DEMS Department of Economics, Management and Statistics

DEMS WORKING PAPER SERIES

An evolutionary Cournot oligopoly model with imitators and perfect foresight best responders

Lorenzo Cerboni Baiardi
and Ahmad Naimzada

No. 407 – May 2019

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

AN EVOLUTIONARY COURNOT OLIGOPOLY MODEL WITH IMITATORS AND PERFECT FORESIGHT BEST RESPONDERS.

Lorenzo Cerboni Baiardi[1], Ahmad K. Naimzada[1]

[1]Dipartimento di Economia, Metodi Quantitativi e Strategie di Impresa, Bicocca University, Milano, Italy

ABSTRACT. We consider the competition among quantity setting players in a linear evolutionary environment. To set their outputs, players adopt, alternatively, the best response rule having perfect foresight or an imitative rule. Players are allowed to change their behavior through an evolutionary mechanism according to which the rule with better performance will attract more followers. The relevant stationary state of the model describes a scenario where players produce at the Cournot-Nash level. Due to the presence of imitative behavior, we find that the number of players and implementation costs, needed to the best response exploitation, have an ambiguous role in determining the stability properties of the equilibrium and double stability thresholds can be observed. Differently, the role of the intensity of choice, representing the evolutionary propensity to switch to the most profitable rule, has a destabilizing role, in line with the common occurrence in evolutionary models. The global analysis of the model reveals that increasing values of the intensity of choice parameter determine increasing dynamic complexities for the internal attractor representing a population where both decision mechanisms coexist.

Keywords: Imitation, heterogeneity, evolutionary game, replicator dynamics, dynamic instability, dynamical systems

1. INTRODUCTION

In the literature concerning oligopoly competition, heuristic behaviors and rule of thumbs are considered as decision mechanisms for players with limited rationality. Among such heuristics, the gradient rule can be mentioned, introduced by G. I. Bischi and Naimzada, 2000, Bischi, Gallegati, and Naimzada, 1999; Gian-Italo Bischi, Kopel, and Naimzada, 2001, Agiza, Hegazi, and El-sadany, 2002 and recently considered by Sameh Askar, 2014; SS Askar, 2014 and Fanti, Gori, and Sodini, 2015, according to which players with limited knowledge on the market demand adjust their outputs in order to increase their profits based on the local estimate of the slope of their profit function at the actual market state. Another important heuristic is the Local Monopolistic Approximation

(LMA), introduced by Tuinstra, 2004, considered in the framework of repeated oligopolies by G. I. Bischi, Naimzada, and Sbragia, 2007, A. K. Naimzada and Sbragia, 2006, A. K. Naimzada and Tramontana, 2009 and in a monopolistic framework by A. Naimzada and Ricchiuti, 2011. According to the LMA rule, players try to optimize their earnings by estimating a linear market demand based on the local knowledge of the true demand obtained at the actual market state. Moreover, the circumstance of heterogeneous degrees of rationality and computational abilities among players lead to consider the simultaneous presence of different decision mechanisms. Various pairings of heterogeneous behaviors, including the best response, the gradient rule and the LMA rule, are considered in Leonard and Nishimura, 1999, Den Haan, 2001, Agiza and Elsadany, 2003, 2004, Angelini, Dieci, and Nardini, 2009, Tramontana, 2010, Cavalli and Naimzada, 2014, Andaluz and Jarne, 2015, Cavalli, Naimzada, and Tramontana, 2015, Cavalli and Naimzada, 2015, Pireddu, 2015, Tramontana, Elsadany, Xin, and Agiza, 2015, A. Naimzada and Tramontana, 2015 and in Andaluz, Elsadany, and Jarne, 2017.

Heterogeneous behaviors have also been considered in evolutionary settings where players choose to behave according to a certain rule by selecting, within a finite set of possible decision mechanisms, the most profitable one evaluated on the basis of the relative performances it has brought in the past. See Droste, Hommes, and Tuinstra, 2002, for the first contribution in this sense, and Kopel, Lamantia, and Szidarovszky, 2014, G. I. Bischi, Lamantia, and Radi, 2015, Cerboni Baiardi, Lamantia, and Radi, 2015 and, also, Radi, 2017 among others. Remarkably, endogenous fluctuations and evolutionary stable heterogeneous configurations, where different behaviors coexist along complex dynamics, are often observed. Relevance of complex dynamics in economics have also been highlighted in Anufriev, Radi, and Tramontana, 2018.

The presence of heterogeneous decision mechanisms among agents is empirically founded in Cournot competitions. Indeed, both the best response and imitation behaviors are typically detected in experiments. In detail, the partial emergence of choices matching the Cournot-Nash equilibrium in experiments gives indirect indications for the presence of a component of the best response behavior (see, for example, Huck, Normann, and Oechssler, 1999, Offerman, Potters, and Sonnemans, 2002 and Bigoni and Fort, 2013). Experiments reveal also that Cournot competitions attain fairly high competitive levels. For example, in Huck et al., 1999 (see also Offerman et al., 2002 and Bigoni and Fort, 2013), the authors show that players boost more competitive outcomes provided they are supplied by information on disaggregated quantities and related profits of competitors. Fairly high competitive levels are also observed in the experimental Cournot oligopolies considered in Apesteguia, Huck, and Oechssler, 2007, Huck, Normann, and Oechssler, 2004, Apesteguia, Huck, Oechssler, and Weidenholzer, 2010, Oechssler, Roomets, and Roth, 2016, Friedman, Huck, Oprea, and Weidenholzer, 2015. Such a circumstance provides indication about the presence of a relevant component of imitative behavior, an interpretation which is supported by the theoretical contribution by Vega-Redondo, 1997, where the author shows that the competitive equilibrium emerges under the so called "imitate the best" rule, exploited by quantity setting agents. A further experimental fact was evidenced in Friedman et al., 2015, where the authors show that when players are allowed to compete for an extended time period, the oligopoly is brought towards the Cournot-Nash equilibrium, after reaching high competitive levels. As the authors pointed out, such an empirical fact can be interpreted in the light of a learning process along which decision mechanisms are adapted over time based on past experience.

In the present work we consider the above mentioned empirical facts and we consider a Cournot model formulated in a linear environment where competitors set their outputs adopting, alternatively, the proportional imitation rule introduced in Cerboni Baiardi and Naimzada, 2018a (and similar to the one considered in Cerboni Baiardi and Naimzada, 2017, Cerboni Baiardi and Naimzada, 2018c, Cerboni Baiardi and Naimzada, 2018b and Cerboni Baiardi and Naimzada, 2019) and the best response rule with perfect foresight exploited at a fixed implementation cost. The evolutionary part of the model accounts for the changing propensity of each agent to adopt one rule over the other, based on his evaluation of past performances coming from each decision mechanism and aims to represent learning processes taking place during the competition.

The model we consider summarizes the dynamics of a population of N agents by means of a two dimensional discrete time dynamical system. The resulting map is characterized by a stationary state that represents heterogeneous population where both the best responders and imitators coexist and produce at the Cournot-Nash level.

The local analysis reveals that the above mentioned stationary state can undergo flip bifurcation. In particular, variations in the number of players have an ambiguous role and a double stability threshold may be observed. Such an event is quite unexpected since, from the Theocharis' result provided in

Theocharis, 1960, most of the literature concerning oligopoly competition highlights the destabilizing role of the number of players. Similarly, we show that the implementation costs influence the stability of the stationary state and, even in this case, a double stability threshold is found. Again, this occurrence represents a remarkable circumstance since implementation costs in evolutionary models usually have only destabilizing effects (see Hommes, 2013, Brock and Hommes, 1997). Both phenomena originate from the particular form of the imitative behavior we consider. Remarkably, such occurrences have also been found in Cerboni Baiardi and Naimzada, 2018a, where imitators and best responders with static expectations have been considered in an evolutionary framework. Due to the simplified mathematical structure of the model here proposed, we show that the stability retrieval of the stationary state can be explained observing that the imitative behavior reduces, under suitable conditions, to an adaptive process where imitators' outputs are adjusted towards their static expectation best response, thus making the equilibrium stable. Differently, the intensity of choice has a destabilizing role, a circumstance which is in line with most of the literature involving evolutionary selection mechanisms as introduced in Brock and Hommes, 1997 (see also Hofbauer and Sigmund, 2003).

The global analysis, mainly performed through numerical simulations, highlights the effects of parameters' variations on the dynamical complexities of trajectories, on the shape of basins of attractions of the existing attracting sets and on the presence of multiple attractors. In particular, the variations in the number of players may make the multi-stability rise. Moreover, increasing values of the intensity of choice determine increasing dynamic complexities for the internal attractor, where different behaviors coexist in the long run. This occurrence comes along with the reduction of the related basin of attraction. From an interpretative point of view, this means that the probability to observe feasible trajectories that converge towards the internal attractor are decreased at increasing values of the intensity of choice.

We point out that the model here formulated mainly differs from those considered in Cerboni Baiardi and Naimzada, 2017, Cerboni Baiardi and Naimzada, 2018c, Cerboni Baiardi and Naimzada, 2018b and Cerboni Baiardi and Naimzada, 2019 because it allows players to change their decision mechanism by means of an evolutionary mechanism, thus making the share of imitators in the population a dynamic variable. In Cerboni Baiardi and Naimzada, 2018a, the authors formulate a model that follows the same evolutionary framework here considered, pairing imitators and best responders with static expectations, thus obtaining a three dimensional discrete time map to represent the population dynamics. Here, we strengthen the latter assumption pairing imitators together with perfect foresight best responders. As a consequence, the Cournot competition is represented by means of a two dimensional discrete time map characterized by a simplified mathematical structure. By doing so it is possible to better understand dynamical scenarios arising from the model and to provide further interpretations on parameters' roles.

The paper is organized as follows. In Section 2 the model is formulated. In Section 3 the stationary states of the model and the related stability features are highlighted in analytic form, when possible. In Section 4, global dynamic scenarios that the model describes are discussed. Section 5 concludes.

2. The model

We consider a Cournot oligopoly where a population of $N \geq 2$ agents compete producing homogeneous goods bearing the same constant marginal production cost c. We will denote by q_k the output of the generic k-th agent and by Q the total supply by all agents, which can be expressed by $Q = \sum_{k=1}^{N} q_k$. We further assume that the oligopoly is characterized by a linear inverse demand function, namely $P(Q) = \max\{a - bQ, 0\}$. The k-th agent profits result

$$\pi_k = P(Q)q_k - cq_k \tag{2.1}$$

From a theoretical perspective, the set of N players, with strategies given by positive productions $q_k \geq 0$, with $k = 1, ..., N$, and whose utilities match their profits, defines a game characterized by the Cournot-Nash equilibrium where each player produces at the level

$$q^* = \frac{a - c}{b(N+1)}$$

Noteworthy, the Cournot-Nash notion of equilibrium is very demanding in terms of rationality and information set owned by players. Because of this, we will consider the possibility for agents to adopt strategies coming from decision mechanisms with less burden requirements. The burden of a rule is related to the costs needed to its implementation. As a result, the performance of an output will be measured through profits, which that specific output has brought, diminished by implementation costs. Then, the performance of agent k will be given by $U_k = \pi_k - C_k$, where $C_k \geq 0$ accounts for the per-period implementation costs associated to the rule adopted by agent k.

In the present paper we consider that players can follow either the best response rule having perfect foresight or the proportional imitation rule introduced in Cerboni Baiardi and Naimzada, 2018a. The best response rule with perfect foresight requires high computational abilities and the hold of relevant information set, which includes the players' profit structure and the market demand. Hence, we assume that such rule is implemented at a constant average per period cost C. On the other hand, the imitation heuristic requires the knowledge of the previous period outputs together with resulting performances. Then it is reasonable to assume that implementation costs associated to the imitative rule are significantly lower than those required to implement the perfect foresight best response. Therefore, we set the imitation heuristic free from implementation costs. From this it follows that

$$U_k = \pi_k - C_k, \quad \text{where } C_k = \begin{cases} C & \text{if } k \text{ is a best responder} \\ 0 & \text{if } k \text{ is an imitator} \end{cases} \tag{2.2}$$

Remark 1*.* The implementation cost C is assumed to be small enough to ensure positive performances at the Cournot-Nash equilibrium. Hence, the upper bound of C will not exceed the profit π^* earned by players when producing at the Cournot-Nash level, namely

$$C < \pi^* := \frac{1}{b}\left(\frac{a-c}{N+1}\right)^2 \tag{2.3}$$

The model is developed in a discrete time framework where each player, at the beginning of each period, chooses which decision mechanism to exploit and, accordingly, he sets his output. If an agent produces following the best response rule having perfect foresight, he adapts immediately and optimally his production to his competitors' aggregate production. Then, at time $t+1$, the generic i-th best responder sets the quantity

$$q_i(t+1) := \arg\max_q \pi_i(q + Q_{-i}(t+1)) = \frac{a - c - bQ_{-i}(t+1)}{2b} \tag{2.4}$$

where $Q_{-i}(t+1)$ denotes the quantity produced by i's competitors at time $t+1$. Otherwise, if an agent adopts the proportional imitation rule, he chooses his production at the weighted average of the previous period outputs. Weights are given by the relative performances coming from the previous period choices. Then, at time $t+1$, the generic j-th imitator sets the quantity

$$q_j(t+1) = \frac{\sum_{l\in\mathcal{A}(t)} U_l q_l}{\sum_{l\in\mathcal{A}(t)} U_l} \tag{2.5}$$

where $\mathcal{A}(t)$ includes outputs in the market at time period t, that is

$$\mathcal{A}(t) := \{q \in \mathbb{R}_+ : \exists\, n \in \mathbb{N} \text{ s.t. } q = q_n(t)\}$$

The rule (2.5) considers that imitators are aware of the presence of strategic interactions and that an action that brought high performances (or the highest performance) in the previous period may not produce so good a result in the present time because of changes in environmental conditions. Because of the indeterminacy of actions' performances, imitators tackle the problem of whom to imitate by considering all previous period choices through the rule 2.5, which can be interpreted as a prudent imitative behavior.

Remark 2. Differently from the proportional imitation rule considered in Cerboni Baiardi and Naimzada, 2017 (and also in Cerboni Baiardi and Naimzada, 2018b and Cerboni Baiardi and Naimzada, 2019), the rule 2.5 accounts for performances as weights instead of profits. This reflects the implicit assumption according to which imitators do not distinguish between competitors exploiting different decision mechanisms, having only access to information on quantities adopted by competitors and their related outcomes, which correspond to performances in the present formulation.

The imitation rule here considered does not provide any performance based selection. This marks the main difference from the rules considered by Vega-Redondo, 1997 and by Schlag, 1998, where players make a careless and incautious selection among the quantities to imitate. Indeed, according to the rule defined by Vega-Redondo, 1997 every player refuses to imitate every strategy that did not produce the best result, while, according to the rule considered by Schlag, 1998, every player refuses to imitate every strategy that has produced worse outcomes than his own.

We reduce the complexities of the model by assuming that players exploiting the same rule are characterized by the same initial conditions. This implies that the outputs of those players using the same rule are equal among them also in subsequent periods. Then, at time t, the best responders' outputs equal the single value $q_1(t)$. The resulting performances coming from using the best response will be denoted by $U_1(t)$. Analogously, the imitators' outputs equal the single value $q_2(t)$ and the performances that result from using the proportional imitation rule will be denoted by $U_2(t)$. The splitting of the population between best responders and imitators can be described by introducing the new variable $\omega(t) \in (0, 1)$ that represents the share of imitators in the population at time t.

From the previous arguments, recurrences (2.4) reduce to the following one-dimensional correspondence

$$q_1(t) := G(q_2(t)) = \frac{a - c - bN\omega(t)q_2(t)}{b(N(1-\omega(t))+1)} \tag{2.6}$$

Equation (2.6) reports the strategy of best responders with perfect foresight and the strategy of the representative imitator agent. Under the same assumptions, recurrences (2.5) reduce to the following one-dimensional map

$$q_2(t+1) = \frac{U_1(t)}{U_1(t)+U_2(t)}q_1(t) + \frac{U_2(t)}{U_1(t)+U_2(t)}q_2(t) \tag{2.7}$$

where

$$U_1(t) = (\max\{0,\ a - bN((1-\omega(t))G(q_2(t)) + \omega(t)q_2(t))\} - c)\, G(q_2(t)) - C$$
$$U_2(t) = (\max\{0,\ a - bN((1-\omega(t))G(q_2(t)) + \omega(t)q_2(t))\} - c)\, q_2(t)$$

The evolutionary part of the model describes how the propensity to adopt a decision mechanism changes over time. Along the line marked by Brock and Hommes, 1997 (see also Hommes, 2013), we consider that the rule with better performance will attract more followers. Then, the fraction $\omega(t)$ of imitators evolves along with the exponential replicator dynamics

$$\omega(t+1) = \frac{e^{\beta U_2(t)}}{e^{\beta U_2(t)} + e^{\beta U_1(t)}} \tag{2.8}$$

Parameter β is the so called "intensity of choice", which measures how sensitive the players are for selecting the previous-time best performer decision mechanism. In the extreme case $\beta = 0$, differences in performances cannot be observed and the fraction of imitators remains fixed over time at the value $1/2$. In the other extreme case $\beta = \infty$, performances are perfectly observed and, in each period, all agents choose the previous-time decision rule which carried out the best performance.

The oligopoly dynamics is driven by a discrete time correspondence

$$(q_2(t), \omega(t)) \xrightarrow{T} (q_2(t+1), \omega(t+1))$$

where T stands for a two dimensional map defined by recurrences (2.7) and (2.8), each of which describes time advancements of imitators' output and their share in the population. In the next Sections, we will provide both local and global analysis of the model limited to those dynamic scenarios along which the price remains positive.

3. LOCAL ANALYSIS

Stationary states of the model are provided in the following proposition.

Proposition 3. The dynamical system defined by map T is characterized by the stationary state E^*

(3.1a) $$E^* = (q^*, \omega^*) \quad \text{where} \quad q_1^* = q_2^* = q^* = \frac{a-c}{b(N+1)}, \quad \omega^* = \frac{1}{1+e^{-\beta C}}$$

In addition, provided that condition (2.3) is met, the dynamical system has a second stationary state E^0 given by

(3.2a)

$$E^0 = (q^0, \omega^0) \text{ where } q_1^0 = \sqrt{C/b},\ q_2^0 = q^0 = \frac{a-c}{bN\omega^0} - \sqrt{\frac{C}{b}}\frac{N(1-\omega^0)+1}{N\omega^0},$$

Here ω^0 is the unique solution in the interval $(0,1)$ of the equation $f(\omega) = 0$ with

$$f(\omega) = \frac{a-c}{bN\omega} - \sqrt{\frac{C}{b}}\frac{N(1-\omega)+1}{N\omega} + \frac{1}{\beta\sqrt{Cb}}\log\left(\frac{1}{\omega}-1\right)$$

Proof. See Appendix 6.1 □

Remark 4. The stationary state E^*, where the production level of best responders equals the imitators' one, is consistent with the Cournot-Nash equilibrium. Moreover, the equilibrium share of imitators is an increasing function with respect to the evolutionary parameter β. Indeed, increasing values of β enhances the capacity of players to distinguish the differences in performances from each rule, thus strengthening the incentive to adopt the imitation rule leading to higher performances at the Cournot-Nash level. Similarly, the equilibrium share of imitators grows along with C. Indeed, increasing C determines the decreasing incentive to adopt the best response rule due to high implementation costs.

We remark also that the stationary state E^0 is consistent with the Walrasian equilibrium in the special case in which $C \to 0$ since, in this limit, players produce at the market clearing price. Indeed, denoting the aggregate production at E^0 by Q^0, it results

$$\lim_{C\to 0} P(Q^0) = \lim_{C\to 0} \max\{0,\ a - bN((1-\omega^0)q_1^0 + \omega^0 q_2^0)\} = c$$

Remark 5. If profits were used instead of performances in the imitation rule (2.5), the first stationary states E^* of the model would still be the same, whereas the second stationary state E^0 would change into the following

$$E^0_\pi = (q^0_{2,\pi}, \omega^0_\pi), \quad q^0_{2,\pi} = \frac{a-c}{bN\omega^0_\pi}\ , \quad \omega^0_\pi = \frac{1}{1+e^{-\beta C}}$$

In addition, at E^0_π, best responders set vanishing productions (namely $q^0_{1,\pi} = 0$). It results that E^0_π can be interpreted as the Walrasian equilibrium at which players produce at the market clearing price. Indeed, denoting the aggregate production at E^0_π by Q^0_π, it results

$$P(Q^0_\pi) = \max\{0,\ a - bN((1-\omega^0)q^0_{1,\pi} + \omega^0_\pi q^0_{2,\pi})\} = c$$

for any value of $C \geq 0$.

The following proposition outlines the stability property of the stationary state E^*.

Proposition 6. The stationery state E^* is locally asymptotically stable provided

$$\omega^* < \omega_f \tag{3.3}$$

where $\omega^* = \dfrac{1}{1+e^{-\beta C}}$ and $\omega_f := \dfrac{N+1}{2N}\dfrac{3\pi^*-C}{2\pi^*-C}$. At $\omega^* = \omega_f$ the fixed point E^* undergoes flip bifurcation.

Proof. See Appendix 6.2. □

Remark 7. Analytical evaluations of the stability property of E^0 cannot be given since the fraction of imitators at E^0 cannot be obtained in analytic form. However, the stationary state E^0 is never found to be stable in numerical simulations performed within a wide range of parameters' values.

In the following, we consider the role of the relevant parameters of the model in determining the stability properties of E^*, that is the number of players, the implementation costs and the intensity of choice.

The stability condition (3.3) can be exploited in order to highlight the role of parameter N. We first note that the above mentioned condition is met at extreme values of N, that is when $N = 2$ and as N approaches the maximum value $N_{\max} := (a-c)/\sqrt{Cb} - 1$ at which the relation (2.3) is no more satisfied. Indeed, in both cases, it is easy to verify that $\omega_f > 1$. On the other hand, the same condition may not be satisfied at intermediate values of N and a double stability threshold can be observed (see bifurcation diagrams in figure 1). Figure 3, left panel, shows that the role of N depends on the values of β. In detail, if the intensity of choice β is small enough, the number of players does not affect the stability of E^*. On the other hand, if the value of β is sufficiently high, a first increase of N's values leads to the loss of stability of E^*, while further increases of the same parameter leads to its stability retrieval. We note that the presence of the double stability threshold at increasing values of N has to be referred to the specific imitative behavior we consider and it represents a remarkable circumstance since, from the Theocharis' result provided in Theocharis, 1960, most of the literature concerning oligopoly competition shows the destabilizing role of the number of players.

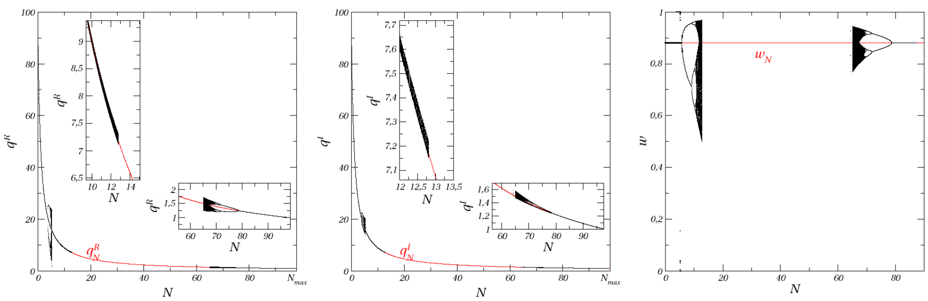

FIGURE 1. Bifurcation diagrams as N varies of q_1 (left), q_2 (center) and ω (right). Parameters are: $a = 100$, $b = c = C = 1$ and $\beta = 2$.

A double stability threshold may also appear at increasing values of C. Indeed, the condition (3.3) is met both at extreme values of C, that is at $C = 0$ and for values of C that approach its the maximum value $C_{\max} := \pi^*$ at which the relation 2.3 is no more satisfied. On the contrary, the same condition may not be satisfied for intermediate values of implementation cost C (see bifurcation diagrams in figure 2[1]). In addition, figure 3, right panel, shows that the destabilizing role of C is conditioned by the values of β. The presence of a double stability threshold is a remarkable circumstance since, usually, implementation costs in evolutionary models have a destabilizing effect (see Hommes, 2013, Brock and Hommes, 1997).

Remark 8. If, in the imitation rule 2.7, profits were used as weights instead of performances, the stability condition of E^* provided in Proposition 6 would boil down to

$$\omega^* < \frac{3}{4}\frac{N+1}{N}$$

[1]We note that in figure 2 left panel, points of the phase space are visited along stable periodic cycles at which the share of imitators vanishes. At those points, only perfect foresight best responders are present, who produce at the Cournot-Nash level even after the stability loss of E^*.

In this case, the stability properties of the stationary state E^* would be determined by parameters β, C and N only, all of which have a destabilizing role. It follows that the double stability threshold that may be observed when N or C vary would disappear.

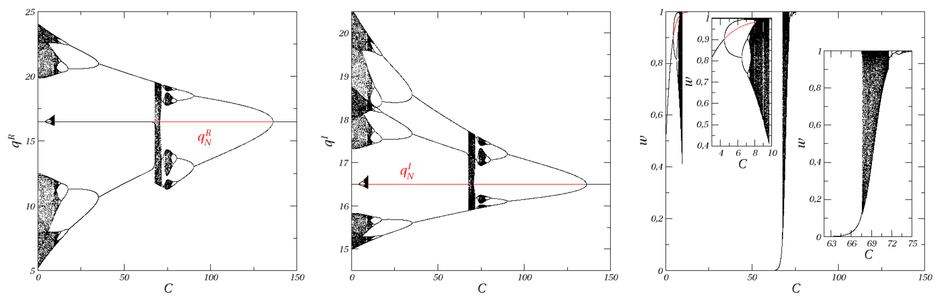

FIGURE 2. Bifurcation diagrams of q_1 (left), q_2 (center) and ω (right) as C varies. Parameters are $N = 5$, $a = 100$, $b = c = 1$ and $\beta = 0.5$.

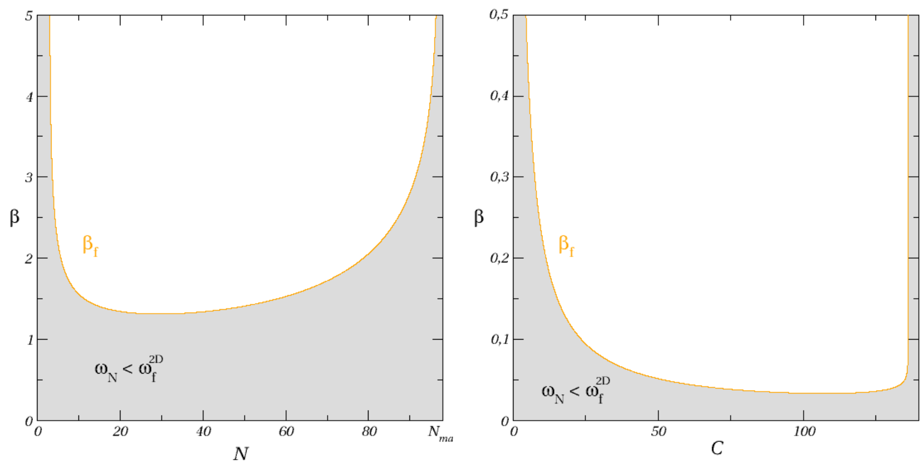

FIGURE 3. Left. Stability regions of E^* in the $N - \beta$ plane (left) at $C = 1$. Right. Stability region in the $C - \beta$ plane at $N = 5$. Other parameters are $a = 100$, $b = c = 1$.

The stability recovery of the stationary state E^*, that occurs as the number of players N or the implementation cost C attain sufficiently high values, is due to the fact that, in these cases, imitators adjust their outputs towards their best response with static expectation, regardless of their share in the population. In detail, taking into account Assumption 2.3, if N approaches the maximum value $N_{\max}$ or if implementation costs C approach the maximum value $C_{\max}$, any single time step adjustment of imitators' output can be approximated by

$$\begin{aligned} q_2(t+1) &\sim \gamma_\epsilon G(q_2(t)) + (1-\gamma_\epsilon)\, q_2(t) \\ &= q_2(t) + \gamma_\epsilon \left(G(q_2(t)) - q_2(t)\right) \end{aligned} \tag{3.4}$$

provided $q_2(t)$ is placed in a suitable neighborhood of the Cournot-Nash production level q^* with radius $\epsilon > 0$. In relation (3.4), γ_ϵ is a positive and small reaction parameter, which represents the speed of velocity with which the quantity $q_2(t)$ is adjusted towards $G(q_2(t))$ (see Appendix 6.3 for more details). The imitators' behavior as described by relation (3.4) can be justified considering that, in the limit $N \to N_{\max}$ or $C \to C_{\max}$, best responders realize small performances with respect to the imitators' ones. Then, when computing their next period production through the weighted average (2.7), imitators assign much less importance to best responders' outputs with respect to their own past choices. Under this point of view, γ_ϵ can be interpreted as the weight that imitators give to the best responders output, while $1-\gamma_\epsilon$ is meant to be the weight given to their own previous period outputs.

The role of the intensity of choice β can be highlighted by rewriting the stability condition (3.3) in order to provide the explicit expression of the threshold value β_f of β, at which the equilibrium point E^* turns to an unstable saddle. Indeed E^* is locally asymptotically stable if

$$\beta < \beta_f := -\frac{1}{C}\log\left(\frac{1}{\omega_f} - 1\right) \tag{3.5}$$

At $\beta = \beta_f$ flip bifurcation occurs. Relation (3.5) clearly highlights the destabilizing role of the intensity of choice (see also the bifurcation diagrams in figures 3). Such a finding is somehow expected and in line with existing literature (see e.g. Brock and Hommes, 1997 and Hommes, 2013). The interpretation is that small values of β result in an important level of inertia of players when choosing the most profitable decision mechanism. Differently, at high values of the intensity of choice, players react to small differences in performances coming from each rule, thus having a high propensity to change their behavior and with the effect of destabilizing the system.

4. Global analysis

The global analysis reveals further interesting dynamic phenomena which cannot be deduced by means of the local stability analysis provided in the previous Section. To this purpose we provide several numerical simulations where

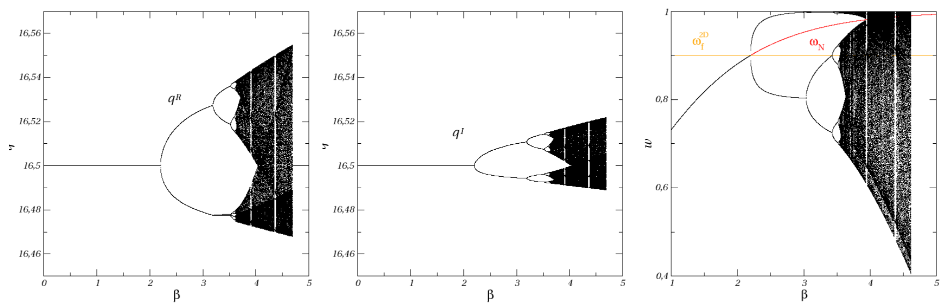

FIGURE 4. Bifurcation diagrams varying β of ω (left), q_2 (center) and q_1 (right) at $N = 5$, $a = 100$, $b = c = C = 1$.

the static game' parameters a, b and c are kept fixed at the values $a = 100$, $b = c = 1$[2]. We first note that the vertical line $\mathcal{L} := \{(q^*, \omega) : \omega \in (0, 1)\}$, where q^* is the production at the Cournot-Nash level, is invariant under the action of map T, that is $T(\mathcal{L}) \subset \mathcal{L}$. Then, initial conditions lying on $\mathcal{L}$ are mapped towards $E^* \in \mathcal{L}$ in one step. This implies that line $\mathcal{L}$ is included in the basin of attraction of E^*. Examples of this occurrence are provided in figure 5, where the invariant line $\mathcal{L}$ is shown. In particular, in the left panel, a dynamic scenario is shown, where $\beta = 1$, $C = 1$ and $N = 12$, at which the stationary state E^* is a stable internal attractor representing a heterogeneous population in which best response and imitative rules coexist. The grey points of the phase space represent the basin of attraction of E^*, while the orange points represent the basin of attraction of unfeasible trajectories. Differently, in the right panel of the same figure, a new scenario is obtained at increased value of $\beta = 2$ beyond the threshold value β_f. The stationary state E^* has lost its stability through the flip bifurcation and an internal chaotic attractor, arising after the usual period doubling cascade, is represented. Its basin of attraction is represented by the green points. As usual, the orange points represent the basin of attraction of unfeasible trajectories. The comparison between the two scenarios provided in figure 5 reveals that increasing values of β produce increasing dynamic complexities in feasible trajectories. Noteworthy, when E^* is unstable, outputs of best responders and imitator players slightly differ from the Cournot-Nash level and feasible trajectories take place in the neighborhood of the invariant line $\mathcal{L}$.

[2]Fundamental parameters are fixed at the values usually adopted within experimental oligopolies with an underlying linear structure (see e.g. Huck et al., 1999 or Oechssler et al., 2016 among others.)

As a consequence, most of the dynamics of the system is due to wide variations of the relative fraction of imitators, taking place above the floor $1/2$. Then, the loss of stability of E^* can be thought as a transition from an initial scenario, where each player produces following a given rule in time, towards a new scenario where players produce at the same level alternating the adoption of either rule or the other. Moreover, the shape of the basins highlights that the possibility for non-diverging dynamics to occur is influenced by initial conditions. Indeed, feasible trajectories are as likely to be observed as initial productions are closer to the Cournot-Nash level. Also, if the initial conditions sufficiently approach the Cournot-Nash production, the convergence towards the internal attractor occurs regardless of the fraction of imitators. At the same time, if the fraction of best responders increases, the initial deviation from the Cournot-Nash production, which still leads to non-divergent paths, grows larger.

Other interesting dynamic scenarios, obtained by keeping $C = 1$ and $N = 5$ fixed while β is increasing ($\beta = 0.162, 1, 2$), are provided in figures 6. With the mentioned values of the parameters, the stationary state E^* is stable and it is represented together with its basin of attraction, denoted by the grey points of the phase space. E^* coexists with a three band chaotic attractor placed at the extreme values of ω, whose basin is denoted by the yellow points.

Such chaotic trajectories describe a scenario where all the players simultaneously switch their behavior from the best response to the imitation rule. The sequence in figure 6 shows that the intensity of choice influences the shape of the basins of attraction (an effect which can also be observed, even if with a lower importance, by comparing the two scenarios in figure 5). Indeed, increasing values of β shrink the basin of the internal attractor E^* around the invariant line $\mathcal{L}$, which makes the basin of chaotic trajectories widen. Then, increasing values of β strengthen the phenomenon according to which the convergence towards the equilibrium occurs provided initial outputs are closer and closer to the invariant line $\mathcal{L}$ as the number of imitators' fraction increases. We finally mention that the scenarios depicted in figures 5 and 6 are distinguished by different values of N, namely $N = 12$ and $N = 5$ respectively. Hence, besides the counter-intuitive effect according to which increasing values in the number of players involved in the competition may give rise to a double stability threshold (as noted in Section 2), variations of N may also produce the rise of multi-stability scenarios.

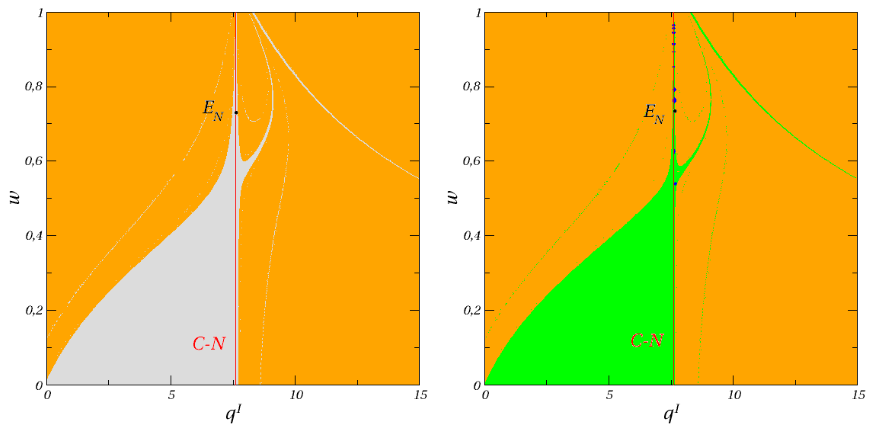

FIGURE 5. Basins of attractions of internal attractors (grey and green points) and of divergent trajectories (orange points). Left. Stable E^* at $\beta = 1$. Right. Chaotic attractor at $\beta = 2$. Other parameters are $N = 12$, $a = 100$ and $b = c = C = 1$.

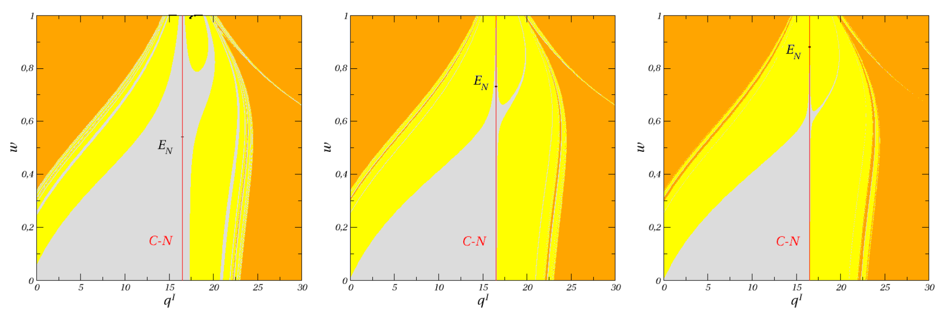

FIGURE 6. Basins of attractions of E^* (grey points), of a three band chaotic attractor placed at extreme values of ω (yellow points) and of unfeasible trajectories (orange points). Left. $\beta = 0.162$. Center. $\beta = 1$. Right. $\beta = 2$. Other parameters are $N = 5$, $a = 100$ and $b = c = C = 1$.

5. CONCLUSION

The dynamics of a population of N quantity setting players that compete in a linear oligopoly framework, iterating their decisions in discrete time periods, is here considered. Players exploit, alternatively, the perfect foresight best response rule or an imitative rule in order to set their outputs. The choice of which decision mechanism to exploit is driven by an evolutionary process according to which the rule that brought the highest performance in the previous time period attracts more followers. The model we formulate is characterized by a relevant stationary state where agents set their outputs at the Cournot-Nash production level, which represents a heterogeneous population where rational and imitative rules coexist. We found that the number of players involved in the competition, considered as a parameter, has an ambiguous role in influencing the stability property of the stationary state and a double stability threshold may be observed. This represents a circumstance in contrast with most of the literature concerning oligopoly competition, which has to be referred to the specific imitative behavior we consider. A similar occurrence can be found when implementation costs, needed to exploit the best response rule, become heavier. Again, this circumstance should be referred to the presence of the imitation heuristic and marks the difference with most part of the literature on evolutionary competition (see Brock and Hommes, 1997 or Hommes, 2013), where implementation costs have an unambiguous destabilizing role. On the contrary, the intensity of choice, which represents the evolutionary propensity to switch to the most profitable rule, is confirmed to be destabilizing, in line with acknowledged evolutionary models (see e.g. Hofbauer and Sigmund, 2003). The global analysis of the model, performed through numerical simulations, reveals that the number of players may also produce the rise of multi-stability. It also shows that the intensity of choice increases the dynamical complexities of feasible trajectories and influences the shape of basins of attraction. Indeed, increasing values of β may produce period doubling cascades and the rise of chaotic dynamics. In addition, increasing values of intensity of choice shrink the basin of the internal attractor, which determines the rise of complex trajectories along which players produce at the Cournot-Nash and, at the same time, undergo cyclic variations in their own behavioral rule.

6. Appendix

6.1. **Proof of Proposition 3.**

Stationary states of (2.7) satisfy the equation

$$(q_2 - q_1)(\pi_1 - C) = 0$$

If $q_1 = q_2 := q^*$, relation (2.6) implies

$$q^* = \frac{a-c}{b(N+1)}$$

By replacing this expression in recurrence (2.8), the stationary fraction of imitators results

$$\omega^* = \frac{1}{1+e^{-\beta C}}$$

Instead, if $\pi_1 - C = 0$, the explicit expression of π_1 and the correspondence (2.6) leads to

$$q_1^0 = \sqrt{Cb}, \quad q_2^0 = \frac{a-c}{bN\omega^0} - \sqrt{\frac{C}{b}} \frac{N(1-\omega^0)+1}{N\omega^0} \tag{6.1}$$

where ω_0 is the solution of the equation

$$f(\omega) = \frac{a-c}{bN\omega} - \sqrt{\frac{C}{b}} \frac{N(1-\omega)+1}{N\omega} + \frac{1}{\beta\sqrt{Cb}} \log\left(\frac{1}{\omega} - 1\right)$$

Provided that condition (2.3) holds, that is

$$\frac{1}{b}\left(\frac{a-c}{N+1}\right)^2 > C, \tag{6.2}$$

it can be showed that $f(\omega)$ vanishes at a unique within $(0,1)$. Indeed, $f(\omega)$ is a monotonically deceasing function

$$\frac{dG}{d\omega} = -\frac{N+1}{N\omega^2}\left(\frac{a-c}{b(N+1)} - \sqrt{\frac{C}{b}}\right) - \frac{1}{\omega(1-\omega)} \frac{1}{\beta\sqrt{Cb}} < 0$$

In addition, the extremes of $f(\omega)$ are of opposite signs at the extremes of the interval $(0,1)$:

$$\lim_{\omega\to 0^+} f(\omega) = \frac{N+1}{N\omega}\left(\frac{a-c}{b(N+1)} - \sqrt{\frac{C}{b}}\right) + \frac{1}{\beta\sqrt{Cb}} \log\left(\frac{1}{\omega} - 1\right) + \sqrt{\frac{C}{b}} = +\infty$$

$$\lim_{\omega\to 1^-} f(\omega) = \frac{N+1}{N\omega}\left(\frac{a-c}{b(N+1)} - \sqrt{\frac{C}{b}}\right) + \frac{1}{\beta\sqrt{Cb}} \log\left(\frac{1}{\omega} - 1\right) + \sqrt{\frac{C}{b}} = -\infty$$

6.2. **Proof of Proposition 6.** The eigenvalues λ_1 and λ_2 of the Jacobian matrix of map (2.6) computed at E^* are root of the characteristic polynomial $P(\lambda)$ given by

$$P(\lambda) = -\lambda\left(\frac{1}{2\pi_* - C}\left(\pi_* - \frac{N\omega^*(\pi_* - C)}{N(1-\omega^*)+1}\right) - \lambda\right) \tag{6.3}$$

The first root is $\lambda_1 = 0$ while the second λ_2 is given by

(6.4)
$$\lambda_2 = \frac{1}{2\pi_* - C}\left(\pi_* - \frac{N\omega^*(\pi_* - C)}{N(1-\omega^*)+1}\right) = \frac{\pi(N(1-2\omega^*)+1) + NC\omega^*}{(2\pi - C)(N(1-\omega^*)+1)}$$

It is easy to see that $\lambda_2 < 1$. Thus, the Cournot-Nash equilibrium E^* is stable provided that $\lambda_2 > -1$, which is satisfied if and only if

$$\omega^* < \frac{N+1}{N} \frac{3\pi_* - C}{2(2\pi_* - C)} \tag{6.5}$$

6.3. **Derivation of formula (3.4).** If N approaches its maximum value $N_{\max}$, the profit π^* at the Cournot-Nash level approaches the implementation cost C. Then, in this limit, it is $C = \pi^* + \epsilon_N$, for some suitably small $\epsilon_N > 0$. Similarly, if the implementation cost C approaches its maximum value $C_{\max} = \pi^*$, then $C = \pi^* - \epsilon_C$, for some suitably small $\epsilon_C > 0$. Denoting by ϵ_0 alternatively ϵ_N or $-\epsilon_C$, the dynamics of imitators can be rewritten as

$$q_2(t+1) = \frac{\pi_1 - (\pi^* + \epsilon_0)}{\pi_1 + \pi_2 - (\pi^* + \epsilon_0)} G(q_2(t)) + \frac{\pi_2}{\pi_1 + \pi_2 - (\pi^* + \epsilon_0)} q_2(t) \tag{6.6}$$

Suppose that, at period t, it results

$$q_2(t) < q^*$$

such that $q_2(t) = q^* - \epsilon$ for some suitable $\epsilon > 0$. On the one hand this implies

$$q_1(t) = G(q_2(t)) = q^* + \frac{N\omega\epsilon}{N(1-\omega)+1} > q^*$$

In addition, the first order approximation of best responders and imitators' profits are respectively

$$\pi_1 \sim \pi^* + 2bq^*k\epsilon$$
$$\pi_2 \sim \pi^* + bq^*(k-1)\epsilon$$

where we used the shortcut $k = N\omega/(N(1-\omega)+1)$. Recurrence 6.6 can be approximated to the first order as follows

$$\begin{aligned}
q_2(t+1) &= \frac{\pi_1 - (\pi^* + \epsilon_0)}{\pi_1 + \pi_2 - (\pi^* + \epsilon_0)} G(q_2(t)) + \frac{\pi_2}{\pi_1 + \pi_2 - (\pi^* + \epsilon_0)} q_2(t) \\
&\sim \frac{2bq^*k\epsilon + \epsilon_0}{\pi^* + bq^*(3k-1)\epsilon + \epsilon_0} G(q_2) + \frac{\pi^* + bq^*(k-1)\epsilon}{\pi^* + bq^*(3k-1)\epsilon + \epsilon_0} q_2(t) \\
&\sim q_2(t) + \frac{2bq^*k}{\pi^*}\epsilon \left(G(q_2(t)) - q_2(t)\right) + \frac{\epsilon_0}{\pi^*}\left(G(q_2(t)) - q_2(t)\right) \\
&= q_2(t) + \left(\frac{2bq^*k}{\pi^*}\epsilon + \frac{1}{\pi^*}\epsilon_0\right)\left(G(q_2(t)) - q_2(t)\right)
\end{aligned}$$

The speed of adjustment

$$\gamma_\epsilon = \frac{2bq^*k}{\pi^*}\epsilon + \frac{1}{\pi^*}\epsilon_0$$

is always positive provided that $\epsilon_0 = \epsilon_N$. Instead, if $\epsilon_0 = -\epsilon_C$, the parameter γ_ϵ remains positive provided that a sufficiently high value of N is selected such that $\pi^* - C = \epsilon_C < 2bq^*k\epsilon$. An analogous result is obtained assuming that, at period t, it results $q_2(t) > q^*$ with $q_2(t) = q^* + \epsilon$ for some suitable $\epsilon > 0$.

Acknowledgement. This work has been developed in the framework of the research project on "Dynamic Models for behavioural economics" financed by DESP, University of Urbino. The authors thank two anonymous referees for their useful comments.

References

Agiza, H., & Elsadany, A. (2003). Nonlinear dynamics in the cournot duopoly game with heterogeneous players. *Physica A: Statistical Mechanics and its Applications*, *320*, 512–524.

Agiza, H., & Elsadany, A. (2004). Chaotic dynamics in nonlinear duopoly game with heterogeneous players. *Applied Mathematics and Computation*, *149*(3), 843–860.

Agiza, H., Hegazi, A., & Elsadany, A. (2002). Complex dynamics and synchronization of a duopoly game with bounded rationality. *Mathematics and computers in Simulation*, *58*(2), 133–146.

Andaluz, J., Elsadany, A., & Jarne, G. (2017). Nonlinear cournot and bertrand-type dynamic triopoly with differentiated products and heterogeneous expectations. *Mathematics and Computers in Simulation*, *132*, 86–99.

Andaluz, J., & Jarne, G. (2015). On the dynamics of economic games based on product differentiation. *Mathematics and Computers in Simulation*, *113*, 16–27.

Angelini, N., Dieci, R., & Nardini, F. (2009). Bifurcation analysis of a dynamic duopoly model with heterogeneous costs and behavioural rules. *Mathematics and Computers in Simulation*, *79*(10), 3179–3196.

Anufriev, M., Radi, D., & Tramontana, F. (2018). Some reflections on past and future of nonlinear dynamics in economics and finance. *Decisions in Economics and Finance*, *41*(2), 91–118.

Apesteguia, J., Huck, S., & Oechssler, J. (2007). Imitation - theory and experimental evidence. *Journal of Economic Theory*, *136*(1), 217–235.

Apesteguia, J., Huck, S., Oechssler, J., & Weidenholzer, S. (2010). Imitation and the evolution of walrasian behavior: Theoretically fragile but behaviorally robust. *Journal of Economic Theory*, *145*(5), 1603–1617.

Askar, S. [Sameh]. (2014). The rise of complex phenomena in cournot duopoly games due to demand functions without inflection points. *Communications in Nonlinear Science and Numerical Simulation*, *19*(6), 1918–1925.

Askar, S. [SS]. (2014). Complex dynamic properties of cournot duopoly games with convex and log-concave demand function. *Operations research letters*, *42*(1), 85–90.

Bigoni, M., & Fort, M. (2013). Information and learning in oligopoly: An experiment. *Games and Economic Behavior*, *81*, 192–214.

Bischi, G.-I., Gallegati, M., & Naimzada, A. (1999). Symmetry-breaking bifurcations and representativefirm in dynamic duopoly games. *Annals of Operations Research*, *89*, 252–271.

Bischi, G. I., Lamantia, F., & Radi, D. (2015). An evolutionary cournot model with limited market knowledge. *Journal of Economic Behavior & Organization*, *116*, 219–238.

Bischi, G. I., & Naimzada, A. (2000). Global analysis of a dynamic duopoly game with bounded rationality. In *Advances in dynamic games and applications* (pp. 361–385). Springer.

Bischi, G. I., Naimzada, A. K., & Sbragia, L. (2007). Oligopoly games with local monopolistic approximation. *Journal of economic behavior & organization*, *62*(3), 371–388.

Bischi, G.-I. [Gian-Italo], Kopel, M., & Naimzada, A. (2001). On a rent-seeking game described by a non-invertible iterated map with denominator. *Nonlinear Analysis: Theory, Methods & Applications*, *47*(8), 5309–5324.

Brock, W. A., & Hommes, C. H. (1997). A rational route to randomness. *Econometrica: Journal of the Econometric Society*, 1059–1095.

Cavalli, F., & Naimzada, A. (2014). A cournot duopoly game with heterogeneous players: Nonlinear dynamics of the gradient rule versus local monopolistic approach. *Applied Mathematics and Computation*, *249*, 382–388.

Cavalli, F., & Naimzada, A. (2015). Nonlinear dynamics and convergence speed of heterogeneous cournot duopolies involving best response mechanisms with different degrees of rationality. *Nonlinear Dynamics*, *81*(1-2), 967–979.

Cavalli, F., Naimzada, A., & Tramontana, F. (2015). Nonlinear dynamics and global analysis of a heterogeneous cournot duopoly with a local monopolistic approach versus a gradient rule with endogenous reactivity. *Communications in Nonlinear Science and Numerical Simulation*, *23*(1), 245–262.

Cerboni Baiardi, L., Lamantia, F., & Radi, D. (2015). Evolutionary competition between boundedly rational behavioral rules in oligopoly games. *Chaos, Solitons & Fractals*, *79*, 204–225.

Cerboni Baiardi, L., & Naimzada, A. K. (2017). Experimental oligopolies modeling: A dynamic approach based on heterogeneous behaviors. *Communications in Nonlinear Science and Numerical Simulation*.

Cerboni Baiardi, L., & Naimzada, A. K. (2018a). An evolutionary model with best response and imitative rules. *Decisions in Economics and Finance*, *41*(2), 313–333.

Cerboni Baiardi, L., & Naimzada, A. K. (2018b). An oligopoly model with best response and imitation rules. *Applied Mathematics and Computation*, *336*, 193–205.

Cerboni Baiardi, L., & Naimzada, A. K. (2018c). Imitative and best response behaviors in a nonlinear cournotian setting. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *28*(5), 055913.

Cerboni Baiardi, L., & Naimzada, A. K. (2019). An oligopoly model with rational and imitation rules. *Mathematics and Computers in Simulation*, *156*, 254–278.

Den Haan, W. J. (2001). The importance of the number of different agents in a heterogeneous asset-pricing model. *Journal of Economic Dynamics and Control*, *25*(5), 721–746.

Droste, E., Hommes, C., & Tuinstra, J. (2002). Endogenous fluctuations under evolutionary pressure in cournot competition. *Games and Economic Behavior*, *40*(2), 232–269.

Fanti, L., Gori, L., & Sodini, M. (2015). Nonlinear dynamics in a cournot duopoly with isoelastic demand. *Mathematics and Computers in Simulation*, *108*, 129–143.

Friedman, D., Huck, S., Oprea, R., & Weidenholzer, S. (2015). From imitation to collusion: Long-run learning in a low-information environment. *Journal of Economic Theory*, *155*, 185–205.

Hofbauer, J., & Sigmund, K. (2003). Evolutionary game dynamics. *Bulletin of the American Mathematical Society*, *40*(4), 479–519.

Hommes, C. (2013). *Behavioral rationality and heterogeneous expectations in complex economic systems*. Cambridge University Press.

Huck, S., Normann, H.-T., & Oechssler, J. (1999). Learning in cournot oligopoly–an experiment. *The Economic Journal*, *109*(454), 80–95.

Huck, S., Normann, H.-T., & Oechssler, J. (2004). Two are few and four are many: Number effects in experimental oligopolies. *Journal of Economic Behavior & Organization*, *53*(4), 435–446.

Kopel, M., Lamantia, F., & Szidarovszky, F. (2014). Evolutionary competition in a mixed market with socially concerned firms. *Journal of Economic Dynamics and Control*, *48*, 394–409.

Leonard, D., & Nishimura, K. (1999). Nonlinear dynamics in the cournot model-without full information. *Annals of Operations Research*, *89*, 165–173.

Naimzada, A. K., & Sbragia, L. (2006). Oligopoly games with nonlinear demand and cost functions: Two boundedly rational adjustment processes. *Chaos, Solitons & Fractals*, *29*(3), 707–722.

Naimzada, A. K., & Tramontana, F. (2009). Controlling chaos through local knowledge. *Chaos, Solitons & Fractals*, *42*(4), 2439–2449.

Naimzada, A., & Ricchiuti, G. (2011). Monopoly with local knowledge of demand function. *Economic Modelling*, *28*(1), 299–307.

Naimzada, A., & Tramontana, F. (2015). Two different routes to complex dynamics in an heterogeneous triopoly game. *Journal of Difference Equations and Applications*, *21*(7), 553–563.

Oechssler, J., Roomets, A., & Roth, S. (2016). From imitation to collusion: A replication. *Journal of the Economic Science Association*, 1–9.

Offerman, T., Potters, J., & Sonnemans, J. (2002). Imitation and belief learning in an oligopoly experiment. *The Review of Economic Studies*, *69*(4), 973–997.

Pireddu, M. (2015). Chaotic dynamics in three dimensions: A topological proof for a triopoly game model. *Nonlinear Analysis: Real World Applications*, *25*, 79–95.

Radi, D. (2017). Walrasian versus cournot behavior in an oligopoly of boundedly rational firms. *Journal of Evolutionary Economics*, 1–29.

Schlag, K. H. (1998). Why imitate, and if so, how?: A boundedly rational approach to multi-armed bandits. *Journal of Economic Theory*, *78*(1), 130–156.

Theocharis, R. D. (1960). On the stability of the cournot solution on the oligopoly problem. *The review of economic studies*, *27*(2), 133–134.

Tramontana, F. (2010). Heterogeneous duopoly with isoelastic demand function. *Economic Modelling*, *27*(1), 350–357.

Tramontana, F., Elsadany, A., Xin, B., & Agiza, H. (2015). Local stability of the cournot solution with increasing heterogeneous competitors. *Nonlinear Analysis: Real World Applications*, *26*, 150–160.

Tuinstra, J. (2004). A price adjustment process in a model of monopolistic competition. *International Game Theory Review*, *6*(03), 417–442.

Vega-Redondo, F. (1997). The evolution of walrasian behavior. *Econometrica*, *65*(2), 375–384.