentalists. In the cobweb model the fundamental forecast requires structural knowledge of demand and supply curves in order to compute the fundamental steady state price* p^*. Since biased agents do not perfectly know the economic fundamentals, we suppose that their information costs are lower than that of unbiased fundamentalists, but still non-negative.

in the share of unbiased fundamentalists makes prices more likely converge towards the steady state (see Hommes, 2013), that recovers its local stability. We stress that the possible destabilization of the steady state occurs via a flip bifurcation, at which a stable period-two cycle emerges, which persists even after the pitchfork bifurcation through which the steady state recovers its local stability. On the other hand, since the map governing the dynamics is monotonically decreasing, like it happened in Hommes and Wagener (2010) in the absence of information costs, no richer dynamics can arise.
In addition to the just described results, along the paper we investigate the effect of the main model parameters on the stability of the steady state. In particular, we find that increasing the information cost of unbiased fundamentalists has a destabilizing effect on the equilibrium, when the latter is not always unstable and if the bias of optimists and pessimists is large enough. Indeed, in such case, raising the information cost of unbiased fundamentalists makes the share of agents opting for such strategy decrease, due to the lower fitness in terms of profits, not only for prices far from the equilibrium, but also in a neighborhood of it, and this may lead to a destabilization of the steady state. On the other hand, increasing the information cost of biased agents has either no effect on the equilibrium stability or it is stabilizing, when their bias is excessively large. Namely, in such case, raising the information cost of optimists and pessimists makes the share of agents opting for those strategies decrease, also due to their lower fitness caused by the scarcely precise predictions when prices are in a neighborhood of the steady state. The consequent increase in the share of unbiased fundamentalists makes prices more likely converge towards the fundamental value. The remainder of the paper is organized as follows. In Section 2 we present the model, that we study analytically in Section 3, discussing the possible scenarios. In Section 4 we describe some extensions of the model.

2 The model

At first we recall the discrete-time evolutionary cobweb setting in Hommes and Wagener (2010), to which we add information costs in the profits (see (2.8)).
The economy is populated by unbiased fundamentalists, that we will call just fundamentalists, and by two types of biased fundamentalists, i.e., optimists and pessimists. In particular, in order to obtain the same steady state as

in Hommes and Wagener (2010), we assume that optimists and pessimists are symmetrically biased, with the former (latter) expecting that the price of the good they produce will always be above (below) the fundamental price. Moreover, for the sake of simplicity, along the paper we focus on the case in which, in addition to unbiased fundamentalists, just one group of optimists and one group of pessimists are present.[2] In the Muthian farmer model, agents have to choose the quantity q of a certain good to produce in the next period and are expected profit maximizers. Assuming a quadratic cost function

$$\gamma(q) = \frac{q^2}{2s}, \tag{2.1}$$

with $s > 0$, the supply curve is given by

$$S(p^e) = sp^e, \tag{2.2}$$

where p^e is the expected price and s describes its slope. The demand function is supposed to be linearly decreasing in the market price, i.e.,

$$D(p) = A - dp, \tag{2.3}$$

with A and d positive parameters, representing respectively the market size and the slope of the demand function. We stress that the demand is positive for sufficiently large values of A.
At the fundamental price $p = p^*$ demand equals supply, i.e.,

$$p^* = \frac{A}{d+s}\,. \tag{2.4}$$

This is also the expression of the unique model steady state in Hommes and Wagener (2010). Like in that paper, we will deal with the case in which the Muthian model is globally eductively stable in the sense of Guesnerie (2002), that is, on the case in which the model is stable under naive expectations, as the slopes of demand and supply satisfy the familiar "cobweb theorem" by

[2]Indeed, our main result, according to which global eductive stability may imply not even local stability under evolutionary learning when information costs are taken into account, holds true in such minimal setting, too. Nonetheless, we are working on extending the present framework by introducing several types of biased fundamentalists facing heterogeneous information costs, inversely proportional to the entity of their bias, in Naimzada and Pireddu (2019), in order to characterize that setting outcomes from a dynamic viewpoint.

Ezekiel (1938) and thus it holds that $s/d < 1$.
Agents have heterogeneous expectations about the price of the good they have to produce. In particular, fundamentalists predict that prices will always be at their fundamental value, while optimists (pessimists) predict that the price of the good will always be above (below) the fundamental price. Hence, assuming a symmetric disposition of the beliefs and characterizing the fundamentalists, pessimists and optimists by subscripts 0, 1, 2, respectively, in symbols we have that their expectations at time t are given by

$$p_{i,t}^e = p^* + b_i, \; i \in \{0, 1, 2\}, \text{ with } b_0 = 0, \, b_1 = -b, \, b_2 = b, \tag{2.5}$$

where $b > 0$ describes the bias of pessimists and optimists. In order to avoid a negative expectation for pessimists, we will restrict our attention to the bias values $b \in (0, p^*)$, with p^* as in (2.4).
Denoting by $\omega_{i,t}$ the share of agents choosing the forecasting rule $i \in \{0, 1, 2\}$ at time t, the total supply is given by $\sum_{i=0}^{2} \omega_{i,t} S(p_{i,t}^e)$ and thus the market equilibrium condition reads as

$$A - dp_t = \sum_{i=0}^{2} \omega_{i,t} S(p_{i,t}^e). \tag{2.6}$$

As concerns the share updating mechanism, Hommes and Wagener (2010) deal with the discrete choice model in Brock and Hommes (1997) for the case without memory, in which only the most recently realized net profits $\pi_{j,t}$, $j \in \{0, 1, 2\}$, are taken into account. In symbols

$$\omega_{i,t} = \frac{\exp(\beta \pi_{i,t-1})}{\sum_{j=0}^{2} \exp(\beta \pi_{j,t-1})}, \; i \in \{0, 1, 2\}, \tag{2.7}$$

where $\beta > 0$ is the intensity of choice parameter.
When considering information costs, net profits $\pi_{j,t}$, $j \in \{0, 1, 2\}$, at time t are defined as

$$\pi_{j,t} = p_t S(p_{j,t}^e) - \gamma(S(p_{j,t}^e)) - c_j, \tag{2.8}$$

with γ and S as in (2.1) and (2.2), respectively, and with the nonnegative parameter c_j representing the information cost deriving by the adoption of forecasting rule j. Since optimists and pessimists do not perfectly know the economic fundamentals and make symmetric errors in estimating them, we may conclude that those agents display the same degree of rationality, and

thus we will assume that $c_1 = c_2 = c$, for a certain $c \geq 0$. On the other hand, unbiased fundamentalists exactly know the formulations of demand and supply functions and they are able to correctly compute the fundamental value. Due to their higher degree of rationality with respect to optimists and pessimists, we will suppose that the information cost c_0 of fundamentalists satisfies $0 \leq c \leq c_0$, i.e., that the information costs are inversely proportional to the entity of the bias. We stress that for $c = c_0 = 0$ we are led back to the framework in Hommes and Wagener (2010).
Introducing, like in that paper, the variable $x_t = p_t - p^*$, we can write our model dynamic equation in deviation from fundamental as

$$x_t = -\frac{s}{d}\sum_{i=0}^{2} \omega_{i,t}\, b_i$$

with

$$\omega_{i,t} = \frac{\exp\left(-\frac{\beta s}{2}(x_{t-1} - b_i)^2 - \beta c_i\right)}{\sum_{j=0}^{2} \exp\left(-\frac{\beta s}{2}(x_{t-1} - b_j)^2 - \beta c_j\right)},$$

or, more explicitly, recalling (2.5), as

$$\begin{aligned} x_t &= \tfrac{sb}{d}(\omega_{1,t} - \omega_{2,t}) \\ &= \tfrac{sb}{d}\frac{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)+\exp\left(-\frac{\beta s}{2}x_{t-1}^2-\beta(c_0-c)\right)}. \end{aligned} \tag{2.9}$$

For the model formulation in terms of x_t, the unique steady state is given by $x^* = 0$ and we will derive the corresponding stability conditions in terms of the intensity of choice parameter in Proposition 3.1. To such aim, it is expedient to rewrite (2.9) as

$$x_t = f(x_{t-1}), \tag{2.10}$$

where the one-dimensional map $f : (-p^*, +\infty) \to \mathbb{R}$ is defined as

$$f(x) = \frac{sb}{d}\frac{\exp\left(-\frac{\beta s}{2}(x+b)^2\right) - \exp\left(-\frac{\beta s}{2}(x-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x+b)^2\right) + \exp\left(-\frac{\beta s}{2}(x-b)^2\right) + \exp\left(-\frac{\beta s}{2}x^2 - \beta(c_0-c)\right)}. \tag{2.11}$$

We stress that f is differentiable and that, recalling the expression of p^* in (2.4), its domain is enlarged by considering increasing values of A. Moreover, when extending its domain to $\mathbb{R}$, the map is odd. Namely, replacing x with

$-x$ leaves the denominator unchanged, while the two terms on the numerator of f are interchanged. Moreover, Hommes and Wagener (2010) prove in their Theorem A that the function in (2.11) is always decreasing when $c = c_0 = 0$, using an argument based on the nonnegativity of the variance in relation to a suitable stochastic process concerning the biases. In a completely analogous manner, it is possible to show that the map f is decreasing also for nonnull information costs. This excludes the possibility of complex dynamics in our framework, too, and indeed at most we observe a period-two cycle, either coexisting with the locally stable steady state, or being the unique attractor. Namely, differently from what obtained in Theorem A in Hommes and Wagener (2010), where in the absence of information costs the fundamental steady state is always (locally or globally) stable for $s/d < 1$, in Proposition 3.1 we show that the intensity of choice parameter β may also be either destabilizing or it may play an ambiguous role on the equilibrium stability when information costs are taken into account.

3 Analytical results and possible scenarios

We start our analysis by investigating in the next result which are the steady states for our model and by studying their local stability.

Proposition 3.1 *Equation* (2.10) *admits* $x = 0$ *as unique steady state. The equilibrium* $x = 0$ *is locally asymptotically stable for map* f *in* (2.11) *if*

$$\beta < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right)}{2b^2 s^2}. \tag{3.1}$$

Hence, if $s/d < 1$, *according to the considered parameter configuration, one of the following possibilities occurs:*

(a) there exists $\bar{\beta} > 0$ *such that* $x = 0$ *is stable for* $\beta \in (0, \bar{\beta})$ *and unstable otherwise;*

(b) there exist $0 < \beta' < \beta''$ *such that* $x = 0$ *is stable for each* $\beta \in (0, \beta') \cup (\beta'', +\infty)$ *and unstable otherwise;*

(c) $x = 0$ *is stable for every* $\beta > 0$.

In particular, (a) occurs when $b < \sqrt{\frac{2(c_0-c)}{s}}$, *(b) occurs when* $b > \sqrt{\frac{2(c_0-c)}{s}}$ *but it is not too large, and (c) occurs for sufficiently large values of* $b > \sqrt{\frac{2(c_0-c)}{s}}$.

Proof. A straightforward check ensures that $x = 0$ solves the fixed-point equation $f(x) = x$, with f as in (2.11).
In order to show that $x = 0$ is the unique steady state it suffices to observe that f is positive if and only if x is negative.
The stability condition follows by imposing that $f'(0) \in (-1, 1)$. By direct computations, we have

$$f'(0) = \frac{-2\beta s^2 b^2 \exp\left(\frac{-\beta b^2 s}{2}\right)}{d\left(2\exp\left(\frac{-\beta b^2 s}{2}\right) + \exp\left(-\beta(c_0 - c)\right)\right)}.$$

Since $f'(0)$ is always negative, the stability of $x = 0$ is guaranteed when $f'(0) > -1$, which is equivalent to (3.1). In particular, setting $g(\beta) = \beta$ and $h(\beta) = \left(d\left(2 + \exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right)\right) / (2b^2 s^2)$, we notice that g is increasing. According to the sign of the exponent $\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)$ in h, such map may be decreasing or increasing for $\beta > 0$. More precisely, the former possibility occurs for $b < \sqrt{\frac{2(c_0-c)}{s}}$, and the graphs of g and h intersect in one point $\bar{\beta} > 0$ with $g(\beta) < h(\beta)$ only for $\beta < \bar{\beta}$, and thus (a) occurs. If instead h is increasing, it is also convex. Since $g(0) < h(0)$ and h tends to $+\infty$ faster than g for $\beta \to +\infty$ due to the presence of the exponential function, the graphs of g and h either intersect twice, when $b > \sqrt{\frac{2(c_0-c)}{s}}$ but it is not too large, so that (b) occurs, or they never intersect, for sufficiently large values of $b > \sqrt{\frac{2(c_0-c)}{s}}$, and thus (c) occurs. This concludes the proof. □

In the numerical simulations we shall perform below, c_0 will be suitably chosen and we will let b free to vary, while the remaining parameters will be fixed as follows: $A = 8$, $s = 0.95$, $d = 1$, $\beta = 10$, $c = 0.1$. In such configuration, setting $c_0 = 0.2$ like in Figure 2, case (a) in Proposition 3.1 occurs for $b < \sqrt{\frac{2(c_0-c)}{s}} = 0.459$, while for $b > 0.459$ cases (b) and (c) take place. Namely, fixing for instance $b = 0.4$ we find that $x = 0$ is stable just for $\beta \in (0, 9.565)$, while for $b = 1$ we have that $x = 0$ is stable for

$\beta \in (0, 2.512) \cup (5.717, +\infty)$ and $x = 0$ is stable for all values of β when $b = 2$.
We stress that when setting $c = c_0 = 0$, i.e., in the absence of information costs, case (a) in Proposition 3.1 can not occur, because map h in the proof above is always increasing. Hence, if we neglected information costs we would not observe, in particular, the most classical effect of the intensity of choice parameter, that is, the local destabilizing scenario. Moreover, also case (b) in Proposition 3.1 would not occur since, as shown in Theorem A in Hommes and Wagener (2010) and as we recalled above, without information costs the steady state is always (locally or globally) asymptotically stable.
Despite such differences with the findings in Hommes and Wagener (2010) concerning the local stability of the steady state, similar conclusions to those drawn in Theorem A therein in regard to the existence of a locally stable period-two cycle can be obtained in the presence of information costs, too. Namely, Proposition 3.2 holds true. In particular, we stress that with the introduction of information costs, when the steady state loses stability, a flip bifurcation occurs at which a globally stable period-two cycle emerges. The period-two cycle becomes locally stable only in case the steady state recovers its stability for increasing values of β, otherwise the period-two cycle remains globally stable. When instead the steady state is always stable like in Hommes and Wagener (2010), the locally stable period-two cycle emerges, together with an unstable period-two cycle, for sufficiently large values of β through a double fold bifurcation of the second iterate of f and both the steady state and one of the two period-two cycles remain locally stable when raising β.

Proposition 3.2 *For $1/2 < s/d < 1$, the dynamical system in* (2.10) *admits a (locally) stable period-two cycle when β is large enough. Moreover, for $\beta \to +\infty$ the values of the period-two cycle approach $\{-sb/d, sb/d\}$.*

Proof. The proof of the existence of the (locally) stable period-two cycle when β is sufficiently large follows the same steps described in the verification of point (ii) in Theorem A in Hommes and Wagener (2010), adapted in order to take into account the information costs. In particular, since f^2 is monotonically increasing and bounded, the desired conclusion follows by rewriting the map f in (2.11) as

$$f(x) = \frac{sb}{d} \frac{\exp(-2\beta sbx) - 1}{1 + \exp(-2\beta sbx) + \exp\left(-\frac{\beta sb}{2}(2x - b) - \beta(c_0 - c)\right)}$$

and checking that $f^2(U) \subset U$, with $U = [b(1+\varepsilon)/2, +\infty)$ where $\varepsilon = s/d - 1/2 > 0$.
As concerns the limit situation with $\beta \to +\infty$, we find that

$$f(x) = \begin{cases} \frac{sb}{d} & \text{for} \quad x < \min\left\{\frac{-b^2+(c_0-c)}{2b}, 0\right\} \\ 0 & \text{for} \quad \min\left\{\frac{-b^2+(c_0-c)}{2b}, 0\right\} < x < \max\left\{\frac{b^2-(c_0-c)}{2b}, 0\right\} \\ \frac{-sb}{d} & \text{for} \quad x > \max\left\{\frac{b^2-(c_0-c)}{2b}, 0\right\} \end{cases}$$

and the values of the period-two cycle for $\beta \to +\infty$ are given by $\{-sb/d, sb/d\}$, as desired.

□

We notice that, although the values of the period-two cycle when $\beta \to +\infty$ are not affected by the introduction of the information costs, the basin of attraction of the period-two cycle and, consequently, that of the steady state, do. Namely, according to the proof of point (iii) of Theorem A in Hommes and Wagener (2010), when information costs are missing and $\beta \to +\infty$ the map governing the dynamics reads as

$$f(x) = \begin{cases} \frac{sb}{d} & \text{for} \quad x < -\frac{b}{2} \\ 0 & \text{for} \quad -\frac{b}{2} < x < \frac{b}{2} \\ \frac{-sb}{d} & \text{for} \quad x > \frac{b}{2} \end{cases}$$

Hence, the interval of values of x for which f vanishes, that is, the basin of attraction of the steady state, is reduced by the introduction of information costs and this induces an enlargement in the basin of attraction of the period-two cycle. In particular, if the bias is small enough with respect to the difference in the information costs, i.e., if $b < \sqrt{c_o - c}$, then being fundamentalists is not profitable for any initial condition as $\beta \to +\infty$, because the losses in the profits of optimists and pessimists deriving by their lower degree of rationality are more than compensated by the reduced information costs they face, when compared to those of fundamentalists. Hence, the basin of attraction of the steady state shrinks to $x = 0$ and the whole population switches from pessimism to optimism in every time period.

Rewriting the stability conditions in Proposition 3.1 in terms of the bias, we obtain the following result:

Corollary 3.1 *The equilibrium $x = 0$ is locally asymptotically stable for map f in* (2.11) *if*

$$b^2 < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right)}{2\beta s^2}. \tag{3.2}$$

Hence, according to the considered parameter configuration, $x = 0$ is stable for every $b > 0$ or there exist $0 < b' < b''$ such that $x = 0$ is stable for each $b \in (0, b') \cup (b'', +\infty)$.

Proof. Condition (3.2) follows by making b explicit in (3.1). Moreover, setting $g(b) = b^2$ and $h(b) = \left(d\left(2 + \exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right)\right) / (2\beta s^2)$, we notice that for $b \geq 0$ both g and h are increasing, convex maps with $g(0) < h(0)$. Since h tends to $+\infty$ faster than g for $b \to +\infty$ due to the presence of the exponential function, the graphs of g and h intersect never or twice according to the considered parameter configuration. The proof is complete. □

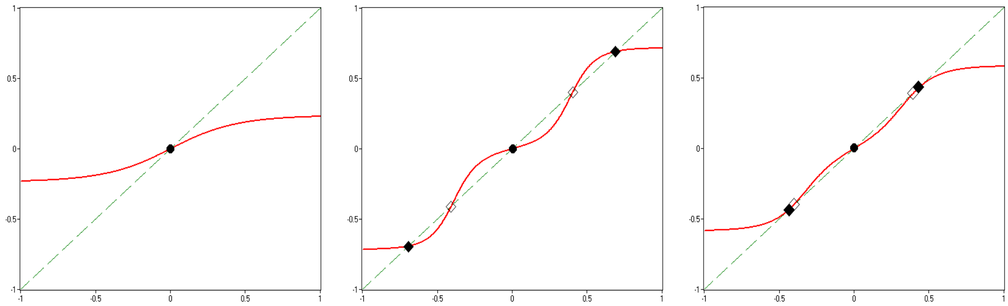
Thus, in Proposition 3.1 and in Corollary 3.1 we found up to two possible stability thresholds for $x = 0$ with respect to β and b, respectively, and $x = 0$ may be locally stable just for sufficiently low and for sufficiently high values of the intensity of choice parameter and of the bias. In particular, this means that the introduction of information costs may not only produce a destabilization of the system for intermediate values of the bias of optimistic and pessimistic agents, but that a sufficiently strong beliefs' heterogeneity may be stabilizing in our setting. As we shall see towards the end of the present section, such counterintuitive result can be easily explained in terms of profits.

We recall that also in the financial market settings considered in Chiarella et. al. (2006), in De Grauwe and Rovira Kaltwasser (2012) and in Naimzada and Pireddu (2015) two stability thresholds for the unique steady state are detected. However, while in those contexts the stability region lies within the two thresholds, in the present framework the stability region lies outside the two thresholds. Moreover, in the numerical simulations performed in Chiarella et. al. (2006) and in De Grauwe and Rovira Kaltwasser (2012) the focus is on the threshold at which the steady state loses stability through a Hopf bifurcation.

We now illustrate in Figures 1 and 2 the possible scenarios found in Corollary 3.1 for increasing values of b, while fixing the other parameters like described just after Proposition 3.1. More precisely, recalling that $c = 0.1$, before considering for c_0 the relatively high value $c_0 = 2$ in Figure 2, we focus in Figure 1 on the case $c_0 = 0.11$, in order to show that for almost coinciding values of the information costs for fundamentalists and for biased agents we obtain just the two frameworks which can occur in the absence of information costs. Namely, in Figure 1 (A) for $b = 0.4$ we find that the steady state $x = 0$ is globally stable and in Figure 1 (B) for $b = 0.8$ we observe, in addition to the locally stable steady state, denoted by a black dot, a stable and an unstable period-two cycles, denoted respectively by black and empty squares, which are born for $b \approx 0.690$ through a double fold bifurcation of the second iterate of f, that we illustrate in Figure 1 (C). Raising the value of the information cost for fundamentalists to $c_0 = 2$, in Figure 2 (A) for $b = 0.3$ we still find that the steady state $x = 0$ is globally stable, but in Figure 2 (B) for $b = 0.5$ we observe that the steady state is now unstable, and it is denoted by an empty dot, being surrounded by a globally stable period-two cycle, born for $b \approx 0.393$ through a pitchfork bifurcation of the second iterate of f, which corresponds to a flip bifurcation of f. In Figure 2 (C) for $b = 1$ the steady state $x = 0$ is again locally stable thanks to a further pitchfork bifurcation of the second iterate of f that has occurred for $b \approx 0.850$ at $x = 0$. The basin of attraction of $x = 0$ is separated by that of the locally stable period-two cycle by an unstable period-two cycle, born through the pitchfork bifurcation. We stress that in the frameworks considered in Figures 1 and 2 the period-two cycle persists for larger values of b and that the distance between the period-two points increases with b. In fact, due to the oddness of map f, those period-two points have opposite values. We also remark that if $b > \sqrt{\frac{2(c_0-c)}{s}}$, but it is not too large, so that case (b) in Proposition 3.1 occurs, then $x = 0$ loses and recovers stability for increasing values of β through a flip bifurcation of f and a pitchfork bifurcation of the second iterate of f, respectively, as it happens in Figure 2 when raising b.

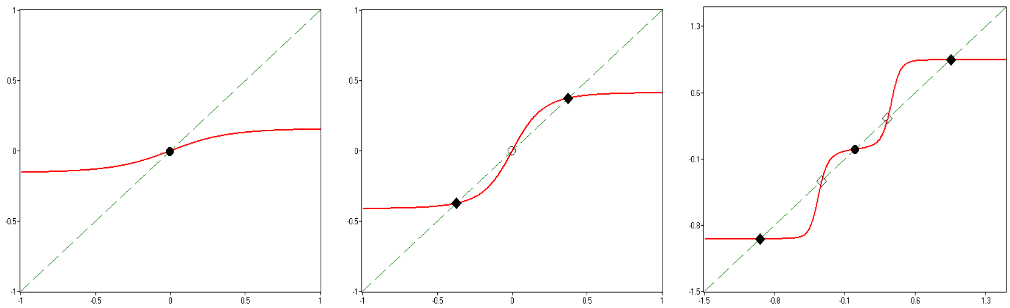
Before concluding the section by deriving in Corollary 3.2 the stability conditions for $x = 0$ with respect to the information costs, we give an economic interpretation of the reason why the introduction of information costs may lead, for increasing values of the bias, to a destabilization of the steady state, possibly followed by a recovery of its local stability.

Let us start noticing that, in the absence of information costs, $x = 0$ is (locally or globally) stable for every value of b because in a neighborhood of the steady state it is relatively advantageous being fundamentalists. More



(A) (B) (C)

Figure 1: The graph of the second iterate of f for $c_0 = 0.11$, and $b = 0.4$ in (A), $b = 0.8$ in (B), and $b = 0.690$ in (C).



(A) (B) (C)

Figure 2: The graph of the second iterate of f for $c_0 = 0.2$, and $b = 0.3$ in (A), $b = 0.5$ in (B) and $b = 1$ in (C).

precisely, if the bias is small, $x = 0$ is globally stable because profits of fundamentalists and biased agents do not differ very much and thus it does not happen that the profits of optimists or pessimists are much higher than those of fundamentalists when the initial condition is far from the steady state. On the other hand, the latter phenomenon does occur when b is large enough, as in this case the price forecast of optimists or pessimists is considerably more precise than that of fundamentalists when prices are far from the equilibrium. Then, in that region, a locally stable period-two cycle arises, along which agents switch between optimism and pessimism, even if $x = 0$ remains locally stable, because in a neighborhood of the steady state it is still rela-

tively profitable being fundamentalists.
When introducing information costs, which are higher for fundamentalists than for biased agents, for intermediate values of the bias it may be relatively more profitable being optimists or pessimists than fundamentalists even for prices in a neighborhood of the steady state.[3] Hence, starting from lower values of b and increasing the bias, the basin of attraction of $x = 0$ shrinks, until a globally stable period-two cycle emerges, with agents switching between optimism and pessimism. However, when the bias is excessively large, for prices close to the equilibrium it becomes again relatively more profitable being fundamentalists, because the forecast error made by biased agents is too big, and thus $x = 0$ recovers its stability.

Rewriting the stability conditions in Proposition 3.1 in terms of the information costs, we obtain the following result:

Corollary 3.2 *The equilibrium $x = 0$ is locally asymptotically stable for map f in (2.11) for every value of the information costs $0 \leq c \leq c_0$ if $b \leq \sqrt{d/(\beta s^2)}$, while if $b > \sqrt{d/(\beta s^2)}$ then $x = 0$ is locally asymptotically stable when*

$$c_0 - c < \log \left(\frac{d \exp\left(\frac{\beta b^2 s}{2}\right)}{2(b^2 \beta s^2 - d)} \right)^{\frac{1}{\beta}} . \tag{3.3}$$

Hence, if $b > \sqrt{d/(\beta s^2)}$, according to the considered parameter configuration, with respect to c one of the following possibilities occurs:

(a) $x = 0$ is unstable for every $0 \leq c \leq c_0$;

(b) there exists $c' \in (0, c_0)$ such that $x = 0$ is unstable for $c \in (0, c')$ and stable for each $c \in (c', c_0)$;

(c) $x = 0$ is stable for every $0 \leq c \leq c_0$,

[3]In this respect, we stress that, for low values of b, when the difference in the information costs is high with respect to the bias, $x = 0$ may be globally asymptotically stable even if the net profits of fundamentalists are lower than the net profits of biased fundamentalists, also in a neighborhood of the steady state. This is indeed what happens for the parameter configuration considered in Figure 2, where for $b = 0.3$ we have that $x = 0$ is globally asymptotically stable and for $x(0) = 0.1$ it holds that $\pi_1 = 8.213$, $\pi_2 = 8.270$, $\pi_3 = 8.184$, while for $b = 0.5$ just the period-two cycle is globally asymptotically stable and for $x(0) = 0.1$ it holds that $\pi_1 = 8.118$, $\pi_2 = 8.213$, $\pi_3 = 8.184$. Hence, the introduction of information costs allows for a large variety of frameworks, whose correct interpretation requires to take into account the values both of the information costs and of the bias.

while with respect to c_0 *one of the following possibilities occurs:*

(d) $x = 0$ *is unstable for every* $c_0 \geq c$*;*

(e) there exists $c_0' > c$ *such that* $x = 0$ *is stable for each* $c_0 \in (c, c_0')$ *and unstable for* $c_0 \in (c_0', +\infty)$*.*

In particular, (a) and (d) occur when $d \leq (2b^2\beta s^2)/\left(\exp\left(\frac{\beta b^2 s}{2}\right)+2\right)$*, (b) occurs when* $d > (2b^2\beta s^2)/\left(\exp\left(\frac{\beta b^2 s}{2}\right)+2\right)$ *and* c_0 *is larger than the right-hand side in* (3.3)*, (c) occurs when* c_0 *is smaller than the right-hand side in* (3.3)*, (e) occurs when* $d > (2b^2\beta s^2)/\left(\exp\left(\frac{\beta b^2 s}{2}\right)+2\right)$*.*

Proof. Condition (3.3) immediately follows by making $c_0 - c$ explicit in (3.1), and (a)–(e) directly follow by observing that the right-hand side in (3.3) is positive if and only if $d > (2b^2\beta s^2)/\left(\exp\left(\frac{\beta b^2 s}{2}\right)+2\right)$. □

Hence, according to Corollary 3.2, increasing the information cost of biased agents has either no effect on the equilibrium stability or it is stabilizing, when their bias is excessively large. Namely, in such case, raising the information cost of optimists and pessimists makes the share of agents opting for those strategies decrease, also due to their lower fitness caused by their scarcely precise predictions when prices are in a neighborhood of the steady state. The consequent increase in the share of unbiased fundamentalists makes prices more likely converge towards the fundamental value (see Hommes, 2013), and this explains the observed stabilizing effect on the equilibrium. On the other hand, Corollary 3.2 tells us that raising the information cost of fundamentalists is destabilizing, when the equilibrium is not always unstable and if the bias of optimists and pessimists is large enough. Indeed, in such case, increasing the information cost of unbiased fundamentalists makes the share of agents opting for such strategy decrease, due to the lower fitness in terms of profits, not only for prices far from the equilibrium, but also in a neighborhood of it, and this may lead to a destabilization of the steady state.

4 Conclusion

We believe the analyzed setting, which despite its simplicity allowed us to give a neat negative answer to the question *does eductive stability always* *imply evolutive stability?* addressed in Hommes and Wagener (2010), can be a starting point for other research developments. Indeed, in addition to investigating, as we are doing in Naimzada and Pireddu (2019), which are the differences in the outcomes deriving from the introduction in the model of several types of biased fundamentalists facing heterogeneous information costs, inversely proportional to the entity of their bias, we recall that the final sentence in Hommes and Wagener (2010) reads as follows: "The study of the stability of evolutionary systems with many trader types in various market settings and with more complicated strategies remains an important topic for future work". In this perspective, the present framework could be further extended by considering a richer set of forecasting rules, including e.g. rational expectations agents, with differentiated information costs, in view of analyzing whether dynamic phenomena more complex than the period-two cycle can emerge when the fundamental steady state loses stability or even when it is stable.

References

Brock, W.A., Hommes, C.H., 1997. A rational route to randomness. Econometrica 65, 1059–1095.

Chiarella, C., He, X.Z., Hommes, C.H., 2006. A dynamic analysis of moving average rules. J. Econ. Dyn. Control 30, 1729–1753.

De Grauwe, P., Rovira Kaltwasser, P., 2012. Animal spirits in the foreign exchange market. J. Econ. Dyn. Control 36, 1176–1192.

Ezekiel, M., 1938. The cobweb theorem. Quart. J. Econ. 52, 255–280.

Guesnerie, R., 2002. Anchoring economic predictions in common knowledge. Econometrica 70, 439–480.

Hommes, C., 2013. Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems. Cambridge University Press, Cambridge.

Hommes, C., Wagener, F., 2010. Does eductive stability imply evolutionary stability? J. Econ. Behav. Organ. 75, 25–39.

Naimzada, A., Pireddu, M., 2015. A financial market model with endogenous fundamental values through imitative behavior. Chaos 25, 073110. DOI: 10.1063/1.4926326

Naimzada, A., Pireddu, M., 2019. Eductive stability may not imply evolutionary stability in the presence of information costs and several types of biased fundamentalists, mimeo.

Naimzada, A., Ricchiuti, G., 2008. Heterogeneous fundamentalists and imitative processes. Appl. Math. Comput. 199, 171–180.

Naimzada, A., Ricchiuti, G., 2009. The dynamic effect of increasing heterogeneity in financial markets. Chaos Solit. Fract. 41, 1764–1772.