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Complex interplay between monetary and fiscal policies in a real economy model

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Abstract

In this paper we consider a nonlinear model for the real economy described by a multiplier-accelerator setup. The model comprises the government sector, which influences the output dynamics by means of the fiscal policy, and the money market, where the money supply depends upon the fluctuations in the economic activity. Through rigorous analytical tools combined with numerical simulations, we investigate the stability conditions of the unique steady state and the emergence of different kinds of endogenous dynamics, which are the results of the action of the fiscal and the monetary policy through their reactivity degrees. Such policies, if properly tuned, can lead the economy toward the desired full employment target but, on the other hand, can also generate endogenous fluctuations in the pace of the economic activity, associated with the occurrence of closed invariant curves and multistability phenomena.

Keywords: nonlinear dynamics; monetary and fiscal policies; bifurcations; multistability.

1 Introduction

It is well-known that macroeconomic variables, such as national income, interest rates, inflation rates, money supply, etc. exhibit persistent and irregular fluctuations. The models for explaining the evolutions of these variables are widespread and come from any field of economics, with the aim of detecting the endogenous sources of such fluctuations. In this respect, *nonlinearity* is a key ingredient which is thought to be the source of endogenous cyclic behavior. Accordingly, the literature that stems from this idea is widespread and developed in the last decades. Among the milestones in the macro-dynamics literature we can mention the paper by Kalecki [1], who stressed the crucial role played by investments in a capitalist system and the issue of finding a well-specified investment function to appease many problems within the considered economy. Samuelson [2] built a multiplier-accelerator model to analyze the business cycle, while Kaldor [3] explicitly adopted a nonlinear investment function based on the profit principle. Finally, Hicks [4] extended the Samuelson's seminal model with the ideas of floor and ceiling to bound the growth of investments. After these contributions embedded in the general economic thought, the original papers have been extended in several directions thanks to the application of the tools from nonlinear dynamics and bifurcation theory, which allowed reconsidering the various existing economic models. Many interesting improvements of business cycle modeling spread during the

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recent decades, frequently exhibiting complex dynamics, nonlocal bifurcations and transitions from order to chaos, as in [5]-[9].

The Samuelsonian multiplier-accelerator modelling approach, which is considered a benchmark to explain business cycle fluctuations, has been developed and extended in different ways (see, e.g., [10]-[17]), where the birth of persistent oscillations is discussed through the analysis of the role of different elements, such as the presence of an income ceiling and an investment floor, or the introduction of delays in the consumption or investments function.

Nonetheless, a usual objection to business cycle models based on the interaction of the multiplier and the accelerator is that they neglect monetary factors and, when the latter are considered, are not an integral part of the model. Only few papers examined the role of the introduction of the monetary policy within a multiplier-accelerator framework, even if the monetary factors may be of relevance for understanding the emergence and evolution of the business cycle. In this regard, we mention the paper by Smith in [18] who introduces money in a simple and linear multiplier-accelerator model, showing that periodic oscillations may arise and increase when monetary factors are considered. In [19], Lovell and Prescott study the role of money supply in determining periods of stability and oscillations in the business cycle within the Samuelsonian multiplier-accelerator setup and investigate the effects of alternative stabilization strategies that might be activated by the monetary authority. The work by Sordi in [20] tackles the issue of business cycle fluctuations in a discrete time multiplier-accelerator model in which a floor and a ceiling are introduced into the accelerator component of the capital stock evolution. Also the paper in [21] studies in depth the multiplier-accelerator framework in a context of a monetary economy where the multiplier effects are closely related to the monetary transactions. Finally, more recently, Karpētis and Varelas in [22] introduced a money market and a balanced government expenditure rule in a discrete-time multiplier-accelerator to study their interaction and how they affect the overall economic stability.

The idea of including the money market in this framework, together with the fiscal policy, fits into the debate on the proper degree of activism in fiscal and monetary policy making. The main question to give an answer is of how much to vary monetary and fiscal instruments to reduce market turmoil and whether a good fiscal policy is more or less effective than a good monetary policy for stabilization purposes. In the present paper we enrich the discrete time multiplier-accelerator setting considered in [23] by taking into account a money market. The aim is to examine the effects of making monetary factors a relevant part of the economic setup under investigation in order to understand the interplay between these factors and the fiscal policy instruments may give rise, reduce or foster the oscillations in the business cycle.

The multiplier-accelerator model encompasses a nonlinear investments function which takes into account the presence of the monetary sector through the interest rate which, in turn, is determined by the equilibrium condition on the money market. The money supply is influenced by the discrepancy between the full employment national income level and the more recent output realizations. Moreover, the public sector may influence the possibility of the economy to reach a full employment output level through a level-adjusting rule. Therefore, it turns out to be relevant to study whether the interplay between these two policies can render the overall system stable or, instead, endogenous fluctuations arise at the ground of the business cycle.

To this end, we analytically obtain the local asymptotic stability conditions of the unique steady state, and, with the help of numerical investigations, we investigate the possible kinds of bifurcations, showing the consequent emergence of periodic, quasi-periodic and chaotic dynamics.

We find that the introduction of the monetary policy is capable of leading the economy toward the targeted level of output.

In general, there is not an unambiguous role played by the two policies since both can be either the source of endogenous fluctuations, arising when they induce overreaction to real economy signals, or lead to a stabilization of the dynamics. Indeed, there is a certain role played by both policies

for the stabilization purposes. For example, instabilities spreading from the money market can be deadened by the activation of a convenient fiscal policy, as well as stability goals can be achieved through an appropriate cautionary monetary policy, grounded on a suitably inertial response to fluctuations in target variables. However, overreactive policies are not beneficial and may generate complex dynamics in the evolution of the national income in the long run. From the mathematical viewpoint, this is described by endogenous fluctuations that arise due to Neimark-Sacker or period doubling bifurcations. In particular, the interplay between fiscal and monetary policies may be the source of quasi-periodic oscillations, resembling the emergence of business cycle, which are not possible, as shown in [23], when the money market is not considered and only a level adjusting fiscal policy is taken into account. Moreover, the effect of endogenizing the money market may imply multistability. Results, containing a variety of dynamic features, are discussed through the analysis of local bifurcations and through numerical examples that give insights about global dynamics.

The rest of the paper is organized as follows: Section 2 introduces the model, Section 3 presents analytical results on the stability of the unique steady state and the conditions for its asymptotic stability, Section 4 reports the numerical simulations showing how the relevant parameters of the two policies may give rise to complex dynamics. Finally Section 5 collects conclusions and some possible future research perspectives.

2 The baseline model

We present a closed economy model consisting of a real sector, described by a multiplier-accelerator setup, and of a monetary sector. The macroeconomic equilibrium condition, at any time t , is given by

$$Y_t = C_t + I_t + G_t, \quad (1)$$

where Y_t represents the national income, C_t denotes consumption, I_t investments and G_t public expenditures. Consumption linearly depends on the last realization of national income, i.e.

$$C_t = \bar{C} + cY_{t-1}, \quad (2)$$

where \bar{C} is the autonomous consumption and $c \in (0, 1)$ is the marginal propensity to consume.

We assume that government intervenes into the real market to stabilize the economy by means of a level-adjusting rule. In other words, the government has to establish a full employment income Y^F and modifies its expenditures according to the gap between the full employment income and the national income, namely

$$G_t = \bar{G} + g(Y^F - Y_{t-1}). \quad (3)$$

Government expenditures depends on an autonomous component $\bar{G} > 0$ and on a discretionary expenditure $g(Y^F - Y_{t-1})$ where $g > 0$ measures the reactivity of the fiscal policy with respect to deviations from the target Y^F .

The principle of *acceleration* determines investments. Precisely, we assume the investments function is made up by three components. Besides an autonomous component, a second component is increasing in the national income variation between period $t - 1$ and $t - 2$ and is described by a bounded S-shaped function according to the hypothesis that investments can not reach too high or too low values (see [23]). Such a function is continuous and differentiable at each point and it makes this second component constant when the national income does not change for two consecutive periods. A third component is also included to highlight the negative dependence of investments to the interest rate. Thus the investments function can be summarized as

$$I_t = \bar{I} + \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(Y_{t-1} - Y_{t-2})} + a_2} - 1 \right) + \varphi R_t, \quad (4)$$

where \bar{I} is the autonomous component of investments, $\gamma > 0$ relates to the accelerator component, a_1 and a_2 are positive parameters that determine the investment function variation range and $\varphi \leq 0$. It is worth noting that a functional form as the one proposed here for the investments function is in line with the classic macroeconomic literature of the 1930s-1950s (see, e.g., [1], [3], [24]). Moreover, the motivation for considering a sigmoid function to model a component of the investments comes from the Hicks' idea of embodying a floor and a ceiling in the evolution of investments, in order to take into account the impossibility of an indefinite growth and disinvestment due to resource and physical constraints (see [12]-[13] and [25]-[26]). Finally, expression (4) also states that investments negatively depend on the interest rate, being $\varphi \leq 0$. In fact, the interest rate reflects the cost of borrowing in order to finance investment projects and, other things being equal, as the interest rates rise, financing new investment projects becomes more expensive.

Let us now introduce the money market, for which we the equilibrium condition reads as

$$\frac{M_t^S}{\bar{P}} = \frac{M_t^D}{\bar{P}}, \quad (5)$$

that is, the real money demand M_t^D/\bar{P} equals the real money supply M_t^S/\bar{P} . In what follows, for simplicity, we set $\bar{P} = 1$ and we consider the real money demand function

$$M_t^D = d_1 Y_{t-1} + d_2 R_t, \quad (6)$$

which is determined, in accordance with the liquidity preference theory (see e.g. [27] and [19]), by the national income and the interest rate R_t , where $d_1 > 0$ and $d_2 < 0$ accounts for the income and the interest rate effects on real money demand, respectively.

As concerns the money supply, the total money quantity at time t is defined by the money creation policy of the monetary authority. For instance, one can suppose that the authority focuses on the money variation with the aim of moderating fluctuations in the economic activity with respect to the benchmark of full employment income Y^F . Accordingly, we shall assume the following target adjusting monetary policy

$$M_t^S = M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Y_{t-2}), \quad (7)$$

where $\mu > 0$ represents the reaction of the monetary policy with respect to deviation of the full employment output to the last observed income variation, while $\theta \in [0, 1)$ weights the past realization of national income. When $\theta = 0$, only the last output realization is considered by the money supply rule in reacting to the deviations from the full employment income, while when $0 < \theta < 1/2$ the two most recent output observations are taken into account, giving more relevance to the closest one. For $\theta = 1/2$, both Y_t and Y_{t+1} are equally weighted while for $\theta > 1/2$ a greater importance is assigned to farther output realizations, accounting for a higher level of inertia or cautionary response¹. Thus, according to (7), the money supply will move cyclically or counter-cyclically in order to moderate adjustments in interest rates as output changes. In particular, when the economic activity is low (namely when $Y^F > (1 - \theta)Y_{t-1} + \theta Y_{t-2}$), the aim of the monetary policy is to stimulate the economy through an increase of the money supply. This has the effect of reducing the interest rate, as also evident by making explicit R_t through (5), (6) and (7), which provides

$$R_t = \frac{1}{d_2}(M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}). \quad (8)$$

¹We would like to remark that the fiscal policy, and thus the changes in the government expenditures, affects income more rapidly than the monetary policy and it is reasonable to assume only one lag in its response to the deviation of the full employment income from the last realization. On the contrary, as argued by Woodford in [28], it is generally optimal for the monetary policy to respond inertially to fluctuations in the target variables and/or their determinants, so that the policy will continue for some time to depend upon past variable realizations, even when these are irrelevant to the determination of the future values of the target variables. See also Gramlich ([29]) for a discussion of monetary versus fiscal policy and the role of time lags in shaping stabilization strategies.

As a consequence, reduced interest rates have a positive effect on investments and, therefore, on the level of the national income.

Plugging (8) in (4) we obtain the expression of investments

$$I_t = \bar{I} + \gamma_t a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(Y_{t-1} - Y_{t-2})} + a_2} - 1 \right) + \frac{\varphi}{d_2} (M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}),$$

which, inserted in (1) together with (2) and (3) and coupled with the monetary policy equation (7), leads to the model

$$\begin{cases} Y_t &= A + Y_{t-1}c - g(Y_{t-1} - Y^F) + a_2\gamma \left(\frac{a_1 + a_2}{a_2 + a_1 e^{-(Y_{t-1} - Y_{t-2})}} - 1 \right) \\ &\quad + \frac{\varphi}{d_2} (M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}) \\ M_t &= M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Y_{t-2}). \end{cases}$$

Then, by defining $\tilde{\varphi} \equiv \varphi/d_2$ to collect the joint effect of the interest rate on investments and money demand, and introducing $Z_t \equiv Y_{t-1}$ in view of the subsequent analysis, we can introduce the function $T = (T_1, T_2, T_3) : \mathbb{R}_+^3 \rightarrow \mathbb{R}^3$, $(Y_t, M_t, Z_t) \mapsto (T_1(Y_t, M_t, Z_t), T_2(Y_t, M_t, Z_t), T_3(Y_t, M_t, Z_t))$, which describes the functioning of the whole economy² as a the three-dimensional system

$$T : \begin{cases} Y_t &= A + Y_{t-1}c - g(Y_{t-1} - Y^F) + a_2\gamma \left(\frac{a_1 + a_2}{a_2 + a_1 e^{-(Y_{t-1} - Z_{t-1})}} - 1 \right) \\ &\quad + \frac{\varphi}{d_2} (M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Z_{t-1}) - d_1 Y_{t-1}) \\ M_t &= M_{t-1}^S + \mu(Y^F - (1 - \theta)Y_{t-1} - \theta Z_{t-1}), \\ Z_t &= Y_{t-1}. \end{cases} \quad (9)$$

3 Analytical results on the existence of the steady state and local stability properties

In this section we shall investigate the existence of steady states for the system in (9) and we shall analyze how the relevant monetary and fiscal policy parameters affect stability. Firstly, we study the number and the analytical expression of possible steady states for the map in (9), ending up with the following:

Proposition 1. *The system in (9) has a unique steady state given by*

$$(Y^*, M^*, Z^*) = \left(Y^F, \frac{d_2 \left(Y^F(1 - c) - A + \frac{Y^F d_1 \varphi}{d_2} \right)}{\varphi}, Y^F \right)$$

to which corresponds the interest rate

$$R^* = \frac{Y^F(1 - c) - A}{\varphi}.$$

Moreover, the values of (Y^*, M^*, Z^*) are positive provided that

$$\frac{A}{(1 - c) + d_1 \varphi / d_2} < Y^F < \frac{A}{(1 - c)}.$$

²We stress that not all the possible parameter configurations and initial conditional give rise to economically significant trajectories. In the following analysis we implicitly limit to the feasible configurations. Moreover, the simulations reported in Section 4 are consistent with this restriction.

The previous Proposition states that, in equilibrium, the system is able to reach the full employment desired level of output while, out of equilibrium, the monetary and fiscal policies should operate to move the system toward the targeted national income level. In other words, the combined action of the two policies can be able to drive the economy toward the full employment national income, which was not reached in [23] where only the fiscal policy was considered. We observe that the steady state values of the money quantity positively depends on the full employment output level Y^F in accordance to the money supply rule, and on the parameter d_1 , signaling a positive relationship with the increase of the output level; moreover, it negatively depends on the parameter $|\varphi|$ since, when it grows, it makes the investments less attractive and, accordingly, the national income grows less with a consequent contraction in the money demand. On the other hand, M^* positively depends on $|d_2|$, since it reduces the speculative component of the money demand (the interest rate) and thus stimulates the economic activity through investments and national income, with a consequent high level of money in equilibrium. We also highlight that the steady state output $Y^* = Y^F$ lies between a value associated with the absence of money $A/((1-c) + d_1\varphi/d_2)$, corresponding to a situation in which no Keynesian functions of money would be considered, and the Samuelsonian steady state $A/(1-c)$, where no policy interventions are taken into account.

In the next propositions we study the stability of the equilibrium with respect to the reactivity of the monetary policy μ . To this end, we say that we are in an unconditionally unstable, mixed and destabilizing scenario if (Y^*, M^*, Z^*) is respectively never locally asymptotically stable for any $\mu > 0$, locally asymptotically stable for $\mu \in (\bar{\mu}_1, \bar{\mu}_2)$ and locally asymptotically stable for $\mu \in (0, \bar{\mu}_2)$, for some $0 < \bar{\mu}_1 < \bar{\mu}_2$. Finally, we do not discuss cases in which stability occurs for single values of the parameters. Before studying the model in (9), it is worth recalling the stability condition of the steady state for the model studied in [23, Prop 3.], in which only the fiscal policy is considered and money is not present. In this regard, henceforth, we shall make use of substitution $\tilde{\gamma} \equiv \gamma(a_1 + a_2)/(a_1a_2)$.

Proposition 2. *When no monetary policy is considered, the steady state (Y^*, Z^*) is locally asymptotically stable provided that $\tilde{\gamma} < 1$ and*

$$g < c + 2\tilde{\gamma} + 1. \quad (10)$$

When the money market is not considered, the fiscal policy should be set in a way such that it does not react too aggressively to deviations of output realizations with respect to the full employment income target, otherwise periodic fluctuations arise in the business cycle.

Now we move the analysis to the study of the monetary policy parameters on the steady state stability. The following Proposition reports the stability conditions for the steady state (Y^*, M^*, Z^*) .

Proposition 3. *The local stability conditions for (Y^*, M^*, Z^*) are given by:*

$$\begin{cases} -\tilde{\varphi}(1-2\theta)\mu + 4\tilde{\gamma} - 2(g + d_1\tilde{\varphi} - c) + 2 > 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1-\theta) + \theta)\mu + (g + d_1\tilde{\varphi} - c) - \tilde{\gamma} - (g + d_1\tilde{\varphi} - c)\tilde{\gamma} + 1 > 0, \\ -\tilde{\varphi}\theta\mu + g + d_1\tilde{\varphi} - c - 2\tilde{\gamma} + 3 > 0. \end{cases}$$

Then, since we are interested in the role that the monetary policy parameter μ plays on the steady state stability, in the next Proposition we make explicit the role of such a parameter. Thus the following result holds:

Proposition 4. *The steady state (Y^*, M^*, Z^*) is locally asymptotically stable provided that $\tilde{\gamma} < 1$ and*

- *when $0 \leq \theta < \frac{1}{2}$ if*

$$0 \leq \mu < s_2 \quad \text{provided that} \quad g < c + 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) - d_1\tilde{\varphi} \quad (11a)$$

$$0 \leq \mu < s_1 \quad \text{provided that} \quad c + 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) - d_1\tilde{\varphi} \leq g < 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi} \quad (11b)$$

- *when $\frac{1}{2} \leq \theta \leq 1$ if*

$$0 \leq \mu < s_2 \quad \text{provided that} \quad g < c + 2\tilde{\gamma} + 1 - d_1\tilde{\varphi} \quad (12a)$$

$$s_1 < \mu < s_2 \quad \text{provided that} \quad 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi} < g < c + 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) - d_1\tilde{\varphi} \quad (12b)$$

where

$$s_1 = \frac{2\tilde{\varphi}(2\tilde{\gamma} - (g - c + d_1\tilde{\varphi}) + 1)}{1 - 2\theta} \quad \text{and} \quad s_2 = \frac{\tilde{\varphi}(g - c + d_1\tilde{\varphi} + 1)(1 - \tilde{\gamma})}{(\tilde{\gamma}(1 - \theta) + \theta)}. \quad (13)$$

Conditions (11) and (12a) give rise to destabilizing scenarios with respect to μ , while condition (12b) gives rise to a mixed scenario. For all the remaining parameters' configurations we have unconditionally unstable scenarios.

Proposition 4 allows us to understand the effect of the accelerator mechanism and of the policies on the stability of (Y^*, M^*, Z^*) . The first comment is about the role of the accelerator parameter. If the reactivity of investments to income variations is too strong ($\tilde{\gamma} > 1$), neither fiscal nor monetary policy intervention is able to stabilize the system, and the source of endogenous fluctuations is ascribed to the accelerator mechanism, in agreement with the setting studied in Proposition 2. Conversely, when $\tilde{\gamma} < 1$ and the money supply is adjusted giving more relevance to the most recent real economy signals, the money market can be the source of instability. This can be understood by comparing the stability condition (10) with those in (11). If (10) holds true, conditions in (11) can be violated either due to an overreaction of the monetary policy (when $\mu > s_i$) or when the parameters characterizing the money market (d_1 and d_2) and its effects on investments (ϕ) are too large, so that $g < 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi}$. This may happen also when more relevance is given to the least recent real economy signal, but in this case there is some space for a stabilizing role of the monetary policy, the steady state can be stable even if (10) is violated (as evident from (12b)) and hence the fiscal policy is the source of endogenous fluctuations. This is a scenario in which the monetary authority, by sufficiently adjusting the level of money supply, is able to stabilize an otherwise unstable scenario up to a certain level, beyond which the steady state becomes unstable again. To summarize, the reaction of the monetary authority in changing the money supply is not always stabilizing since when such a reactivity is sufficiently large, the steady state may turn unstable.

We now comment more in detail the joint role of μ, g and θ on the stability of (Y^*, M^*, Z^*) with the help of Figures 1 and 2. In Figure 1 we report the stability regions of the steady state in the (μ, g) plane, obtained by setting $c = 0.7$, $\tilde{\gamma} = 0.5$, $d_1 = 1$ and $\tilde{\varphi} = 1$, for different values of θ . In the left panel of Figure 1 the stability region is represented using yellow color and shows, in accordance with the result of Proposition 4, how the steady state become unstable as μ sufficiently increases. Moreover, for small values of μ , increasing g has a destabilizing effect, as in [23] while if μ increases, we find that introducing a fiscal policy has an initial stabilizing effect, which is thwarted by increasing the degree of its reactivity g . In other words, there exists a double stability threshold, which arises due to the intervention of the monetary policy and makes the fiscal policy able to stabilize the economy if its reactivity is not too strong. The central and right panels of Figure 1 highlight the evolution of the stability region as long as an additional weight is assigned to the past level of national income on determining the money supply. If

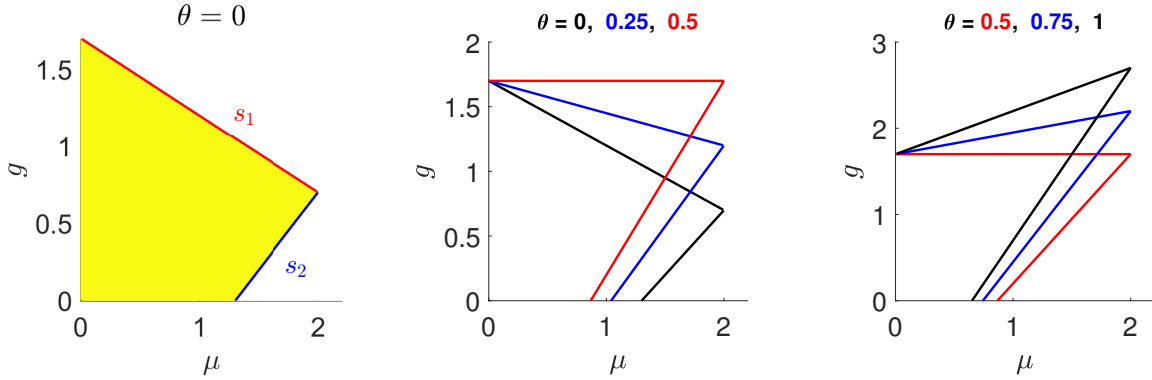


Figure 1: The left panel depicts the stability region (yellow color) of (Y^*, M^*, Z^*) for $\theta = 0$, on varying μ and g . The stability thresholds s_1 and s_2 , defined in (13), are represented by a red and blue line, respectively. The central and the right panels represent the evolution of the stability region, bounded by the vertical axis and by the lines with the same color, for increasing values of θ .

we compare the stability regions when $0 \leq \theta \leq 1/2$ and when $1/2 \leq \theta \leq 1$, we note that the two stability thresholds, s_1 (lower lines) and s_2 (upper lines), move upward, signaling that, in order to get stability when a higher weight is assigned to the past levels of national income, a stronger reaction of the fiscal policy is required. However, when $0 \leq \theta \leq 1/2$ the unique possible scenarios on increasing μ are those destabilizing (for small enough reactivity of the fiscal policy) and unconditionally unstable, while when $1/2 < \theta \leq 1$ a sufficiently reactive monetary policy can counterbalance the destabilizing effects of the fiscal policy, provided that it is not too overreactive. In this case we have a mixed scenario for μ .

In Figure 2 we report the stability region of the steady state in the (θ, μ) parameter plane for different values of g , where the red and the blue curves are related to the first two stability conditions of Proposition 3. The parameters are the same used for Figure 1, with the exception of $d_1 = 0.1$. These panels allow us to appraise the role that the two parameters of the monetary policy exert on the stability of (Y^*, M^*, Z^*) . In the left panel the case of a weakly reactive fiscal policy is depicted and the region of parameters that guarantees the stability of (Y^*, M^*, Z^*) is represented by yellow color. In this case, when μ is sufficiently small, the steady state is always stable while it can be destabilized for increasing values of θ as long as the monetary authority reacts slightly more to the deviations of the income realizations to its full employment level. On the other hand, if the reaction parameter μ grows more, there are no means of stabilization even if a growing weight is assigned to the past income levels. Such behavior occurs for all values of g that preserve the stability of the economy when no monetary policy is used (i.e. when $\mu = 0$), which, in the present setting, corresponds³ to $g < 2.6$, as noticeable also from the middle panel of Figure 2, where, for $g = 0.5$ and $g = 2$, stability is achieved in the regions below the solid and the dashed lines, respectively. When the degree of the fiscal policy reactivity increases, if μ is not too large, the steady state is stable for any value of θ , while there exists a double stability threshold on increasing the reactivity of the monetary policy μ . Moreover, also recalling the left panel of Figure 2, we can note that the stability region increases in size due to the impact played by the fiscal policy.

Conversely, if the reactivity of the fiscal policy is too large, (i.e. $g \geq 2.6$), as in the right panel of Figure 2, it turns out to be useful to introduce a form of inertia in the response of the monetary policy ($\theta > 1/2$) in order to gain stability. In this case, the size of the stability region diminishes as g increases and a certain degree of reactivity is needed to preserve stability, even if we note that increasing values of μ are not always beneficial since the stability region may shrink as long

³The threshold is obtained from condition (11a) when $\theta = 0$.

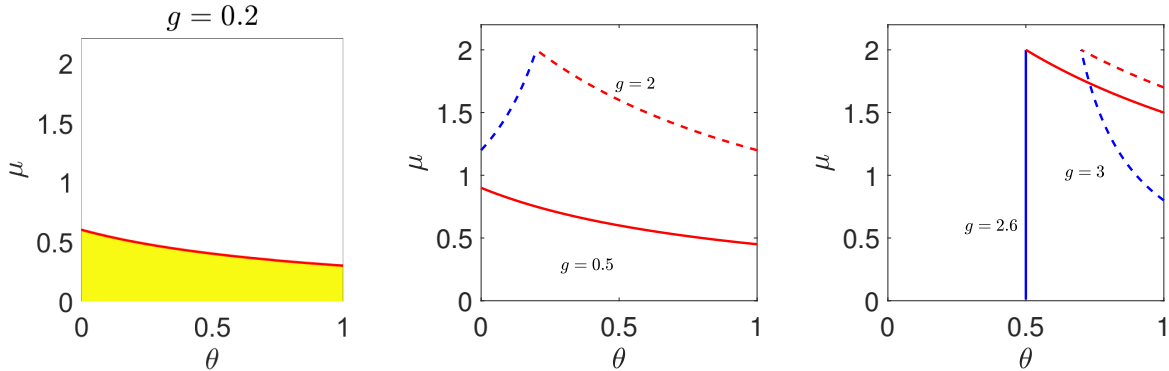


Figure 2: The left panel depicts the stability region (yellow color) of (Y^*, M^*, Z^*) for $g = 0.2$, on varying θ and μ . Middle and right panels show the evolution of stability regions for couples of increasing values of g . The red and blue colors are related to the thresholds defined by the first and the second condition in Proposition 3, respectively. Stability is guaranteed for θ and μ belonging to the regions lying below (middle panel) or between (right panel) stability thresholds.

as the reactivity of the monetary policy gets more aggressive.

4 Numerical simulations

In this section we present some numerical simulations in order to complement the previously conducted analysis and check whether complex dynamics may arise when the steady state turns unstable, as a consequence of the joint actions of the two policies. Unless differently stated, we shall make use of these parameter values: $\gamma = 1.11$, $A = 100$, $Y^F = 205$, $\varphi = -20$, $d_1 = 1$, $d_2 = -20$ and, according to what done in [23], we set $a_1 = 4.5$, $a_2 = 0.5$ and $c = 0.7$. We point out that in the two dimensional bifurcation diagrams reported in Figures 3-7 the white color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. Moreover, hatched regions correspond to parameter configurations characterized by divergence or unfeasibility. Finally, the initial datum is chosen in a suitably small neighborhood of the steady state.

The left panels of Figures 3-5 report the two-dimensional bifurcation diagrams in the (μ, g) parameter plane for different values of θ .

The first consideration that can be inferred from such diagrams is the different kind of dynamics arising when stability is lost. In fact, either on increasing g or μ , when the white region is crossed through its upper border (corresponding to the line defined by s_1 in Proposition 4, see also Figure 1) convergence to the steady state is replaced by convergence to a period two cycle, as we enter the red region. Conversely, when instabilities occur passing over the lower bound of the white region (corresponding in this case to the line defined by s_2), the steady state is immediately replaced by an attractor consisting of several points.

To emphasize how the reactivity of the monetary policy affects the overall dynamics, we can observe the two examples of one-dimensional bifurcation diagrams with respect to μ , reported in the middle and right panels of Figure 3, obtained for two different values of g . In the middle panel of Figure 3, there is no intervention of the fiscal policy nor degree of inertia in the monetary policy. Observing the black bifurcation diagram, we can see that when μ increases, the steady state loses stability via Neimark-Sacker bifurcation. However, when the reactivity μ of the monetary policy is not too strong, the steady state also coexists with a cycle of period 3 (red bifurcation diagram⁴).

⁴Throughout this section, red bifurcation diagrams are computed following the attractor along a sequence $\{x_i\}$ of parameter values. This means that the initial datum for the simulation related to parameter value x_{i+1} is chosen

In this case, the actual final outcome of the economy depends not only on its parameters but also on the initial level of national income, whose evolution is then significantly path dependent. When the steady state becomes unstable, an attracting closed invariant curve emerges, whose diameter quickly increases as μ increases. When this curve collides with the basin boundary of the period 3 cycle, it disappears and trajectories converge to the periodic attractor, which then evolves through a cascade of period doubling bifurcations leading to complex dynamics. We will provide more details about such global phenomena at the end of this section.

In the right panel of Figure 3 we account for a certain level of intervention of the fiscal authority. In this case, the steady state remains stable for a larger interval of μ until it loses stability via period doubling bifurcation, after which the occurrence of a secondary Neimark-Sacker bifurcation evolving into complex dynamics.

The previous phenomena are robust with respect to the introduction of a small degree of inertia ($\theta \leq 0.5$) in the money supply rule, as exemplified by the two-dimensional bifurcation diagram obtained for $\theta = 0.25$ and reported in the left panel of Figure 4. It is worth to remark that, when a certain degree of intervention of the government is encompassed, the intervention of the public authority in reaction to the deviation of the income from the full employment level is able to drive the economy towards the objective of the policy, even for large values of the monetary policy reactivity μ . However, we note that the parameter g has an ambiguous effect on the stability of the steady state, for a given value of μ . In fact, an increase of g has the effect of reducing the complexity of the orbits, and lets the steady state gain stability; however, as g keeps increasing, we cross the upper bound of the stability region, associated with the white color, and the steady state undergoes a period doubling bifurcation. With this respect we consider the right panel of Figure 4, in which the one dimensional bifurcation diagram is computed with respect to g . When the fiscal policy is not sufficiently strong, the dynamics of the output are not convergent towards the steady state, being the latter associated with the occurrence of a Neimark-Sacker bifurcation. On the other hand, when the policy acts sufficiently strong, it is able to stabilize the dynamics, leading the economy to the desired output level or, at least, to a reduction of the qualitative complexity of the trajectories, with the occurrence of a cycle of period two. We stress that when the money market is considered, the interplay of both monetary and fiscal policies gives rise to possibly high levels of complexity, differently from [23], in which it is shown that on changing the reactivity of a level targeting fiscal policy the steady state could lose stability just giving rise to a period 2 cycle.

The same qualitative behaviors are observed also when the parameter θ is further increased (left panel, Figure 5). In particular, comparing the two dimensional bifurcation diagrams in Figures 3- 5, we can observe that when the two bifurcation curves move upward, the size of the region associated with complex dynamics increases. It is worth to note that, when the inertia in the money supply grows, a sufficiently large degree of the fiscal policy reaction is necessary to get the stability of the steady state if the monetary authority overreacts to the deviations of the full employment income to its recent realizations. Moreover, in the case reported in Figure 5, as predicted by Proposition 4, if we consider a high level of reactivity in the fiscal policy, we have the appearance of a double stability threshold, as already commented before, and the steady state is locally asymptotically stable only for intermediate values of the policy parameter μ . This is also evident from the one dimensional bifurcation diagram reported in the right panel of Figure 5, from which we can additionally observe a situation of multistability for increasing values of μ , where the steady state coexists with cycles, making again crucial the choice of the national income to consider when setting the policy intervention.

Figures 6 and 7 represent the two-dimensional bifurcation diagram in the (θ, μ) parameter plane for different values of the fiscal policy parameter g . In Figure 6 we consider the situation in which the fiscal policy does not intervene. We observe that the steady state is locally stable if the

suitably close to the attractor toward which convergence occurred for parameter value x_i

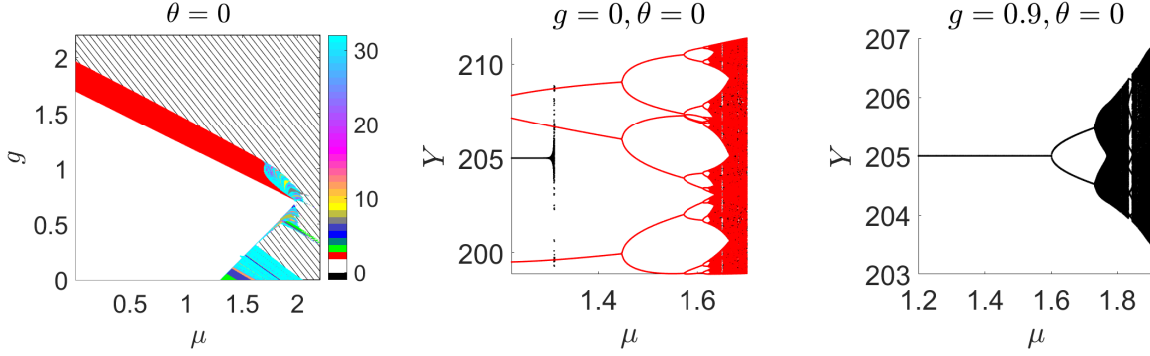


Figure 3: Two-dimensional bifurcation diagram in the (μ, g) plane for $\theta = 0$ (left panel). White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables. The central and the left panels report the bifurcation diagrams with respect to μ for different values of g , showing the possibility of attractors coexistence, even when the steady state is locally stable (central panel). For the red bifurcation diagram we modified the initial value of variables Y and Z , setting $Y_0 = Z_0 = 210$.

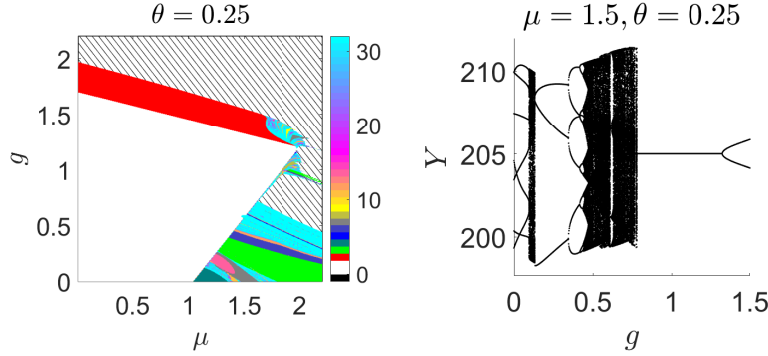


Figure 4: Two-dimensional bifurcation diagrams in the (μ, g) plane for $\theta = 0.25$ (left panel). White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables. The right panel reports the bifurcation diagram on varying g .

degree of the monetary policy reaction is not too large. In fact, for increasing values of μ , and for any θ , the steady state turns unstable and complex dynamics arise with consequent endogenous fluctuations that characterize the course of the business cycle. In the right panel of Figure 6 we report a bifurcation diagram on increasing parameter θ in order to show how the degree of inertia in the money supply rule may generate different dynamic scenarios. In this case, when there is no space for the fiscal policy and the reaction of the monetary authority is set at an intermediate level, the steady state is locally asymptotically stable when more weight is assigned to the most recent output observations while the steady state loses stability via a Neimark-Sacker bifurcation when θ increases, with the consequent emergence of complex dynamics. Hence, in this case, an increase in the degree of inertia in the money supply rule is not necessarily an advantage in terms of reaching the desired output level.

When the economy accounts for a certain level of fiscal policy intervention (left panel, Figure 7), we note a general increase in the white region, which is associated with the stability of the steady state. Moreover, there exists a double stability threshold when the monetary policy aggressively reacts. In fact, on one hand, when the degree of inertia θ is very low, the dynamics are periodic while a sufficient degree of inertia in the response of the monetary authority is able to stabilize the economy; on the other hand, when the policy assigns more weights to the past output realization,

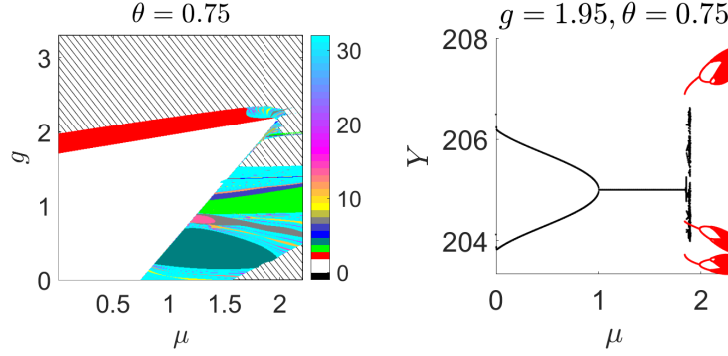


Figure 5: Two-dimensional bifurcation diagram in the (μ, g) plane for $\theta = 0.75$ (left panel). White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables. The right panel reports the bifurcation diagram with respect to μ and reveals the possibility of different coexisting business cycles for a sufficiently high level of the reactivity parameter μ . The red bifurcation diagram is computed following the attractor along a decreasing sequence of parameter values ranging from $\mu = 2.3$ to $\mu = 1.84$.

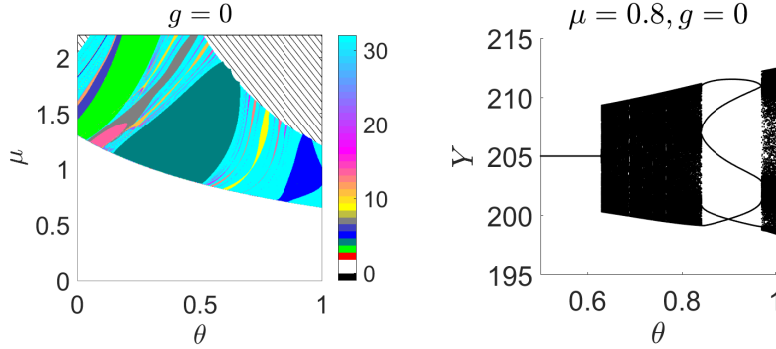


Figure 6: Two-dimensional bifurcation diagram in the (θ, μ) plane for $g = 0$ (left panel). White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of variables. The right panel shows the bifurcation diagram with respect to θ in the absence of fiscal policy.

the coupling with the reaction to the deviation of the output from its full employment level renders the steady state unstable and complex dynamics arise again. Looking at the bifurcation diagram in the right panel in Figure 7, we can see that θ is able to stabilize the dynamics by reducing the complexity of the orbits through a reverted Neimark-Sacker bifurcation followed by a period halving bifurcation. Nonetheless, the range of parameters θ for which the steady state remains locally stable is quite narrow and a Neimark-Sacker bifurcation of the steady state occurs, with dynamics that start oscillating in a complex and intricate manner.

Finally, in Figure 8 we show a bifurcation diagram with respect to the parameter μ when the fiscal policy is not considered ($g = 0$) and there is no inertia in the money supply rule ($\theta = 0$). As it is clearly visible from the picture, when the reactivity of the monetary policy increases, the steady state loses stability confirming all the previous analytical results; moreover, when it is still locally stable, it may coexist with a closed invariant curve associated with orbits that largely oscillates above and below the steady state output level. It is worth stressing the evidence that the introduction of the monetary policy in this simple multiplier-accelerator setting is the responsible for generating the business cycle with fluctuations in the national income dynamics, in a context in which the same qualitative dynamics would not occur if only the fiscal policy were considered (see [23]). This may be due to the fact that an excessive tightening of monetary policy may lead

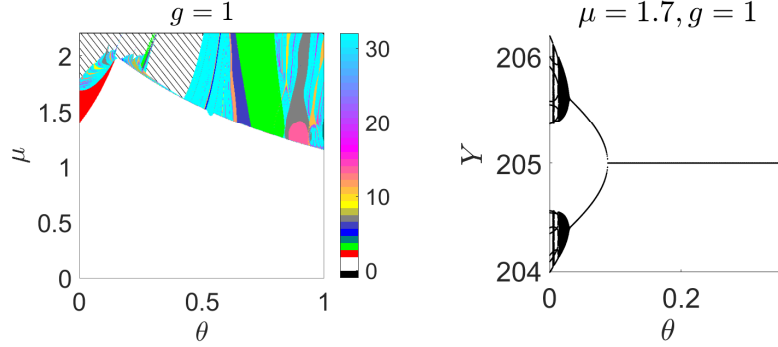


Figure 7: Two-dimensional bifurcation diagram in the (θ, μ) plane for $g = 1$ (left panel). White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of variables. The right panel depicts the bifurcation diagram with respect to θ when also the fiscal policy is present, and shows the double stability threshold for the steady state.

to instability in other sectors of the economy, with a negative effect on economic actors' behavior by weakening their assessment of the future state of the economy. Moreover, as evident from the bifurcation diagram and the times series reported in Figure 8, there could be the possibility of coexisting business cycles in which, being the national income different from its desired level, the final state of the economy can be characterized by persistent higher or lower level of output.

The corresponding basins of attraction of this multistability situation is depicted in Figure 9 where we highlight the evolution of the two coexisting attractors as long as μ increases. In particular, the basin of the steady state is colored in blue while the basin of the closed curve is represented in yellow. In the first row, moving from left to right, which corresponds to an increase of the value of the reactivity parameter μ , we observe a quite sharp shrink in the basin of the steady state. Such a shrinking is even more evident in the bottom-left panel where a higher value of μ is considered. For this value, the state state is unstable and another closed curve coexists with the previously existing one. The enlargement of the bottom-right panel of Figure 8 allows us to appraise the smallest of the two coexisting curves and the corresponding basins. This situation, in which one basin is extremely small, is associated with the consequent unpredictability of the asymptotic state of the economy in a wide region of the state variables.

5 Concluding remarks

In this paper we have shown how a rich variety of dynamical outcomes may arise in a real economy described by a nonlinear multiplier-accelerator model when the public authority influences the dynamics of the national income either via fiscal or monetary policy. The consideration of the two policy instruments fits into the debate on which of the two instruments is better able to pursue the stabilization objective. The investigation of the effects of these policies is pertinent not only for deepening the discussion on the potential implications of the different fiscal and monetary programs, but also with respect to the dynamical outcomes that they can give rise. The dynamics of the specified model basically depends on two parameters, one related to the reactivity of the fiscal policy in adjusting the deviation of the output realizations from its full employment level, and another linked to the responsiveness of the money supply with respect to the variation between the full employment income and the two last output realizations.

It is firstly shown that, from a static viewpoint, the introduction of the monetary policy is able to lead the economy to the desired output equilibrium level. Secondly, from a dynamic point of view, the interaction of the two policy instruments causes a variety of local bifurcation scenarios

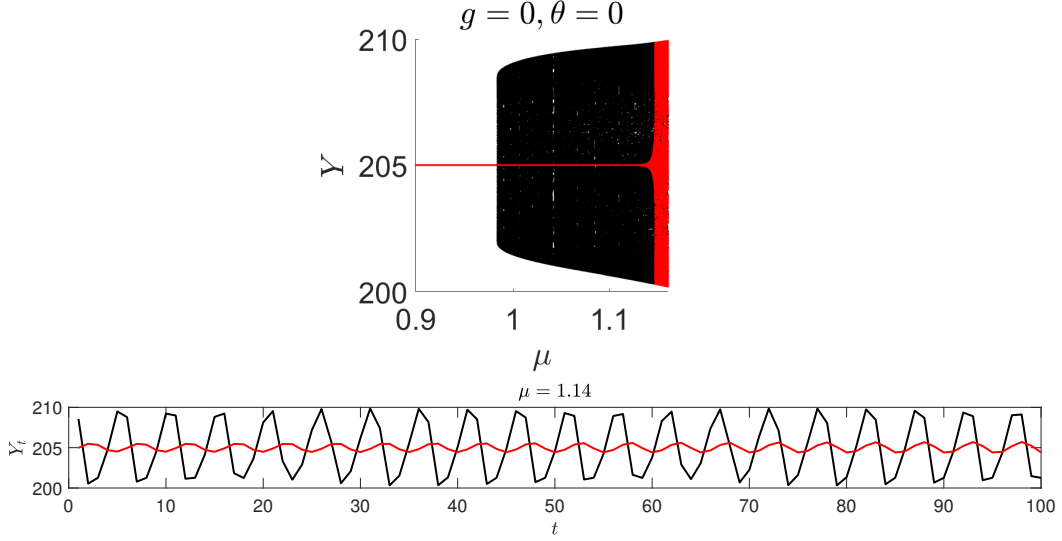


Figure 8: Top panel: bifurcation diagram on varying μ showing the coexistence of different attractors. Bottom panel: times series of Y_t showing coexisting business cycles. Parameters are set as declared at the beginning of the section except for $\gamma = 1.13$. The red diagram is obtained with initial datum suitably close to the steady state while the black diagram is obtained with $Y_0 = Z_0 = 220, M_0 = 88$.

(Neimark-Sacker and period doubling), multistability, as well as complex dynamics. In particular, the introduction of the monetary policy can have beneficial effects in reducing the complexity of the orbits and in leading the economy toward the full employment income, if it is not too aggressive. Otherwise endogenous oscillations in the national income dynamics can take place along an attractive closed invariant curve, which is interpreted as a business cycle in economics. Both the government and the monetary authority are able to influence the size and the persistence of the oscillations by properly tuning their policy instruments. Finally, from a global analysis perspective, we have shown the coexistence of different attractors, occurring even when the national income steady state is locally stable, and thus making the choices of policy makers crucial to shift the output in the desired direction. Within the present framework, the introduction of the monetary policy allowed us to thoroughly investigate the dynamics of the economic activity, which can exhibit interesting dynamic features that would not be present in a context where only the fiscal policy is present. In particular, the role of the monetary policy has to be read in terms of the possibility of stabilizing the national income dynamics but, at the same time, it can also be responsible for the generation of the business cycle. This confirms the importance of the role played by the monetary sector and the relevance of studying how manipulating and targeting the money supply (and, ultimately, the interest rate) influences the real economic variables.

The present setting can be extended in several directions that take into account the role of the monetary policy: firstly, the introduction of the financial sector would add realism in order to contribute to the debate on whether the monetary authority should respond to financial factors, such as asset prices, in monetary policy rules; secondly, the introduction of an asset market would allow to account for different agents' expectations, whose interaction would affect the overall economic stability; thirdly, the role of expectation can also be considered on the real side, as the consumption choices can also be affected by individual perceptions on the output level which, in turn, may contribute to the amplification of business cycle fluctuations.

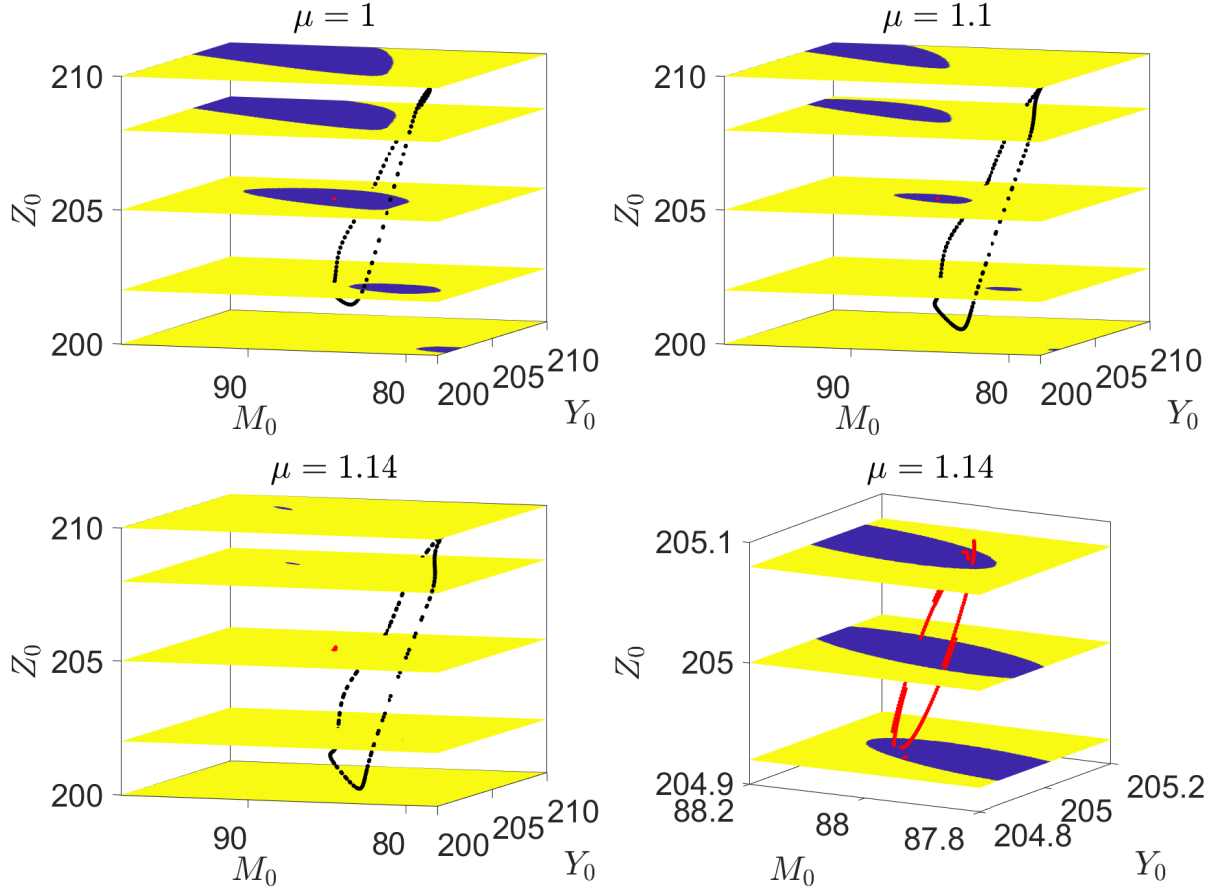


Figure 9: Basins of attraction for different values of the reactivity parameter μ . The yellow color refers to the basin of the closed invariant curve surrounding the steady state (first row), whose basin is depicted in blue, and the other closed curve (second row) arising when the steady state loses stability, whose basin is still represented in blue.

Appendix

Proof of Proposition 1. The assert follows by setting $Y_{t-1} = Y_t = Y^*$, $Z_t = Z_{t-1} = Z^*$ and $M_t = M_{t-1} = M^*$. From the second equation in (9) we immediately obtain $Y^* = Y^F$, which indeed provides $Z^* = Y^F$ from the third equation in (9). Setting $Y^* = Z^* = Y^F$ in the first equation in (9) we find M^* .

Then, setting $M_{t-1} = M^*$ and $Y_{t-1} = Y^*$ in (8) we obtain R^* . Recalling that $d_2 < 0$ and $\varphi < 0$, imposing $M^* > 0$ and $R^* > 0$ and solving with respect to Y^F we easily find the positivity conditions. \square

Proof of Proposition 2. See [23]. \square

Proof of Propositions 3-4. The Jacobian matrix of the system is given by

$$J = \begin{pmatrix} c - g - \tilde{\varphi}(d_1 - \mu(\theta - 1)) + \frac{\tilde{\gamma}e^{Z-Y}(a_1+a_2)^2}{(a_2+a_1e^{Z-Y})^2} & \tilde{\varphi} & -\tilde{\varphi}\mu\theta - \frac{\tilde{\gamma}e^{Z-Y}(a_1+a_2)^2}{(a_2+a_1e^{Z-Y})^2} \\ \mu(\theta - 1) & 1 & -\mu\theta \\ 1 & 0 & 0 \end{pmatrix},$$

which, evaluated at the steady state (Y^*, M^*, Z^*) , becomes

$$J^* = \begin{pmatrix} c - g + \tilde{\gamma} - \tilde{\varphi}(d_1 + \mu(1 - \theta)) & \tilde{\varphi} & -\tilde{\gamma} - \tilde{\varphi}\mu\theta \\ -\mu(1 - \theta) & 1 & -\mu\theta \\ 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial $P(\lambda) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3$ associated to matrix J^* is defined by

$$C_1 = g - c - \tilde{\gamma} + d_1\tilde{\varphi} + \tilde{\varphi}\mu - \tilde{\varphi}\mu\theta - 1, \quad C_2 = c - g + 2\tilde{\gamma} - d_1\tilde{\varphi} + \tilde{\varphi}\mu\theta, \quad C_3 = -\tilde{\gamma}.$$

The stability conditions (see [30]) require

$$\begin{cases} 1 + C_1 + C_2 + C_3 > 0, \\ 1 - C_1 + C_2 - C_3 > 0, \\ 1 - C_2 + C_1C_3 - (C_3)^2 > 0, \\ C_2 < 3, \end{cases}$$

namely

$$\begin{cases} \tilde{\varphi}\mu > 0, \\ 2c - 2g + 4\tilde{\gamma} - 2d_1\tilde{\varphi} - \tilde{\varphi}\mu + 2\tilde{\varphi}\mu\theta + 2 > 0, \\ g - c - \tilde{\gamma} + c\tilde{\gamma} + d_1\tilde{\varphi} - g\tilde{\gamma} - d_1\tilde{\varphi}\tilde{\gamma} - \tilde{\varphi}\tilde{\gamma}\mu - \tilde{\varphi}\mu\theta + \tilde{\varphi}\tilde{\gamma}\mu\theta + 1 > 0, \\ g - c - 2\tilde{\gamma} + d_1\tilde{\varphi} - \tilde{\varphi}\mu\theta + 3 > 0. \end{cases}$$

The first condition is always true. Introducing $\alpha = g - c + d_1\tilde{\varphi}$ we can rewrite the last three condition of the previous system as

$$\begin{cases} -\tilde{\varphi}(1 - 2\theta)\mu + 4\tilde{\gamma} - 2\alpha + 2 > 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1 - \theta) + \theta)\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} + 1 > 0, \\ -\tilde{\varphi}\theta\mu + \alpha - 2\tilde{\gamma} + 3 > 0. \end{cases} \quad (14)$$

Before making the stability conditions explicit from System (14), we collect some identities and inequalities that will be used in the rest of the proof. Let us define

$$s_3 = \frac{\alpha - 2\tilde{\gamma} + 3}{\tilde{\varphi}\theta}.$$

and note that since $c < 1$, we have $\alpha > -1$. We have

$$\begin{aligned} s_1 > 0 &\Leftrightarrow \begin{cases} 0 \leq \theta < 1/2, \\ \alpha < 2\tilde{\gamma} + 1, \end{cases} \cup \begin{cases} 1/2 < \theta \leq 1, \\ \alpha > 2\tilde{\gamma} + 1, \end{cases} \\ s_2 > 0 &\Leftrightarrow \tilde{\gamma} < 1, \\ \theta \neq 0 \text{ and } s_2 > 0 &\Rightarrow s_3 > 0, \end{aligned} \quad (15)$$

where the last implication is due to the fact that, when $\theta \neq 0$, $s_3 > 0$ is equivalent to $\alpha > 2\tilde{\gamma} - 3$, but $2\tilde{\gamma} - 3 < -1$ when $\tilde{\gamma} < 1$.

Moreover, we have

$$s_1 - s_2 = \frac{(\tilde{\gamma} + 1)(-\alpha + 4\tilde{\gamma} + 4\theta - 4\tilde{\gamma}\theta - 1)}{\tilde{\varphi}(1 - 2\theta)(\tilde{\gamma}(1 - \theta) + \theta)}, \quad (16)$$

and

$$\theta \neq 0 \text{ and } s_2 > 0 \Rightarrow s_2 - s_3 < 0 \quad (17)$$

since

$$s_2 - s_3 = \frac{-3\tilde{\gamma} - 2\theta - \alpha\tilde{\gamma} + 4\tilde{\gamma}\theta - 2\tilde{\gamma}^2\theta + 2\tilde{\gamma}^2}{\tilde{\varphi}\theta(\tilde{\gamma}(1 - \theta) + \theta)} < 0$$

is equivalent to

$$\alpha > \frac{-3\tilde{\gamma} - 2\theta + 4\tilde{\gamma}\theta - 2\tilde{\gamma}^2\theta + 2\tilde{\gamma}^2}{\tilde{\gamma}} = \frac{2\gamma^2 - 3\tilde{\gamma} - 2\theta(1 - \tilde{\gamma})^2}{\tilde{\gamma}},$$

in which the rightmost term is smaller than -1 , since when $s_2 > 0$ we have $\tilde{\gamma} < 1$ and consequently

$$\frac{2\gamma^2 - 3\tilde{\gamma} - 2\theta(1 - \tilde{\gamma})^2}{\tilde{\gamma}} < -1 \Leftrightarrow -2\tilde{\gamma}(1 - \tilde{\gamma}) - 2\theta(1 - \tilde{\gamma})^2 < 0.$$

We can now make (14) explicit with respect to μ . In particular, we distinguish different cases depending on the values of μ .

• $\theta = 0$

System (14) becomes

$$\begin{cases} -\tilde{\varphi}\mu + 4\tilde{\gamma} - 2\alpha + 2 > 0, \\ -\tilde{\varphi}\tilde{\gamma}\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} + 1 > 0, \\ \alpha - 2\tilde{\gamma} + 3 > 0, \end{cases} \quad (18)$$

in which the last condition requires $\alpha > 2\tilde{\gamma} - 3$, while the first and the second one respectively become $\mu < s_1$ and $\mu < s_2$. Hence we can write

$$\begin{cases} \theta = 0, \\ \alpha > 2\tilde{\gamma} - 3, \\ 0 \leq \mu < \min\{s_1, s_2\}. \end{cases}$$

To have a non-empty stability interval we need $s_1 > 0$ and $s_2 > 0$, which, setting $\theta = 0$ in (15), respectively provide $\alpha < 2\tilde{\gamma} + 1$ and $\tilde{\gamma} < 1$ (so $\alpha > 2\tilde{\gamma} - 3$ holds true). Moreover, setting $\theta = 0$ in (16) we have that $s_1 < s_2$ when $\alpha > 4\tilde{\gamma} - 1$. System (18) then becomes

$$0 \leq \mu < s_2 \text{ when } \begin{cases} \theta = 0, \\ -1 < \alpha < 4\tilde{\gamma} - 1, \\ \tilde{\gamma} < 1, \end{cases} \quad (19)$$

and

$$0 \leq \mu < s_1 \text{ when } \begin{cases} \theta = 0, \\ 4\tilde{\gamma} - 1 \leq \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \quad (20)$$

• $0 < \theta < 1/2$

Conditions in (14) respectively become $\mu < s_1, \mu < s_2$ and $\mu < s_3$. Thus, to have a non-empty stability interval, we need $s_1 > 0$, i.e. $\alpha < 2\tilde{\gamma} + 1$ and $s_2 > 0$, i.e. $\tilde{\gamma} < 1$. Recalling (15) and (17), this last condition guarantees that $s_3 > 0$ and $s_2 < s_3$. Hence we can write

$$\begin{cases} 0 < \theta < 1/2, \\ -1 < \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1, \\ 0 \leq \mu < \min\{s_1, s_2\}. \end{cases}$$

Since from (16) we have $s_1 < s_2$ when $\alpha > 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma})$ and noting that

$$4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) < 2\tilde{\gamma} + 1 \Leftrightarrow 2(1 - \tilde{\gamma})(2\theta - 1) \quad (21)$$

is negative for $\tilde{\gamma} < 1$ and $0 < \theta < 1/2$, we can conclude

$$0 \leq \mu < s_2 \text{ when } \begin{cases} 0 < \theta < 1/2, \\ -1 < \alpha < 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}), \\ \tilde{\gamma} < 1, \end{cases} \quad (22)$$

and

$$0 \leq \mu < s_1 \text{ when } \begin{cases} 0 < \theta < 1/2, \\ 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) \leq \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \quad (23)$$

Combining (19) and (22) we obtain condition (11a), while (20) and (23) together provides (11b).

• $\theta = 1/2$

The first condition in (14) is $\alpha < 2\tilde{\gamma} + 1$, the second one requires $\mu < s_2$ and the third one requires $\mu < s_3$. Thus, to have a non-empty stability interval we need $s_2 > 0$, i.e. $\tilde{\gamma} < 1$, which guarantees $s_3 > 0$ and $s_2 < s_3$. The system (14) then becomes

$$0 \leq \mu < s_2 \text{ when } \begin{cases} \theta = 1/2, \\ -1 < \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \quad (24)$$

• $1/2 < \theta \leq 1$

The first condition in (14) is $\alpha > s_1$, the second one requires $\mu < s_2$ and the third one requires $\mu < s_3$. Thus, to have a non-empty stability interval we need $s_2 > 0$, i.e. $\tilde{\gamma} < 1$, which again guarantees that $s_3 > 0$ and $s_2 < s_3$. The system (14) becomes

$$\begin{cases} 1/2 < \theta \leq 1, \\ \tilde{\gamma} < 1, \\ s_1 < \mu < s_2. \end{cases} \quad (25)$$

To have a non-empty stability interval we need $s_1 < s_2$. Recalling (16), we have

$$\alpha < 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}),$$

where it is easy to see that the right hand side is greater than -1 .

In particular, a destabilizing scenario occurs when $s_1 \leq 0$, i.e. from (15) when $\alpha < 2\tilde{\gamma} + 1$, which allows writing

$$0 \leq \mu < s_2 \text{ when } \begin{cases} 1/2 < \theta \leq 1, \\ -1 < \alpha \leq 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1, \end{cases}$$

which combined with (24) provides condition (12a). Finally, we have a mixed scenario when $s_1 > 0$, i.e. from (15) when $\alpha > 2\tilde{\gamma} + 1$, which allows rewriting System (25) as (12b). We stress that, thanks to (21), the second condition in the rightmost system is always fulfilled by some α . \square

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