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The first fundamental theorem of welfare in a general equilibrium evolutionary setting

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Abstract

In the present note we prove the first fundamental theorem of welfare economics, according to which all equilibrium allocations are Pareto optimal, for the standard pure exchange model with shares. In this context the social interaction among agents enters the definition of equilibrium only through the market clearing conditions, but it does not affect the agents' maximization problem. We show that the first fundamental theorem of welfare holds true also when introducing stationary equilibria in relation to a share updating mechanism.

Keywords: general equilibrium; stationary equilibria; Pareto optimality.

1 Introduction

Although in the standard pure exchange model all equilibrium allocations are Pareto optimal, it is well known that externalities in general equilibrium settings typically lead to inefficiencies, and the first theorem of welfare may not hold anymore (see e.g. Mas-Colell et al. 1995).

We here consider a framework with shares, in which the social interaction among agents enters the definition of equilibrium only through the market clearing conditions, but it does not affect the agents' maximization problem. We show that this is not enough to impair the efficiency of the equilibrium allocations, also when introducing stationary equilibria in relation to a share updating mechanism.

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This work completes the analysis of the discrete-time exchange economy evolutionary model in Naimzada and Pireddu (2019b), in which two groups of agents are characterized by possibly different preference structures, described within a class of utility functions. The reproduction level of a group is there related to its attractiveness degree, which depends on the social visibility level, determined by the consumption choices of the agents in that group. The attractiveness of a group is described by a generic bell-shaped function, increasing for low visibility levels, but decreasing when the visibility of the group exceeds a given threshold value, due to a congestion effect. Thanks to the combined action of the price mechanism and of the share updating rule, the model is able to reproduce the recurrent dynamic behavior typical of the fashion cycle, presenting booms and busts both in the agents' consumption choices and in the population shares.
We recall that the setting in Naimzada and Pireddu (2019b) is a generalization of that in Naimzada and Pireddu (2019a), in which only Cobb-Douglas utility functions were considered. The context in Naimzada and Pireddu (2019a) is in turn an extension of the framework in Naimzada and Pireddu (2018), where just one particular formulation of the attractiveness was taken into account.
The remaider of the paper is organized as follows. In Section 2 we present the model, which coincides with the setting in Naimzada and Pireddu (2019b), as well as the needed definitions. In Section 3 we prove the first theorem of welfare economics, in relation to both the static and the dynamic notion of equilibrium, and we make some concluding comments.

2 Definitions and set-up of the model

2.1 The Walrasian equilibria with shares

Let us consider an exchange economy with a continuum of agents, which may be of type α or of type β.[1] There are two consumption goods, x and y, and agents' preferences, as in most literature on smooth economies (see e.g. Villanacci et al. 2002), are described by the class of utility functions introduced in the following definition.

Definition 2.1 *For $i \in \{\alpha, \beta\}$, we define $\mathcal{U}_i$ as the set of utility functions $u_i : (0, +\infty)^2 \to \mathbb{R}$ such that*

(A1) $u_i \in \mathcal{C}^2((0, +\infty)^2)$;

(A2) u_i *is differentiably strictly increasing, i.e.,* $Du_i(x, y) >> 0$, $\forall (x, y) \in (0, +\infty)^2$;

(A3) u_i *is differentiably strictly quasiconcave, i.e.,* $Du_i(x, y)v = 0$ *implies* $vD^2u_i(x, y)v < 0$ $\forall (x, y) \in (0, +\infty)^2$, $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$;

(A4) $\forall (\overline{x}, \overline{y}) \in (0, +\infty)^2$, $\{(x, y) \in (0, +\infty)^2 : u_i(x, y) \geq u_i(\overline{x}, \overline{y})\}$ *is closed in the topology of* $\mathbb{R}^2$.

[1]We keep the notation used in Naimzada and Pireddu (2018, 2019a), where $i \in \{\alpha, \beta\}$ was the weight assigned to good x in the Cobb-Douglas utility function by agents in group i, with $0 < \alpha < \beta < 1$.

Assumption $(A1)$ allows to perform computations and to employ tools from Calculus such as the implicit function theorem. Assumption $(A2)$ says that households always prefer a bundle with slightly more of anything, no matter what they are consuming. Assumption $(A3)$ says that households prefer bundles in which commodities are fairly distributed and allows to obtain a unique solution to the household maximization problem. Assumption $(A4)$ implies that indifference curves of utility functions do not touch the axes and thus that the solution is interior.

We notice that, by Definition 2.1, we have $\mathcal{U}_\alpha = \mathcal{U}_\beta$. Hence, we will denote the set of utility functions simply by $\mathcal{U}$.

In our model we assume that time is discrete, i.e., that $t \in \mathbb{N}$. The quantity of good x (y) consumed by an agent of type $i \in \{\alpha, \beta\}$ at time t is denoted by $x_{i,t}$ $(y_{i,t})$. Both kinds of agents have the same positive endowments of the two goods, denoted respectively by w_x and w_y. We define the set of economies as $\mathcal{E} = \mathcal{U}^2 \times (0, +\infty)^2$, with generic element $E = (u_\alpha, u_\beta, w_x, w_y)$. We denote by $p_{x,t} > 0$ and $p_{y,t} > 0$ the prices at time t for goods x and y, respectively. The size of the population of kind α (β) at time t is denoted by A_t (B_t). The normalized variable $a_t = A_t/(A_t + B_t) \in [0, 1]$ represents the population fraction composed by the agents of type α and $b_t = 1 - a_t = B_t/(A_t + B_t) \in [0, 1]$ represents the population fraction composed by the agents of type β.

We are now in position to provide the definition of market equilibrium.

Definition 2.2 *Given the economy $E \in \mathcal{E}$ and the population share $a_t \in [0, 1]$, a market equilibrium at time t is a vector $(p^*_{x,t}, p^*_{y,t}, x^*_{i,t}, y^*_{i,t})$, with $i \in \{\alpha, \beta\}$, such that:*

- *every kind of agent i chooses a utility-maximizing consumption bundle $(x^*_{i,t}, y^*_{i,t})$, given $(p^*_{x,t}, p^*_{y,t})$, i.e., the agents of group $i \in \{\alpha, \beta\}$ at time t solve*

$$\begin{gathered} \max_{(x_{i,t}, y_{i,t}) \in (0,+\infty)^2} u_i(x_{i,t}, y_{i,t}) \quad s.t. \\ p_{x,t}\, x_{i,t} + p_{y,t}\, y_{i,t} \leq p_{x,t}\, w_x + p_{y,t}\, w_y \end{gathered} \tag{2.1}$$

- *the markets for the two goods clear, i.e., at time t for good $j \in \{x, y\}$ it holds that*

$$a_t\, j_{\alpha,t} + (1 - a_t)\, j_{\beta,t} = a_t\, w_j + (1 - a_t) w_j = w_j.$$

We notice that, since utility functions are differentiably strictly increasing, problem (2.1) may be rewritten as

$$\begin{gathered} \max_{(x_{i,t}, y_{i,t}) \in (0,+\infty)^2} u_i(x_{i,t}, y_{i,t}) \quad \text{s.t.} \\ p_{x,t}\, x_{i,t} + p_{y,t}\, y_{i,t} = p_{x,t}\, w_x + p_{y,t}\, w_y \end{gathered} \tag{2.2}$$

and from the budget constraint we obtain $x_{i,t} = w_x + p_t w_y - p_t\, y_{i,t}$, where we set $p_t = p_{y,t}/p_{x,t}$. Hence, (2.2) simply becomes

$$\max_{y_{i,t} \in (0,+\infty)} u_i(w_x + p_t w_y - p_t\, y_{i,t}, y_{i,t}).$$

From here, thanks to the fact that u_i is differentiably strictly quasiconcave, there exists a unique optimal consumption choice for good y, depending on p_t, that we call $y^*_{i,t}(p_t)$. Hence, the optimal consumption choice of agent i for good x, depending on p_t, that we call $x^*_{i,t}(p_t)$, is given by $x^*_{i,t}(p_t) = w_x + p_t\, w_y - p_t\, y^*_{i,t}(p_t)$. The equilibrium price p^*_t can then be determined by using one of the two market clearing conditions, since by Walras' law the other market clearing condition is redundant. In this manner p^*_t will be influenced by the population share a_t, so that we can write $p^*_t(a_t)$. Inserting $p^*_t(a_t)$ into $x^*_{i,t}(p_t)$ and $y^*_{i,t}(p_t)$, we find the equilibrium consumption choices $x^*_{i,t}$ and $y^*_{i,t}$, which will depend on a_t, as well. Indeed, using the extended approach based on first order conditions and market clearing conditions to characterize equilibria (cf. Paragraph 8.4 in Villanacci et al. 2002), it is possible to prove that in any time period, for all $E \in \mathcal{E}$ and $a_t \in (0, 1)$, there exists at least a market equilibrium (see Proposition 1 in Naimzada and Pireddu 2019b) and that, for all population shares, generically in the set of economies, market equilibria are finite and regular, i.e., they depend in a smooth manner on economies and population shares (cf. Proposition 2 in Naimzada and Pireddu 2019b). Moreover, in order to avoid indeterminacy issues, we checked in Proposition 3 in Naimzada and Pireddu (2019b) that a unique equilibrium exists when dealing with utility functions that yield individual demand functions with the gross substitute property, such as the Cobb-Douglas utility functions. If this is the case, for every economy $E \in \mathcal{E}$ and for every $t \in \mathbb{N}$, it holds that the equilibrium price p^*_t and the equilibrium allocation $(x^*_{i,t}, y^*_{i,t})_{i\in\{\alpha,\beta\}}$ are uniquely determined by the value of the population share $a_t \in (0, 1)$. Actually, the argument above suggests that a unique equilibrium exists even when $a = 0$ and $a = 1$, although such extreme cases are not encompassed in Propositions 1 and 3 in Naimzada and Pireddu (2019b) due to the differential topology tools used in the proofs of those results.

We stress that, even if by now no dynamic aspects have been introduced, and thus we are just considering a variation of the classical exchange economy setting with two consumers, in which we take into account population shares in market clearing conditions, we need to check that all the steps in the original proofs of existence, generic regularity and uniqueness of equilibria still hold true in our framework. In particular, such verification cannot be performed on the Edgeworth box due to the fact that the two groups of agents in general do not have the same numerosity.

2.2 The stationary equilibria

In Definition 2.3 below we shall introduce the dynamic notion of equilibrium, i.e., of market stationary equilibrium, that is a market equilibrium in which population shares, and consequently prices and optimal consumption choices, are constant over time. We stress that shares are constant because in every period they solve the dynamic equation governing the share updating rule. For the brevity's sake, we will not report any equation describing the share updating rule, referring the interested reader e.g. to (2.4) in Naimzada and Pireddu (2018), or to (15) in Naimzada and Pireddu (2019b), where a more general formulation of the objects involved is proposed. We recall that the same general formulation of the evolutionary mechanism can be found also in (5) in Naimzada and Pireddu (2019b), where however, like in Naimzada and Pireddu (2018), just the Cobb-Douglas utility functions were considered. In all cases, if shares are constant, the equilibrium price determined through the market clearing condition is constant, too, and consequently also the equilibrium consumption choices are constant.

Definition 2.3 *Given the economy* $E \in \mathcal{E}$*, the vector* (a^*, p^*, x_i^*, y_i^*)*,* $i \in \{\alpha, \beta\}$*, is a market stationary equilibrium if* $a^* \in [0,1]$ *is constant and if, given* a^**,* (p^*, x_i^*, y_i^*)*,* $i \in \{\alpha, \beta\}$*, is a market equilibrium in every time* t*.*

We remark that, in order not to overburden notation and terminology, although a^* is not part of the market equilibrium vector introduced in Definition 2.2, we call the objects described in Definition 2.3 (market stationary) equilibria, and we use the symbol * even for the shares. In fact, as done in Naimzada and Pireddu (2018, 2019a, 2019b), when for all population shares each economy admits a unique market equilibrium, it is possible to identify market stationary equilibria just with the population share a^*, since it univocally determines all other equilibrium components. Namely, according to what explained in Subsection 2.1, when dealing e.g. with utility functions that yield individual demand functions with the gross substitute property, it holds that, for every economy, a^* determines a unique equilibrium price p^*, which in turns determines a unique equilibrium allocation $(x_i^*, y_i^*)_{i \in \{\alpha,\beta\}}$. We also notice that in Definition 2.2 there are time subscripts, missing in Definition 2.3, as the latter describes a stationary, time-unvarying, situation.

3 The first theorem of welfare economics

Before stating and proving the first theorem of welfare, according to which all equilibrium allocations are Pareto optimal, we have to explain how the definition of Pareto efficient allocation reads in our context with population shares. Accordingly, we provide the following:

Definition 3.1 *Given the population share* $a_t \in [0,1]$*, an allocation* $(x_{i,t}^*, y_{i,t}^*)_{i \in \{\alpha,\beta\}} \in (0, +\infty)^4$ *is Pareto optimal at time* t *if there does not exist* $(x'_{i,t}, y'_{i,t})_{i \in \{\alpha,\beta\}} \in (0, +\infty)^4$ *such that*

(*i*) *at time* t *for good* $j \in \{x, y\}$ *it holds that*

$$a_t\, j_{\alpha,t}^* + (1 - a_t)\, j_{\beta,t}^* = a_t\, j'_{\alpha,t} + (1 - a_t)\, j'_{\beta,t}$$

(*ii*) $u_\alpha(x'_{\alpha,t}, y'_{\alpha,t}) \geq u_\alpha(x_{\alpha,t}^*, y_{\alpha,t}^*)$ *and* $u_\beta(x'_{\beta,t}, y'_{\beta,t}) > u_\beta(x_{\beta,t}^*, y_{\beta,t}^*)$ *or vice versa* $u_\alpha(x'_{\alpha,t}, y'_{\alpha,t}) > u_\alpha(x_{\alpha,t}^*, y_{\alpha,t}^*)$ *and* $u_\beta(x'_{\beta,t}, y'_{\beta,t}) \geq u_\beta(x_{\beta,t}^*, y_{\beta,t}^*)$*.*

Also of the definition of Pareto efficient allocation it is possible to furnish a dynamic counterpart, starting from the definition of market stationary equilibrium, which reads as follows:

Definition 3.2 *The vector* (a^*, x_i^*, y_i^*)*, with* $i \in \{\alpha, \beta\}$*, is stationary Pareto optimal if* $a^* \in [0,1]$ *is constant and if, given* a^**,* $(x_i^*, y_i^*)_{i \in \{\alpha,\beta\}}$ *is a Pareto optimal allocation in every time* t*.*

We stress that, differently from the case of market stationary equilibria, it is not possible to identify stationary Pareto optimal vectors just with the population share a^*, since the latter does not univocally determine the components of the Pareto optimal allocations, even when each economy admits for all population shares a unique market equilibrium. The same remark applies to the concept of Pareto optimal allocation, which is not univocally determined by the population shares. Namely, even when the two groups of agents have the

same numerosity, the Pareto set, i.e., the set of Pareto optimal allocations, in the classical exchange economy is a curve joining in the Edgeworth box the bottom-left edge to the top-right edge.
Recalling that, according to Proposition 1 in Naimzada and Pireddu (2019b), in any time period, for all economies and population shares $a_t \in (0,1)$, there exists at least a market equilibrium, we are now in position to state our main result:

Proposition 3.1 (The first fundamental theorem of welfare economics) *Given the economy $E \in \mathcal{E}$ and the population share $a_t \in (0,1)$, let $(p_{x,t}^*, p_{y,t}^*, x_{i,t}^*, y_{i,t}^*)$, with $i \in \{\alpha, \beta\}$, be a market equilibrium at time t. Then the allocation $(x_{i,t}^*, y_{i,t}^*)_{i\in\{\alpha,\beta\}} \in (0,+\infty)^4$ is Pareto optimal.*

Proof. Since the arguments we shall employ are independent of the considered time period, in order not to overburden notation, we will omit the subscript t along the proof.
Assume by contradiction that, given $E \in \mathcal{E}$ and $a \in (0,1)$, $(p_x^*, p_y^*, x_i^*, y_i^*)$, with $i \in \{\alpha, \beta\}$, is a market equilibrium, but that $(x_i^*, y_i^*)_{i\in\{\alpha,\beta\}}$ is not Pareto optimal. Then, there exists $(x_i', y_i')_{i\in\{\alpha,\beta\}} \in (0,+\infty)^4$ such that conditions (i) and (ii) in Definition 3.1 hold true.
Recalling (2.2), we claim that from (ii) it follows that

$$p_x\, x_i' + p_y\, y_i' \geq p_x\, x_i^* + p_y\, y_i^* = p_x\, w_x + p_y\, w_y, \text{ for } i \in \{\alpha, \beta\}. \tag{3.1}$$

Namely, if for instance $p_x\, x_\alpha' + p_y\, y_\alpha' < p_x\, x_\alpha^* + p_y\, y_\alpha^* = p_x\, w_x + p_y\, w_y$, then agent α could choose as consumption bundle $(\widetilde{x}_\alpha, \widetilde{y}_\alpha) = (x_\alpha', y_\alpha') + ((p_x\, x_\alpha^* + p_y\, y_\alpha^* - p_x\, x_\alpha' - p_y\, y_\alpha')/p_x, 0)$, since $(\widetilde{x}_\alpha, \widetilde{y}_\alpha)$ would belong to the budget set of agent α. Indeed,

$$p_x\, \widetilde{x}_\alpha + p_y\, \widetilde{y}_\alpha = p_x\, x_\alpha' + p_y\, y_\alpha' + p_x\, x_\alpha^* + p_y\, y_\alpha^* - p_x\, x_\alpha' - p_y\, y_\alpha' = p_x\, x_\alpha^* + p_y\, y_\alpha^* = p_x\, w_x + p_y\, w_y.$$

However, since $\widetilde{x}_\alpha > x_\alpha'$ and $\widetilde{y}_\alpha = y_\alpha'$, by condition $(A2)$ in Definition 2.1 it would follow that $u_\alpha(\widetilde{x}_\alpha, \widetilde{y}_\alpha) > u_\alpha(x_\alpha', y_\alpha') \geq u_\alpha(x_\alpha^*, y_\alpha^*)$, contradicting the fact that (x_α^*, y_α^*) solves the maximization problem (2.2). This shows that (3.1) is fulfilled.
To fix ideas let us assume that (ii) in Definition 3.1 reads as $u_\alpha(x_\alpha', y_\alpha') \geq u_\alpha(x_\alpha^*, y_\alpha^*)$ and $u_\beta(x_\beta', y_\beta') > u_\beta(x_\beta^*, y_\beta^*)$, and let us show that this implies that

$$p_x\, x_\beta' + p_y\, y_\beta' > p_x\, x_\beta^* + p_y\, y_\beta^* = p_x\, w_x + p_y\, w_y. \tag{3.2}$$

Namely, if it were $p_x\, x_\beta' + p_y\, y_\beta' \leq p_x\, x_\beta^* + p_y\, y_\beta^* = p_x\, w_x + p_y\, w_y$, agent β could choose (x_β', y_β') as consumption bundle, since it would belong to his/her budget set. However, since $u_\beta(x_\beta', y_\beta') > u_\beta(x_\beta^*, y_\beta^*)$, this would violate the fact that (x_β^*, y_β^*) solves the maximization problem (2.2). Thus (3.2) holds true.
Summing condition (3.1) for agent α multiplied by the share a with condition (3.2) multiplied by the share $1-a$, we obtain

$$p_x\left(ax_\alpha' + (1-a)x_\beta'\right) + p_y\left(ay_\alpha' + (1-a)y_\beta'\right) > p_x\left(ax_\alpha^* + (1-a)x_\beta^*\right) + p_y\left(ay_\alpha^* + (1-a)y_\beta^*\right),$$

which contradicts condition (i) in Definition 3.1.
Hence $(x_i^*, y_i^*)_{i\in\{\alpha,\beta\}}$ must be Pareto optimal, and this concludes the proof. □

From the previous result, we immediately obtain the following:

Corollary 3.1 *Given the economy* $E \in \mathcal{E}$*, let* (a^*, p^*, x_i^*, y_i^*)*,* $i \in \{\alpha, \beta\}$*, be a market stationary equilibrium with* $a^* \in (0,1)$*. Then the vector* (a^*, x_i^*, y_i^*)*, with* $i \in \{\alpha, \beta\}$*, is stationary Pareto optimal.*

Proof. Since (a^*, p^*, x_i^*, y_i^*), $i \in \{\alpha, \beta\}$, is a market stationary equilibrium, by Definition 2.3, $a^* \in (0,1)$ is constant and, given a^*, (p^*, x_i^*, y_i^*), $i \in \{\alpha, \beta\}$, is a market equilibrium for every t. Hence, by Proposition 3.1, the allocation $(x_i^*, y_i^*)_{i \in \{\alpha, \beta\}}$ is Pareto optimal in any time period t. By Definition 3.2 the vector (a^*, x_i^*, y_i^*), with $i \in \{\alpha, \beta\}$, is then stationary Pareto optimal.
The proof is complete. □

We conclude by recalling that at the end of Subsection III.A in Naimzada and Pireddu (2018) we gave evidence of the decreasing behavior, with respect to the corresponding population share, for each group of consumers, of the (Cobb-Douglas) utility level computed in correspondence to the optimal consumption quantities (cf. Figure 12 in Naimzada and Pireddu 2018). From this observation, due to the inverse proportionality between the two groups' shares, we drew the wrong conclusion that all stationary equilibria are Pareto inefficient, since each change in shares which made a group better off, would make the other group worse off.
The mistake in such argument lies in the fact that, according to Definition 2.2, the concept of equilibrium allocation is given *for a fixed value of the population shares*, and the same remark applies to Definition 2.3 of market stationary equilibrium, as well as to Definition 3.1 of efficient allocation and to Definition 3.2 of stationary Pareto optimal vector. Hence, it makes no sense to investigate the Pareto optimality of equilibrium allocations when the population shares vary. In particular, it is not correct to wonder whether some of the stationary equilibria are Pareto efficient considering them all together, as they are characterized by different population shares and thus they are not comparable one with the other.
In fact, it is not true that in our model all stationary equilibria are Pareto inefficient. On the contrary, we checked above the validity of the first theorem of welfare, according to which every (stationary) equilibrium allocation is (stationary) Pareto optimal.

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