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No. 435 – March 2020

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http://dems.unimib.it/

COMPLETE AND COMPETITIVE FINANCIAL MARKETS IN A COMPLEX WORLD

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ABSTRACT. We investigate the possibility of completing financial markets in a model with no exogenous probability measure and market imperfections. A necessary and sufficient condition is obtained for such extension to be possible.

JEL Classification: G10, G12.

1. INTRODUCTION.

Since the seminal contributions of Arrow [4] and of Radner [24], market completeness and the no arbitrage principle have played a prominent rôle in financial economics. Market completeness, as first noted by Arrow, is a crucial property as it permits the optimal allocation of risk bearing among risk averse agents. In fact the equilibria of an economy under conditions of uncertainty but with competitive and complete financial markets are equivalent to those of an ordinary static economy so that classical welfare theorems apply. The equilibrium analysis on which this conclusion rests requires that financial markets are free of arbitrage opportunities.

General equilibrium theory with financial markets, however, is traditionally cast in the framework of a finite state space (or at least of an infinite sequence economy with finitely many states at each date) in which an appropriate justification of market incompleteness is more difficult. Our model will assume a completely arbitrary set Ω as the sample space – a situation to which we shall refer as *complexity*. We believe that, despite the fast pace of financial innovation, the complexity of modern economic systems seems to be growing as fast which makes market completion an ongoing process. A natural consequence of this analysis is the assumption that in a complex world financial markets are incomplete. Given this general premise, the main questions we address in the paper are: (*a*) can an incomplete set of financial markets be extended to a complete one while preserving the basic economic principle of absence of arbitrage opportunities? (*b*) if so, can such an extension be supported by a competitive market mechanism?

Our answer is that this need not be the case. Competition on financial markets may in principle produce two distinct outcomes. On the first hand it lowers margins on currently traded assets and results thus in lower prices. On the other hand, competition involves the design and issuance of new securities. We argue that lower prices on the existing securities may destroy the possibility to obtain complete markets free of arbitrage opportunities. In principle the net effect of competition on collective welfare may be unclear. Second, we argue that the completion of financial markets in respect of the no arbitrage principle may not be possible under linear pricing (which we take as synonymous of perfect competition). We actually

Date: March 4, 2020.

Key words and phrases. Market completeness; No arbitrage; Market power; Sublinear pricing.

provide an explicit example. On the other hand we show that if such an extension is possible with a limited degree of market power, it is then possible under perfect competition as well.

We should make clear that, although it is indeed natural and appropriate on a general ground, to interpret the extension of markets as the effect of financial innovation, we do not model the strategic behaviour of intermediaries, as done, e.g., by Allen and Gale [2] or Bisin [7]. We rather study the properties of pricing functions described as a sublinear functional on the space of traded assets' payoffs. The non linearity of prices captures the non competitive nature of financial markets as well as the role of other market imperfections.

In addition to market power, our model departs from traditional financial literature inasmuch as it lacks of any particular mathematical structure, topological or measure theoretic. In particular, following the thread of our previous papers [10] and [13], we do not assume the existence of any exogenously given probability measure. Although this choice implies giving up the powerful artillery of stochastic analysis, particularly in continuous time, it permits, we believe, a better understanding of how financial markets work in a context of unrestricted complexity. A thorough discussion of the reasons supporting this choice may be found in [10].

In recent years there have been several papers in which the assumption of a given reference probability is relaxed, if not abandoned. Riedel [25] (and more recently Burzoni, Riedel and Soner [9]) suggests that an alternative approach to finance should be based on the concept of Knightian uncertainty. A typical implication of this approach is that a multiplicity of probability priors is given – rather than a single one. Some authors, including Bouchard and Nutz [8], interpret this multiplicity as an indication of *model uncertainty*, a situation in which each prior probability corresponds to a different model that possesses all the traditional properties but in which it is unknown which of the models should be considered the correct one. An exemplification is the paper by Epstein and Ji [20] in which model uncertainty simply translates into ambiguity concerning the volatility parameter. Other papers, among which the ones by Davis and Hobson [16] and by Acciaio et al [1], take the sample space to consist of all of the trajectories of some underlying asset and study the prices of options written thereon based on a path by path or *model-free* definition of arbitrage.

In our model, and similarly to Arrow's setting, contingent claims are described simply as *functions* of the sample space Ω. Differently from the papers mentioned above, this need not be a space of trajectories (and thus a Polish space) and the functions describing securities payoffs need not be continuous in any possible sense. Moreover, we do not adopt the pointwise definition of arbitrage suggested in [1], as this would implicitly correspond to assuming a form of rationality on economic agents even more extreme than probabilistic sophistication. Our starting point is rather a criterion of economic rationality embodied in a partial order which describes on what all agents agree when saying that "f is more than g". This modeling of economic rationality, first introduced in [13], is referred to as *common order* in [9].

In section 2 we describe the model in all details, we introduce the notion of arbitrage and prove some properties of prices. In section 3 we characterize the set of pricing measures and in the following section

4 we prove one of our main results, Theorem 1, in which the existence of market extensions is fully characterized. Then, in section 5 we establish a second fundamental result, Theorem 2, in which we give exact conditions for such an extension to be competitive. Several additional implications are proved. Given its importance in the reference literature, in section 6 we examine the question of countable additivity and eventually, in section 7 we return on the interpretation of the common order as a probabilistic ranking.

2. The Economy.

We model the market as a triple, $(\mathscr{X}, \geq_*, \pi)$, in which $\mathscr{X}$ describes the set of payoffs generated by the traded assets, $\geq_*$ the criterion of collective rationality used in the evaluation of investment projects and π is the price of each asset as a function of its payoff. Each of these elements will now be described in detail.

Before getting to the model we introduce some useful notation. Throughout Ω will be an arbitrary, non empty set that we interpret as the sample space so that the family $\mathfrak{F}(\Omega)$ of real valued functions on Ω will be our ambient space. If $A \subset \mathfrak{F}(\Omega)$, we write $\overline{A}^u$ to denote its closure in the topology of uniform distance. A class of special importance in $\mathfrak{F}(\Omega)$ is the family $\mathfrak{B}(\Omega)$ of bounded functions. The symbol $\mathbb{P}(\Omega)$ designates the collection of finitely additive, probability functions defined on the power set of Ω. All probabilities in this paper will be considered to be just finitely additive, unless explicitly indicated, in which case the symbol $\mathbb{P}$ is replaced with $\mathbb{P}_{ca}$. General references for the theory of finitely additive set functions and integrals are [18] and [5].

2.1. Economic Rationality.

A natural order to assign to $\mathfrak{F}(\Omega)$ is pointwise order, to wit $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$, also written as $f \geq g$. The lattice symbols $|f|$ or f^+ will always refer to such natural order.

Natural as it may appear, pointwise order is not an adequate description of how economic agents rank random quantities according to their magnitude, save when the underlying sample space is particularly simple, such as a finite set. For example, it is well documented that investors base their decisions on a rather incomplete assessment of the potential losses arising from the selected portfolios, exhibiting a sort of asymmetric attention that leads them to neglect some scenarios, in contrast with a pointwise ranking of investment projects[1]. In a complex world, in which the attempt to formulate a detailed description of Ω is out of reach, rational inattention is just one possible approach to deal with complexity. A different approach is the one followed in probability theory to reduce complexity by restricting to measurable quantities.

In this paper, following the thread of [13], we treat monotonicity as a primitive economic notion represented by a further transitive, reflexive binary relation on $\mathfrak{F}(\Omega)$. To distinguish it from the pointwise order $\geq$, we us the symbol $\geq_*$.

Let $\succeq_\alpha$ represent the preference system of agent α over all acts $\mathfrak{F}(\Omega)$ and assume that

$$(i).\quad 1 \succ_\alpha 0 \quad \text{and that} \quad (ii).\quad f \geq g \quad \text{implies} \quad f \succeq_\alpha g \tag{1}$$

i.e. that $\succeq_\alpha$ is non trivial and pointwise monotonic. Then an implicit, subjective criterion of monotonicity (or rationality), $\geq_\alpha$, may be deduced by letting

$$f \geq_\alpha g \quad \text{if and only if} \quad b(f-g) + h \succeq_a h \qquad b, h \in \mathfrak{F}(\Omega),\ b \geq 0. \tag{2}$$

[1] See [10] for a short discussion of some inattention phenomena relevant for financial decisions.

A mathematical criterion $\geq_*$ describing collective rationality may then be defined as the meet of all such individual rankings, i.e. as

$$f \geq_* g \quad \text{if and only if} \quad f \geq_\alpha g \quad \text{for each agent} \quad \alpha \tag{3}$$

the asymmetric part of $\geq_*$ will be written as $>_*$. One easily deduces the following, useful properties:

$$(i). \quad 1 >_* 0 \quad \text{and} \quad (ii). \quad f \geq g \quad \text{implies} \quad f \geq_* g, \tag{4a}$$

$$\text{if} \quad f \geq_* g \quad \text{then} \quad bf + h \geq_* bg + h, \qquad b \in \mathfrak{B}(\Omega)_+, h \in \mathfrak{F}(\Omega) \tag{4b}$$

$$\text{if} \quad f >_* 0 \quad \text{then} \quad f \wedge 1 >_* 0 \tag{4c}$$

which will be the basis for what follows[2].

It will be useful to remark that if $f \geq_* 0$ then, by property (4b), $f\mathbb{1}_{\{f\leq 0\}} \geq_* 0$ so that $f \geq_* f - 2f\mathbb{1}_{\{f\leq 0\}} = |f|$. Associated with $\geq_*$ is the collection of negligible sets

$$\mathscr{N}_* = \{A \subset \Omega : 0 \geq_* \mathbb{1}_A\} \tag{5}$$

together with the subset $\mathbb{P}(\Omega, \mathscr{N}_*) \subset \mathbb{P}(\Omega)$ which consists of probability measures which vanish on $\mathscr{N}_*$. Every subset of Ω not included in $\mathscr{N}_*$ will be called non negligible.

It is immediate to note that any *exogenously* given probability measure P (countably or finitely additive) induces a corresponding ranking defined as

$$f \geq_P h \quad \text{if and only if} \quad \inf_{\varepsilon>0} P(f > h - \varepsilon) = 1 \qquad f, h \in \mathfrak{F}(\Omega), \tag{6}$$

which satisfies the above axioms (4). The same would be true if P were replaced with a family $\mathscr{P} \subset \mathbb{P}(\Omega)$ and if we defined accordingly

$$f \geq_{\mathscr{P}} h \quad \text{if and only if} \quad f \geq_P h \quad \text{for all} \quad P \in \mathscr{P}. \tag{7}$$

The ranking $\geq_{\mathscr{P}}$ defined in (7) arises in connection with the *model uncertainty* approach mentioned in the Introduction and exemplified by the paper by Bouchard and Nutz [8]. In this approach each element P of the given collection $\mathscr{P}$ is a *model*[3].

A more interesting question concerns the conditions under which the ranking $\geq_*$ coincides with the ranking $\geq_P$ for some *endogenous* probability measure P. In this case we shall say that $\geq_*$ is *represented* by P. We shall address this question in the last section of this paper.

[2]The first paper to treat monotonicity in an axiomatic way was, of course, Kreps [22]. In a recent paper, Burzoni et al [9] adopt an approach quite similar to the present one. In [13, Theorem 1] we show that $\geq_*$ may arise from a cash sub additive risk measure.

[3]It should be noted that the choice of Bouchard and Nutz to take $\mathscr{P}$ to be a set of *countably additive* probabilities has considerable implications on $\mathscr{N}_*$ which needs e.g. be closed with respect to countable unions.

2.2. **Assets.** We posit the existence of an asset whose final payoff and current price are used as *numéraire* of the payoff and of the price of all other assets, respectively. Each asset is identified with its payoff expressed in units of the *numéraire* and is modelled as an element of $\mathfrak{F}(\Omega)$. The market is then a convex set $\mathscr{X} \subset \mathfrak{F}(\Omega)$ containing the origin as well as the function identically equal to 1 (that will be simply indicated by 1). Notice that we do not assume that investments may be replicated on any arbitrary scale, i.e. that $\mathscr{X}$ is a *convex cone*, as is customary in this literature.

We assume in addition that (*i*) each $f \in \mathscr{X}$ satisfies $f \geq_* a$ for some $a \in \mathbb{R}$ and (*ii*) that

$$f + \lambda \in \mathscr{X} \qquad f \in \mathscr{X},\ \lambda \geq 0. \tag{8}$$

The first of these assumptions constraints the assets traded on the market to bear a limited risk of losses and may be interpreted as a restriction imposed by some regulator; the second one permits agents, which in principle may only form convex portfolios, to invest into the *numéraire* asset an unlimited amount of capital. Notice that, since the *numéraire* cannot be shorted, the construction of zero cost portfolios – or *self-financing strategies* – is not possible. We are not assuming that the market prohibits short positions but rather that, in the presence of credit risk, long and short positions even if permitted should be regarded as two different investments as they bear potentially different levels of risk. In other words, when taking short positions, investors affect the implicit counterparty risk and modify *de facto* the final payoff of the asset shortened.

The issuance of new securities may result in the extension of the set $\mathscr{X}$ of traded assets. We may consider to this end several possibilities, varying from one another by the degree of completeness. A minimal extension is obtained when, along with each asset in $\mathscr{X}$, investors are permitted to take a short position in the corresponding call option (a strategy very common on the market and known as *call overwriting*). The resulting set of assets is

$$\mathscr{X}_1 = \big\{X \wedge k : X \in \mathscr{X},\ k \in \mathbb{R}_+ \cup \{+\infty\}\big\}. \tag{9}$$

To the other extreme, we have the case of complete markets. In a model with poor mathematical structure such as the one considered here, the definition of market completeness is not entirely obvious. The idea to define completeness as a situation in which all functions $f \in \mathfrak{F}(\Omega)$ are traded is indeed too ambitious, as it would be difficult to define a price function on such a large domain. We rather identify market completeness with the set

$$L(\mathscr{X}) = \big\{g \in \mathfrak{F}(\Omega) : \lambda X \geq_* |g| \text{ for some } \lambda > 0,\ X \in \mathscr{X}\big\} \tag{10}$$

which may be loosely interpreted as the set of superhedgeable claims. It is easily seen that $L(\mathscr{X})$ is a vector lattice containing $\mathscr{X}$ as well as $\mathfrak{B}(\Omega)$.

2.3. **Prices.** In financial markets with frictions and limitations to trade, normalized prices are best modelled as positively homogeneous, subadditive functionals of the asset payoff, $\pi : \mathscr{X} \to \mathbb{R}$, satisfying $\pi(1) = 1$ and the monotonicity condition

$$X, Y \in \mathscr{X},\ X \geq_* Y \quad \text{imply} \quad \pi(X) \geq \pi(Y). \tag{11}$$

We also require that prices be free of arbitrage opportunities, a property which we define as[4]

$$X \in \mathscr{X},\ X >_* 0 \quad \text{imply} \quad \pi(X) > 0. \tag{12}$$

Of course, (12) implies that $\pi(X) \geq 0$ whenever $X \geq_* 0$ while (11) need not follow from (12) if short selling is not permitted. We notice that the situation $X >_* \pi(X) > 0$, exceptional as it appears, does not represent in our model an arbitrage opportunity because of the infeasibility of short positions in the *numéraire* asset. A firm experiencing difficulties in raising funds for its projects and competing with other firms in a similar position may offer abnormally high returns to those who accept to purchase its debt.

A functional satisfying all the preceding properties – including (12) – will be called a *price function* and the corresponding set will be indicated with the symbol $\Pi(\mathscr{X})$. We thus agree that market prices are free of arbitrage by definition and we shall avoid recalling this crucial property. At times, though, it will be mathematically useful to consider pricing functionals for which the no arbitrage property (12) may fail. These will be denoted by the symbol $\Pi_0(\mathscr{X})$.

The non linearity of financial prices is a well known empirical feature documented in the microstructure literature (see e.g. the exhaustive survey by Biais et al. [6]) and essentially accounts for the auxiliary services that are purchased when investing in an asset, such as liquidity provision and inventory services. Subadditivity captures the idea that these services are imperfectly divisible.

Another important property of price functions is cash additivity, defined as[5]

$$\pi(X+a) = \pi(X) + a \qquad X \in \mathscr{X},\ a \in \mathbb{R} \quad \text{such that} \quad X + a \in \mathscr{X}. \tag{13}$$

(the collection of cash additive price functions will be denoted by $\Pi^a(\mathscr{X})$). Although $\pi \in \Pi(\mathscr{X})$ may fail to be cash additive, it always has a cash additive part π^a, i.e. the functional

$$\pi^a(X) = \inf_{\{t \in \mathbb{R}:\ X+t \in \mathscr{X}\}} \pi(X+t) - t \qquad X \in \mathscr{X}. \tag{14}$$

It is routine to show that π^a is the greatest element of $\Pi^a(\mathscr{X})$ dominated by π.

Notice that the same quantity defined in (14) may be computed for each element of the set $\hat{\mathscr{X}} = \mathscr{X} - \mathbb{R}$. If we denote by $\hat{\pi}^a$ the corresponding extension, $\hat{\pi}^a \in \Pi(\hat{\mathscr{X}})$ if and only if there exists no $\hat{X} \in \hat{\mathscr{X}}$ such that $\hat{X} >_* \hat{\pi}^a(\hat{X})$. This corresponds to the classical definition of absence of arbitrage, as given in the literature. This remark further clarifies the differences with our definition.

3. Pricing Measures

Associated with each price $\pi \in \Pi(\mathscr{X})$ is the space[6]

$$\mathscr{C}(\pi) = \big\{ g \in \mathfrak{F}(\Omega) : \lambda[X - \pi(X)] \geq_* g \text{ for some } \lambda > 0 \text{ and } X \in \mathscr{X} \big\}. \tag{15}$$

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[4] See [13] for a short discussion of alternative definitions of arbitrage in an imperfect market.

[5] This property, defined in slightly different terms, is discussed at length relatively to risk measures in [19]. In the context of non linear pricing cash additivity is virtually always assumed in a much stronger version, namely for all $X \in \mathscr{X}$ and all $a \in \mathbb{R}$, see e.g. [3, Definition 1].

[6] It is easily seen that $\mathscr{C}(\pi)$ is a convex cone containing $\mathscr{X}$.

and, more importantly, the collection of *pricing measures*[7]

$$\mathscr{M}_0(\pi)=\Big\{m\in\mathbb{P}(\Omega):L(\mathscr{X})\subset L^1(m)\text{ and }\pi(X)\geq\int Xdm\text{ for every }X\in\mathscr{X}\Big\}. \tag{16}$$

The following is a very basic result illustrating the role of cash additivity and of the set $\mathscr{M}_0(\pi)$.

Lemma 1. *For given $\pi\in\Pi_0(\mathscr{X})$ the set $\mathscr{M}_0(\pi)$ is non empty and each $m\in\mathscr{M}_0(\pi)$ satisfies*

$$\int fdm\geq\int gdm\qquad f,g\in L(\mathscr{X}),f\geq_* g. \tag{17}$$

Moreover, $\mathscr{M}_0(\pi)=\mathscr{M}_0(\pi^a)$ and

$$\pi^a(X)=\sup_{m\in\mathscr{M}_0(\pi)}\int Xdm\qquad X\in\mathscr{X}\cap\mathfrak{B}(\Omega). \tag{18}$$

Eventually, the set $\mathscr{M}_0(\pi)$ is convex and compact in the topology induced by $L(\mathscr{X})$.

Proof. We simply use Hahn-Banach and the representation

$$\phi(f)=\phi^\perp(f)+\int fdm_\phi\qquad f\in L(\mathscr{X}) \tag{19}$$

established in [14, Theorem 3.3] and valid for positive linear functionals on a vector lattice of real valued functions. In (19), $\phi^\perp$ is a positive linear functional on $L(\mathscr{X})$ with the property that $\phi^\perp(f)=0$ whenever $f\in\mathfrak{B}(\Omega)$ while m_ϕ is a positive, finitely additive measure on the power set of Ω such that $L(\mathscr{X})\subset L^1(m_\phi)$. Then, $m_\phi\in\mathbb{P}(\Omega)$ if and only if $\phi(1)=1$.

We easily realize that the functional defined by

$$\overline{\pi}(g)=\inf\big\{\lambda\pi(X):\lambda>0,\ X\in\mathscr{X},\ \lambda X\geq_* g\big\}\qquad g\in L(\mathscr{X}) \tag{20}$$

is an element of $\Pi_0(L(\mathscr{X}))$ extending π. By Hahn-Banach, we can find a linear functional ϕ on $L(\mathscr{X})$ such that $\phi\leq\overline{\pi}$ and $\phi(1)=1$. Necessarily, $f\geq_* 0$ implies $\phi(f)\geq 0$ so that, by (4b), ϕ is $\geq_*$-monotone and thus $m_\phi\in\mathbb{P}(\Omega)$. To show that $m_\phi\in\mathscr{M}_0(\pi)$ observe that, by assumption, each $X\in\mathscr{X}$ admits $a\in\mathbb{R}$ such that $X\geq_* a$ so that $\phi^\perp(X)\geq\phi^\perp(a)=0$ and thus $\pi(X)=\overline{\pi}(X)\geq\phi(X)\geq\int Xdm_\phi$. Suppose now that $m\in\mathscr{M}_0(\pi)$ and that $f,g\in L(\mathscr{X})$ and $f\geq_* g$. Then, by (4b), $\{f-g\leq-\varepsilon\}\in\mathscr{N}_*$ for all $\varepsilon>0$ so that

$$\int(f-g)dm=\int_{\{f-g>-\varepsilon\}}(f-g)dm\geq-\varepsilon.$$

We deduce (17) from $f,g\in L^1(m)$.

Concerning the claim $\mathscr{M}_0(\pi)=\mathscr{M}_0(\pi^a)$, it is clear that the inequality $\pi^a\leq\pi$ induces the inclusion $\mathscr{M}_0(\pi^a)\subset\mathscr{M}_0(\pi)$. However, if $X\in\mathscr{X}$, $t\in\mathbb{R}$ and $m\in\mathscr{M}_0(\pi)$ then $X+t\in\mathscr{X}$ implies

$$\pi(X+t)-t\geq\int(X+t)dm-t=\int Xdm$$

so that $m\in\mathscr{M}_0(\pi^a)$.

The cash additive part $\overline{\pi}^a$ of $\overline{\pi}$, obtained as in (14), is easily seen to be an extension of π^a to $L(\mathscr{X})$. Of course, $\overline{\pi}^a$ is the pointwise supremum of the linear functionals ϕ that it dominates so that (18) follows if

[7] In [13] a pricing measure was defined to be a positive, finitely additive measures dominated by π without restricting it to be a *probability*. The focus on probabilities will be clear after Theorem 1

we show that $m_\phi \in \mathbb{P}(\Omega)$ for all such ϕ. But this is clear since $\overline{\pi}^a \geq \phi$ implies that ϕ is positive on $L(\mathscr{X})$. Moreover,

$$\overline{\pi}^a(f) = \overline{\pi}^a(f+t) - t \geq \phi(f+t) - t = \phi(f) - t(1 - \|m_\phi\|) \qquad t \in \mathbb{R}$$

which contradicts the inequality $\overline{\pi}^a \geq \phi$ unless $m_\phi \in \mathbb{P}(\Omega)$.

The last claim is an obvious implication of Tychonoff theorem [18, I.8.5]. It is enough to note that $L(\mathscr{X})$ contains $\mathfrak{B}(\Omega)$ so that a cluster point of $\mathscr{M}_0(\pi)$ in the topology induced by $L(\mathscr{X})$ is necessarily represented by a finitely additive probability. □

Pricing measures closely correspond to the risk-neutral measures which are ubiquitous in the traditional financial literature since the seminal paper of Harrison and Kreps [21]. We only highlight that the existence of pricing measures and their properties are entirely *endogenous* here and do not depend on any special mathematical assumption – and actually not even on the absence of arbitrage. In traditional models, the condition $\mathscr{M}_0(\pi) \neq \varnothing$ is obtained via Riesz representation theorem (here replaced with (19)) and requires an appropriate topological structure. Also notice that (18) may be considered as our version of the superhedging Theorem. Upon re reading the preceding proof one deduces that a version of (18) may be obtained for π (rather than π^a) by replacing $\mathscr{M}_0(\pi)$ with the collection of finitely additive, positive *set functions* satisfying (16).

It is customary to interpret the integral

$$\int X dm \tag{21}$$

as the asset *fundamental value*, although the values obtained for each $m \in \mathscr{M}_0(\pi)$ chosen may differ significantly from one another. We cannot at present exclude the extreme situation of an asset $X \in \mathscr{X}$ such that $X >_* 0$,

$$\pi^a(X) > 0 \quad \text{but} \quad \sup_{m \in \mathscr{M}_0(\pi)} \int X dm = 0. \tag{22}$$

This case, which, in view of (18), requires X to be unbounded, describes an asset with no intrinsic value (no matter how computed) which still receives a positive market price. For this reason it would be natural to interpret such price as a pure bubble. In Theorem 1 we shall provide necessary and sufficient conditions which exclude pure bubbles.

Notice that bubbles need not always be pure. We define a bubble as the quantity:

$$\beta_\pi(X) \equiv \pi^a(X) - \lim_{k\to\infty} \pi^a(X \wedge k) = \pi^a(X) - \sup_{m \in \mathscr{M}_0(\pi)} \int (X \wedge k) dm \tag{23}$$

where we used (18) and the fact that

$$\lim_{k\to\infty} \sup_{m \in \mathscr{M}_0(\pi)} \int (X \wedge k) dm = \sup_{m \in \mathscr{M}_0(\pi)} \int (X \wedge k) dm.$$

Our definition of a bubble is thus quite conservative as it amounts to the minimum spread of the price over the fundamental value of the asset, no matter how computed.

Another possible failure of pricing measures under the current assumptions is that it may not be possible to support the view expressed in many microstructure models and according to which the *ask* price $\pi(X)$ of an asset is obtained by applying some mark-up to its fundamental value, such as

$$\pi(X) = [1+\alpha(X)]\int X dm \quad \text{with} \quad \alpha(X) \geq 0 = \alpha(1) \qquad X \in \mathscr{X}. \tag{24}$$

In fact (24) not only requires the absence of pure bubbles, but also the existence of a pricing measure with the property that $X >_* 0$ implies $\int X dm > 0$. This further property will be discussed at length in section 5.

4. Market Completeness.

Competition among financial intermediaries may involve existing assets and/or the launch of new financial claims. As a consequence it may produce two different effects: (*a*) a reduction of intermediation margins, and thus lower asset prices, and (*b*) an enlargement of the set $\mathscr{X}$ of traded assets, thus contributing to complete the markets. This short discussion justifies our interest for the set[8]

$$\mathrm{Ext}(\pi) = \left\{\pi' \in \Pi\left(L(\mathscr{X})\right) : \pi'|\mathscr{X} \leq \pi\right\}. \tag{25}$$

In this section we want to address the following question: *under what conditions is it possible to extend the actual markets to obtain an economy with complete financial markets without violating the no arbitrage principle?* This translates into the mathematical condition $\mathrm{Ext}(\pi) \neq \varnothing$ and if $\pi' \in \mathrm{Ext}(\pi)$ we speak of $(L(\mathscr{X}), \geq_*, \pi')$ as a completion of $(\mathscr{X}, \geq_*, \pi)$.

We obtain the following complete characterisation for the case of cash additive completions.

Theorem 1. *For a market $(\mathscr{X}, \geq_*, \pi)$ the following properties are mutually equivalent:*

(a). π satisfies

$$\overline{\mathscr{C}(\pi)}^u \cap \{f \in \mathfrak{F}(\Omega) : f >_* 0\} = \varnothing, \tag{26}$$

(b). the market $(\mathscr{X}, \geq_*, \pi)$ admits a cash-additive completion,

(c). the set $\mathscr{M}_0(\pi)$ is such that

$$\sup_{m \in \mathscr{M}_0(\pi)} \int (f \wedge 1) dm > 0 \qquad f >_* 0. \tag{27}$$

Proof. Assume that (26) holds and define the functional

$$\rho(f) = \inf\left\{\lambda\pi(X) - a : a \in \mathbb{R}, \lambda > 0, X \in \mathscr{X} \text{ such that } \lambda X \geq_* f + a\right\} \qquad f \in L(\mathscr{X}). \tag{28}$$

Property (4b) and $\geq_*$ monotonicity of π imply that $\rho \in \mathrm{Ext}_0(\pi^a)$. Moreover, it is easily seen that

$$\rho(f + x) = \rho(f) + x \qquad f \in L(\mathscr{X}),\ x \in \mathbb{R} \tag{29}$$

i.e. that ρ is cash additive. To prove that $\rho \in \Pi(L(\mathscr{X}))$, fix $f \in L(\mathscr{X})$ and $k \in \mathbb{N}$ such that $f_k = f \wedge k >_* 0$. Observe that $f_k \in L(\mathscr{X})$ and, in search of a contradiction, suppose that $\rho(f_k) \leq 0$. Then for each $n \in \mathbb{N}$

[8]Dropping the index 0 or adding the superscript a to Π will result in a similar transformation of Ext.

there exist $a_n \in \mathbb{R}$, $\lambda_n > 0$ and $X_n \in \mathscr{X}$ such that $\lambda_n X_n \geq_* f_k + a_n$ but $\lambda_n \pi(X_n) < 2^{-n} + a_n$. This clearly implies

$$\lambda_n[X_n - \pi(X_n)] \geq_* f_k - 2^{-n} \tag{30}$$

and $f_k \in \overline{\mathscr{C}(\pi)}^u$, contradicting (26). It follows that $\rho(f_k) > 0$ and that (*a*)⇒(*b*).

Choose $\rho \in \mathrm{Ext}^a(\pi)$ and let f_k be as above. Consider the linear functional

$$\hat{\phi}(x + bf_k) = x + b\rho(f_k) \qquad x, b \in \mathbb{R} \tag{31}$$

defined on the linear subspace $L_0 \subset L(\mathscr{X})$ spanned by $\{1, f_k\}$. Given that ρ satisfies (29), $\hat{\phi}$ is dominated by ρ on L_0 so that we can find an extension ϕ of $\hat{\phi}$ to the whole of $L(\mathscr{X})$ still dominated by ρ. As in Lemma 1, given that ϕ is a positive linear functional on a vector lattice, we obtain the representation (19) with $m_\phi \in \mathscr{M}_0(\rho) \subset \mathscr{M}_0(\pi)$. Moreover, $\phi(f_+^+) \leq \rho(f_k^+) = \rho(f_k) = \phi(f_k)$ so that $\phi(f_k) = \phi(f_k^+)$ by positivity. Eventually observe that, again by Lemma 1,

$$0 < \rho(f_k) = \rho(f_k^+) = \int f_k^+ dm_\phi = \int f_k dm_\phi.$$

Thus (*c*) follows from (*b*).

Assume now (*c*). Let $f \in \overline{\mathscr{C}(\pi)}^u$ be such that $f \geq_* 0$ and choose $n \in \mathbb{N}$ arbitrarily. Then there exist $\lambda^n > 0$ and $X^n \in \mathscr{X}$ such that $2^{-n} + \lambda^n[X^n - \pi(X^n)] \geq_* f$. Notice that this implies the inclusion $\overline{\mathscr{C}(\pi)}^u \cap \{f \in \mathfrak{F}(\Omega) : f \geq_* 0\} \subset L(\mathscr{X})$. But then, for every $m \in \mathscr{M}_0(\pi)$, Lemma 1 and (16) imply

$$\int f dm \leq 2^{-n} + \lambda^n \Big[\int X^n dm - \pi(X^n) \Big] \leq 2^{-n}$$

so that (*a*) follows. □

It is immediate to recognize a very close relationship between (26) and the *No-Free-Lunch-with-Vanishing-Risk* (*NFLVR*) notion formulated long ago by Delbaen and Schachermayer [17] in a highly influential paper. This similarity is quite surprising in view of the deep differences in the starting assumptions of the present model with theirs[9]. The main point is that, in our setting the elements of the form $\lambda[X - \pi(X)]$ with $\lambda > 0$ and $X \in \mathscr{X}$ cannot be interpreted as net payoffs of a corresponding trading strategy since the possibility of borrowing funds by shorting the *numéraire* is precluded as well as the strategy of replicating a given investment on an arbitrary scale. Notice also that in [17] the *NFLVR* condition was formulated in purely mathematical terms (and with reference to an exogenously given probability measure) while its economic content has remained largely unexplained.

Upon relating condition (26) with the existence of a strictly positive, cash additive extension of the pricing functional, Theorem 1 characterizes the economic role of *NFLVR* as a condition necessary and sufficient for financial markets to admit an extension that, while completing the family of assets traded, preserves the absence of arbitrage opportunities. The focus on the extension property of financial prices was clear in the papers by Harrison and Kreps [21] and Kreps [22] (see also [10, Theorem 8.1]) but has

[9] For example, one may remark in mathematical terms that although (26) makes use of the uniform topology (perhaps the one closest to the $L^\infty(P)$ one adopted in [17]) the set $\mathscr{C}(\pi)$ does not consist of bounded functions.

been somehow neglected in the following literature. Notice that a strictly positive extension may still exist even when (26) fails. In this case, however, it cannot be cash additive.

Notice that, in the light of the discussion following (22), the above condition (27) corresponds to a *No-Pure-Bubble* (*NPB*) condition while it does not exclude more general bubbles defined as in (23).

Incidentally we remark that, from the equality $\mathscr{M}_0(\pi) = \mathscr{M}_0(\pi^a)$, it follows that π satisfies condition (26) if and only if so does π^a. More precisely,

Lemma 2. *Let* $\pi \in \Pi_0(\mathscr{X})$. *Then: (a)* $\mathscr{C}(\pi) \subset \mathscr{C}(\pi^a) \subset \overline{\mathscr{C}(\pi)}^u$ *and (b) for every* $X \in \mathscr{X}$, $\pi^a(X) \leq 0$ *if and only if* $X \in \overline{\mathscr{C}(\pi)}^u$. *Therefore,* $\pi^a \in \Pi(\mathscr{X})$ *if and only if*

$$\overline{\mathscr{C}(\pi)}^u \cap \{X \in \mathscr{X} : X >_* 0\}. \tag{32}$$

Proof. (*a*). For each $X \in \mathscr{X}$ it is obvious that $X - \pi(X) \leq X - \pi^a(X)$. However, $X - \pi^a(X)$ is the limit, uniformly as $t \to +\infty$, of $X + t - \pi(X + t) \in \mathscr{C}(\pi)$. (*b*). $X \in \mathscr{X}$ and $\pi^a(X) \leq 0$ imply that for each $n \in \mathbb{N}$ and for $t_n > 0$ sufficiently large

$$X \leq X - \pi^a(X) \leq 2^{-n} + [X + t_n - \pi(X + t_n)]$$

so that $X \in \overline{\mathscr{C}(\pi)}^u$. Viceversa, if $X \leq 2^{-n} + \lambda_n[X_n - \pi(X_n)]$ for some $X_n \in \mathscr{X}$ and $\lambda_n \geq 0$, then, moving $\lambda_n \pi(X_n)$ to the left hand side if positive and using cash additivity, we conclude $\pi^a(X) \leq 2^{-n} + \lambda_n[\pi^a(X_n) - \pi(X_n)] \leq 2^{-n}$. □

To highlight the role of competition in financial markets, consider two pricing functions $\pi, \pi' \in \Pi(\mathscr{X})$. If $\pi \leq \pi'$ then $\mathscr{C}(\pi') \subset \mathscr{C}(\pi)$. Thus, lower financial prices are less likely to satisfy (26) and thus to admit an extension to a complete financial market free of arbitrage. Competition among market makers, producing lower spreads, may thus have two contrasting effects on economic welfare. On the one side it reduces the well known deadweight loss implicit in monopolistic pricing while, on the other, it imposes a limitation to financial innovation and its benefits in terms of the optimal allocation of risk. It may be conjectured that fully competitive pricing, interpreted as the pricing of assets by their fundamental value, may not be compatible with the extension property discussed here. We investigate this issue in the following section.

Eventually, we give a mathematical reformulation of Theorem 1.

Corollary 1. *For a market* $(\mathscr{X}, \geq_*, \pi)$ *the condition* (26) *is equivalent to the following:*

$$\bigcap_{m \in \mathscr{M}_0(\pi)} \overline{\mathscr{C}(\pi)}^{L^1(m)} \cap \{f \in \mathfrak{F}(\Omega) : f >_* 0\} = \varnothing. \tag{33}$$

Proof. Of course, the topology of uniform distance is stronger than the one induced by the $L^1(m)$ distance, for any $m \in ba(\Omega)$. Thus, (33) implies (26). On the other hand, if $m \in \mathscr{M}_0(\pi)$ then necessarily

$$\int f dm \leq 0 \qquad f \in \overline{\mathscr{C}(\pi)}^{L^1(m)}. \tag{34}$$

By Theorem 1, if (26) holds then the inequality (34) excludes the existence of $f \in \bigcap_{m \in \mathscr{M}_0(\pi)} \overline{\mathscr{C}(\pi)}^{L^1(m)}$ such that $f >_* 0$. □

Corollary 1 suggests the choice of a topology, $\tau(\pi)$, weaker than the one induced by the uniform distance and generated by the family of open sets[10]

$$O^\varepsilon_m(h) = \left\{f \in \mathfrak{F}(\Omega) : f - h \in L^1(m), \int |f - h| dm < \varepsilon\right\} \qquad h \in \mathfrak{F}(\Omega),\ m \in \mathscr{M}_0(\pi),\ \varepsilon > 0. \tag{35}$$

This topology has the advantage of being endogenously generated by market prices.

To close this section, we observe that a strictly related problem is whether markets my be extended, even if remaining incomplete. Much of what precedes remains true and we thus only give some hints for the case $\mathscr{X}_1$ defined in (9).

Corollary 2. *The market* $(\mathscr{X}, \geq_*, \pi)$ *admits a cash-additive extension* $(\mathscr{X}_1, \geq_*, \pi_1)$ *if and only if*

$$\overline{\mathscr{C}(\pi)}^u \cap \{f \in \mathscr{X}_1 : f >_* 0\} = \varnothing, \tag{36}$$

or, equivalently,

$$\sup_{m \in \mathscr{M}_0(\pi)} \int (f \wedge 1) dm > 0 \qquad f \in \mathscr{X}_1,\ f >_* 0. \tag{37}$$

5. Competitive Complete Markets.

In this section we investigate the conditions under which the set $\mathscr{M}_0(\pi)$ contains a strictly positive element, i.e. some m such that

$$\int (f \wedge 1) dm > 0 \qquad f >_* 0. \tag{38}$$

If such $m \in \mathscr{M}_0(\pi)$ may be found it is then clear that pricing each payoff in $L(\mathscr{X})$ by its fundamental value results in an arbitrage free price function. Given our preceding discussion this condition may be rightfully interpreted as the possibility of a fully competitive market completion. The subset of those $m \in \mathscr{M}_0(\pi)$ which satisfy (38) will be indicated by $\mathscr{M}(\pi)$[11]. The set $\mathscr{M}(\pi; \mathscr{X}_1)$ can be defined by restricting (38) to elements $f \in \mathscr{X}_1$.

Notice that if m satisfies (38) it is then strictly positive on any non negligible set while the converse is, in general, not true. In fact if $f >_* 0$ then axioms (4) imply that the set $\{f > 0\}$ is necessarily non negligible, but, since $\mathscr{N}_*$ need not be closed with respect to countable unions, they are not sufficient to exclude that $\{f > \varepsilon\} \in \mathscr{N}_*$ for all $\varepsilon > 0$.

In the preceding sections we interpreted the sublinearity of the pricing functional π as an indication of market imperfections, e.g. the market power of market makers. If $\rho \in \Pi_0(L(\mathscr{X}))$ a possible measure of market power is defined as follows (with the convention $0/0 = 0$):

$$\mathfrak{m}(\rho; f_1, \ldots, f_N) = \frac{\sum_{i \leq N} \rho(f_i) - \rho\big(\sum_{i \leq N} f_i\big)}{\sum_{i \leq N} \rho(f_i)} \quad \text{and} \quad \mathfrak{m}(\rho) = \sup \mathfrak{m}(\rho; f_1, \ldots, f_N) \tag{39}$$

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[10] It is in fact obvious that $\bigcap_{m \in \mathscr{M}_0(\pi)} \overline{A}^{L^1(m)} = \overline{A}^{\tau(\pi)}$ for all $A \subset \bigcap_{m \in \mathscr{M}_0(\pi)} L^1(m)$.

[11] In [25] a set function satisfying (38) was said to have *full support* and the emergence of measures of full support follows in that paper from the assumption that Ω is a complete, separable metric space and $\mathscr{X}$ consists of continuous functions defined thereon. See also [9].

the supremum in (39) being over all finite sequences of positive and bounded functions $f_1, \ldots, f_N \in \mathfrak{B}(\Omega)_+$. If $\rho \in \Pi(\mathscr{X}_1)$ we can define the quantity $\mathfrak{m}(\rho; \mathscr{X}_1)$ as in (39) but with $f_1, \ldots, f_N \in \mathscr{X}_1 \cap \mathfrak{B}(\Omega)_+$.

Clearly, $0 \leq \mathfrak{m}(\rho) \leq 1$ and the two extrema correspond to the polar cases of perfect competition and full monopoly. If, e.g., the price of each asset is set by applying a mark-up to its fundamental value, as in (24), then the maximum mark-up on the market provides an upper bound to $\mathfrak{m}(\rho)$ which is then strictly less than unity. On the other side, if prices include fixed costs, then $\mathfrak{m}(\rho)$ may well reach 1. As we shall see, the case $\mathfrak{m}(\rho) = 1$ is an extreme case of special importance.

The question we want to address next is: *given a market, is it possible to find a completion that permits some degree of competitiveness?* In symbols, this translates into the question of whether there exists $\rho \in \mathrm{Ext}(\pi)$ such that $\mathfrak{m}(\rho) < 1$. This condition has in fact far reaching implications.

Theorem 2. *A market $(\mathscr{X}, \geq_*, \pi)$ satisfies $\mathscr{M}(\pi) \neq \varnothing$ if and only if admits a cash additive completion $(L(\mathscr{X}), \geq_*, \rho)$ with $\mathfrak{m}(\rho) < 1$.*

Proof. Necessity is immediate. If $m \in \mathscr{M}(\pi)$, define $\rho : L(\mathscr{X}) \to \mathbb{R}$ by

$$\rho(f) = \int f dm \qquad f \in L(\mathscr{X}). \tag{40}$$

Then, ρ is cash additive and $\mathfrak{m}(\rho) = 0$, by linearity. Conversely, assume that $\rho \in \mathrm{Ext}(\pi)$ is cash additive and that $\mathfrak{m}(\rho) < 1$. Define for each $n \in \mathbb{N}$ the set

$$\mathscr{B}_n = \left\{ b \in L(\mathscr{X}) : 1 \geq b >_* 0 \text{ and } \rho(b) > 1/n \right\} \tag{41}$$

and notice that $\{f \in L(\mathscr{X}) : 1 \geq f >_* 0\} = \bigcup_n \mathscr{B}_n$, because $\rho \in \mathrm{Ext}(\pi)$. Denote by $\mathrm{co}(\mathscr{B}_n)$ the convex hull of $\mathscr{B}_n$. If $f = \sum_{i=1}^N w_i b_i \in \mathrm{co}(\mathscr{B}_n)$ then,

$$\rho(f) \geq \big(1 - \mathfrak{m}(\rho)\big) \sum_{i=1}^N w_i \rho(b_i) \geq \big(1 - \mathfrak{m}(\rho)\big)/n.$$

In view of the properties of $\mathscr{M}_0(\pi)$ proved in Lemma 1, we can then apply Sion minimax Theorem [26, Corollary 3.3] and obtain from (18)

$$0 < \inf_{f \in \mathrm{co}(\mathscr{B}_n)} \rho(f) = \inf_{f \in \mathrm{co}(\mathscr{B}_n)} \sup_{\mu \in \mathscr{M}_0(\rho)} \int f d\mu = \sup_{m \in \mathscr{M}_0(\rho)} \inf_{f \in \mathrm{co}(\mathscr{B}_n)} \int f d\mu.$$

Therefore, for each $n \in \mathbb{N}$ there exists $\mu_n \in \mathscr{M}_0(\rho)$ such that $\inf_{f \in \mathrm{co}(\mathscr{B}_n)} \int f d\mu_n \geq (1 - \mathfrak{m}(\rho))/2n > 0$. Define $m = \sum_n 2^{-n} \mu_n$. Then, $m \in \mathscr{M}_0(\rho) \subset \mathscr{M}_0(\pi)$ and, as a consequence, $L(\mathscr{X}) \subset L^1(m)$. But then, if $f >_* 0$ we conclude that $f \wedge 1 \in L(\mathscr{X})$, that $f \wedge 1 >_* 0$ and that $\int (f \wedge 1) dm > 0$ so that $m \in \mathscr{M}(\pi)$. □

What the preceding Theorem 2 asserts in words is that if a complete, arbitrage free market is possible under limited market power, it is then possible under perfect competition – i.e. with assets priced according to their fundamental value. This does not exclude, however, the somewhat paradoxical situation in which the only possibility to complete the markets is by admitting unlimited market power by financial intermediaries. As noted above, this describes the terms of a potential conflict between the effort of regulating the market power of intermediaries and the support to a process of financial innovation that does not disrupt market stability by introducing arbitrage opportunities.

Let us remark that the condition $\mathfrak{m}(\rho) < 1$, although economically sound, is not a trivial one, at least when the structure of non negligible sets is sufficiently rich i.e. when uncertainty is a complex phenomenon. Consider, e.g., the case in which an uncountable family of possible, alternative scenarios is given. In mathematical terms we can model this situation via an uncountable, pairwise disjoint collection $\{A_\alpha : \alpha \in \mathfrak{A}\}$ of non negligible subsets of Ω. Then, if $\rho \in \mathrm{Ext}(\pi)$ and $f_\alpha = \mathbb{1}_{A_\alpha}$ it must be that $\rho(f_\alpha) > 0$ for each $\alpha \in \mathfrak{A}$ and thus, for some appropriately chosen $\delta > 0$ and $\alpha_1, \alpha_2, \ldots \in \mathfrak{A}$,

$$\inf_n \rho(f_{\alpha_n}) > \delta. \tag{42}$$

But then, $\sum_{1\le n\le N}(1/N)\rho(f_{\alpha_n}) > \delta$ while $\rho\big(\sum_{1\le n\le N}(1/N)f_{\alpha_n}\big) \le 1/N$ so that $\mathfrak{m}(\rho) = 1$. Thus, in the case under consideration $\mathfrak{m}(\rho) = 1$. Let us also remark that in the probabilistic approach, in which $\mathscr{N}_*$ coincides with the collection of null sets of some *a priori* given probability, the existence of the collection $\{A_\alpha : \alpha \in \mathfrak{A}\}$ above is not possible[12].

Lemma 3. *Assume the existence of uncountably many, pairwise disjoint non negligible sets. Then for each market* $(\mathscr{X}, \ge_*, \pi)$ *(and with* $\inf \varnothing = 1$*),*

$$\inf_{\rho\in\mathrm{Ext}(\pi)} \mathfrak{m}(\rho) = 1 \tag{43}$$

The cash additive extensions ρ of π that satisfy the condition $\mathfrak{m}(\rho) < 1$ have special mathematical properties even if not free of arbitrage.

Theorem 3. *Let* $\rho \in \Pi_0(L(\mathscr{X}))$ *be cash additive and such that* $\mathfrak{m}(\rho) < 1$*. Then there exists* $\mu \in \mathscr{M}_0(\rho)$ *such that*

$$\lim_{\mu(|f|)\to 0} \rho(|f| \wedge 1) = 0. \tag{44}$$

Proof. For each α in a given set $\mathfrak{A}$, let $\langle A_n^\alpha\rangle_{n\in\mathbb{N}}$ be a decreasing sequence of subsets of Ω satisfying the following properties: (*i*) for each $\alpha, \beta \in \mathfrak{A}$ there exists $n(\alpha, \beta) \in \mathbb{N}$ such that

$$A_n^\alpha \cap A_n^\beta = \varnothing \qquad n > n(\alpha, \beta) \tag{45}$$

and (*ii*) for each $\alpha \in \mathfrak{A}$ there exists $m_\alpha \in \mathscr{M}_0(\rho)$ such that $\lim_n m_\alpha(A_n^\alpha) > 0$. If the set $\mathfrak{A}$ is uncountable, then, as in the preceding Lemma 3, we can fix $\delta > 0$ and extract a sequence $\alpha_1, \alpha_2, \ldots \in \mathfrak{A}$ such that

$$\inf_{i\in\mathbb{N}} \lim_{n\to+\infty} \rho\big(\mathbb{1}_{A_n^{\alpha_i}}\big) > \delta. \tag{46}$$

[12] In mathematics the condition that no uncountable, pairwise disjoint collection of non empty sets may be given, is known as the *countable chain* (CC) condition and was first formulated by Maharam [23]. See the comments in [15]. It is clear that in the following statement the collection $\{A_\alpha : \alpha \in \mathfrak{A}\}$ may be chosen to meet a weaker condition, namely that the pairwise intersections are negligible sets.

For each $k \in \mathbb{N}$ define $n(k) = 1 + \sup_{\{i,j \le k: i \ne j\}} n(\alpha_i, \alpha_j)$ and $f_i^k = \mathbb{1}_{A_{n(k)}^{\alpha_i}}$ for $i = 1, \ldots, k$. Then $f_1^k, \ldots, f_k^k \in \mathfrak{B}(\Omega)$ are pairwise disjoint functions with values in $[0,1]$ and such that

$$\inf_{1 \le i \le k} \rho(f_i^k) > \delta. \tag{47}$$

But then, taking $w_i = 1/k$, we obtain

$$\sum_{i=1}^{k} \rho(w_i f_i^k) > \delta \quad \text{while} \quad \rho\Big(\sum_{i=1}^{k} w_i f_i^k\Big) = \frac{1}{k}\rho\Big(\sum_{i=1}^{k} f_i^k\Big) \le \frac{1}{k} \tag{48}$$

so that $\mathfrak{m}(\rho) = 1$. We thus reach the conclusion that $\mathfrak{A}$ must be countable and deduce from this and from [15, Theorem 2] that $\mathscr{M}_0(\rho)$ is dominated by some of its elements, μ. In addition, $\mathscr{M}_0(\rho)$ is weak* compact as a subset of $ba(\Omega)$, as proved in Lemma 1. It follows from [28, Theorem 1.3] that $\mathscr{M}_0(\rho)$ is weakly compact. If μ does not dominate $\mathscr{M}_0(\rho)$ uniformly, we can then find a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ of subsets of Ω, a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{M}_0(\rho)$ and some constant $d > 0$ such that $\mu(E_n) \to 0$ while $m_n(E_n) > d$. Passing to a subsequence, we can assume that $\langle m_n \rangle_{n\in\mathbb{N}}$ is weakly convergent and so, by the finitely additive version of the Theorem of Vitali, Hahn and Saks (see e.g. [5, Theorem 8.7.4]), that the set $\{m_n : n \in \mathbb{N}\}$ is uniformly absolutely continuous with respect to $m_0 = \sum_n 2^{-n} m_n$ and, since $\mu \gg m_0$, with respect to μ as well which is contradictory. We conclude that

$$\lim_{\mu(A)\to 0} \sup_{m \in \mathscr{M}_0(\rho)} m(A) = 0. \tag{49}$$

Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence in $L(\mathscr{X})$ that converges to 0 in $L^1(\mu)$ in therefore in μ measure. Then, by (18)

$$\lim_n \rho(|f_n| \wedge 1) \le \lim_n \rho(\mathbb{1}_{\{|f_n|>c\}}) + c = \lim_n \sup_{m \in \mathscr{M}_0(\rho)} m(|f_n| > c) + c$$

so that the claim follows. □

Notice that Theorem 3 does not require the no arbitrage property and may thus be adapted to the case in which $L(\mathscr{X})$ is a generic vector lattice of functions on Ω containing the bounded functions and ρ a monotonic, subadditive and cash additive function, such as the Choquet integral with respect to a sub modular capacity.

Another characterization of the condition $\mathscr{M}(\pi) \ne \varnothing$ may be obtained as follows:

Theorem 4. *A market $(\mathscr{X}, \ge_*, \pi)$ satisfies the condition $\mathscr{M}(\pi) \ne \varnothing$ if and only if there exists $\mu \in \mathbb{P}(\Omega)$ such that*

$$\mathscr{X} \subset L^1(\mu) \quad \textit{and} \quad \overline{\mathscr{C}(\pi)}^{L^1(\mu)} \cap \{f \in \mathfrak{F}(\Omega) : f >_* 0\} = \varnothing. \tag{50}$$

In this case one may choose $\mu \in \mathscr{M}(\pi)$.

Proof. If $\mu \in \mathscr{M}(\pi)$ then, by definition, $\mathscr{X} \subset L^1(\mu)$ and $\int f d\mu \le 0$ for each $f \in \overline{\mathscr{C}(\pi)}^{L^1(\mu)}$ which rules out $f >_* 0$. Conversely, if $\mu \in \mathbb{P}(\Omega)$ satisfies (50) and $h >_* 0$, then $h \wedge 1 \in L^1(\mu)$ and $h \wedge 1 >_* 0$. There exists then a positive and continuous linear functional ϕ_h on $L^1(\mu)$ such that

$$\sup\Big\{\phi_h(f) : f \in \overline{\mathscr{C}(\pi)}^{L^1(\mu)}\Big\} \le 0 < \phi_h(h \wedge 1). \tag{51}$$

Given that necessarily $\phi_h(1) > 0$, (51) remains unchanged if we replace ϕ_h by its normalization so that we can assume $\phi_h(1) = 1$. This implies that $\phi_h \in \mathrm{Ext}_0(\pi)$ and, by [11, Theorem 2], that ϕ_h admits the representation

$$\phi_h(f) = \int f dm_h \quad f \in L^1(\mu) \tag{52}$$

for some $m_h \in \mathscr{M}_0(\pi)$ such that $L^1(\mu) \subset L(m_h)$ and $m_h \ll \mu$. Moreover, by exploiting the finitely additive version of Halmos and Savage theorem, [12, Theorem 1], we obtain that the set $\{m_h : h >_* 0\}$ is dominated by some $m_0 \in \mathscr{M}_0(\pi)$. It is then clear that $m_0(f \wedge 1) > 0$ for all $f >_* 0$ and thus that $m_0 \in \mathscr{M}(\pi)$. □

Let us mention that under finite additivity the existence of a strictly positive element of $\mathscr{M}_0(\pi)$ established in Theorem 2, is not sufficient to imply that the set $\mathscr{M}_0(\pi)$ is dominated, i.e. that each of its elements is absolutely continuous with respect to a given one. It rather induces the weaker conclusion that there is a given pricing measure m_0 such that $m_0(A) = 0$ implies $m(A) = 0$ for all $m \in \mathscr{M}_0(\pi)$.

On the other hand, if such a dominating element exists then, by weak compactness, it dominates $\mathscr{M}_0(\pi)$ uniformly. A similar conclusion is not true in the countably additive case treated in the traditional approach. In that approach, the set of risk neutral measures is dominated as an immediate consequence of the assumption of a given, reference probability measure but such set is not weakly* compact when regarded as a subset of the space of finitely additive measures. This special feature illustrates a possible advantage of the finitely additive approach over the countably additive one.

As above, we can formulate a version of the preceding results valid for partial extensions. Again, the proofs remain essentially unchanged. Let $\mathscr{X}_1$ be defined as in (9). Specializing definitions (39) and (38) by replacing $\mathfrak{F}(\Omega)$ with $\mathscr{X}_1$ we obtain the definitions of $\mathfrak{m}(\rho; \mathscr{X}_1)$ replacing $\mathfrak{m}(\rho)$ when $\rho \in \Pi(\mathscr{X}_1)$ and of $\mathscr{M}(\pi; \mathscr{X}_1)$ replacing $\mathscr{M}(\pi)$.

Corollary 3. *The market* $(\mathscr{X}, \geq_*, \pi)$ *satisfies* $\mathscr{M}(\pi; \mathscr{X}_1) \neq \varnothing$ *if and only if it admits a cash additive extension* $(\mathscr{X}_1, \geq_*, \rho_1)$ *with* $\mathfrak{m}(\rho_1; \mathscr{X}_1) < 1$.

6. Countably Additive Markets

Given the emphasis on countable additivity which dominates the traditional financial literature, it is natural to ask if it possible to characterise those markets in which the set $\mathscr{M}_0(\pi)$ contains a countably additive element. A more ambitious question is whether such measure is strictly positive, i.e. an element on $\mathscr{M}(\pi)$.

Not surprisingly, an exact characterisation may be obtained by considering the fairly unnatural possibility of forming portfolios which invest in countably many different assets. This induces to modify the quantity appearing in (39) into the following (again with the convention $0/0 = 0$):

$$\mathfrak{m}_c(\rho; f_1, f_2, \ldots) = \lim_{k \to +\infty} \frac{\sum_{n \leq k} \rho(f_n) - \rho\big(\sum_n f_n\big)}{\sum_{n \leq k} \rho(f_n)} \qquad \rho \in \Pi(L(\mathscr{X})) \tag{53}$$

for all sequences $\langle f_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{B}(\Omega)_+$ such that $\sum_n f_n \in \mathfrak{B}(\Omega)$. Notice that in principle, the inequality $\mathfrak{m}_c(\rho; f_1, f_2, \ldots) \geq 0$ is no longer valid while, of course, the inequality $\mathfrak{m}_c(\rho; f_1, f_2, \ldots) \leq 1$ is still true. It may at first appear obvious that, upon buying separately each component of a given portfolio, the investment cost results higher, but considered more carefully, this is indeed correct only if the infinite sum $\sum_n \rho(f_n)$ corresponds to an actual cost, i.e only if such a strategy of buying separately infinitely many assets is feasible on the market.

Define then the functionals

$$\mathfrak{m}_c(\rho) = \sup \mathfrak{m}_c(\rho; f_1, f_2, \ldots) \quad \text{and} \quad \mathfrak{n}_c(\rho) = -\inf \mathfrak{m}_c(\rho; f_1, f_2, \ldots) \tag{54}$$

where both the supremum and the infimum are computed with respect to all sequences in $\mathfrak{B}(\Omega)_+$ with bounded sum. Notice that $\mathfrak{m}_c(\rho) = \mathfrak{m}(\rho)$.

Theorem 5. *Let $\pi \in \Pi_0(\mathscr{X})$. Then:*

(a). $\mathscr{M}_0(\pi) \cap \mathbb{P}_{ca}(\Omega) \neq \varnothing$ if and only if there exists $\rho \in \mathrm{Ext}_0(\pi)$ such that $\mathfrak{n}_c(\rho) < +\infty$ and that

$$\sum_n \rho(f_n) < \infty \quad \textit{for all} \quad f_1, f_2, \ldots \in \mathfrak{B}(\Omega)_+ \quad \textit{with} \quad \sum_n f_n \in \mathfrak{B}(\Omega); \tag{55}$$

(b). $\mathscr{M}(\pi) \cap \mathbb{P}_{ca}(\Omega) \neq \varnothing$ if and only if there exists $\rho \in \mathrm{Ext}(\pi)$ such that $\mathfrak{n}_c(\rho) < +\infty$ and $\mathfrak{m}_c(\rho) < 1$.

Proof. Of course any element $m \in \mathscr{M}_0(\pi) \cap \mathbb{P}_{ca}(\Omega)$ when considered as a pricing function is an element of $\mathrm{Ext}_0(\pi)$ such that $\mathfrak{n}_c(m) = \mathfrak{m}_c(m) = 0$. This proves necessity for both claims. To prove sufficiency, let $\rho \in \Pi_0(L(\mathscr{X}))$ and choose a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}(\Omega)_+$ with $\sum_n f_n \in \mathfrak{B}(\Omega)$. If $\mathfrak{m}_c(\rho) < 1$, then

$$\sum_n \rho(f_n) \leq \frac{1}{1 - \mathfrak{m}_c(\rho)} \rho\Big(\sum_n f_n\Big) < \infty \tag{56}$$

so that (55) is satisfied. It is therefore enough to show that if ρ meets the conditions listed under (*a*) then $\mathscr{M}_0(\rho) \subset \mathbb{P}_{ca}(\Omega)$. Assume to this end that $\rho \in \mathrm{Ext}_0(\pi)$ is such a function, choose $m \in \mathscr{M}_0(\rho)$ and let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a disjoint sequence of subsets of Ω. Then, upon setting $f_n = \mathbb{1}_{A_n}$, we get $\sum_n \rho(f_n) < \infty$ and therefore

$$\begin{aligned} m\Big(\bigcup_n A_n\Big) &= \sum_n m(A_n) + \lim_k m\Big(\bigcup_{n>k} A_n\Big) \\ &\leq \sum_n m(A_n) + \lim_k \rho\Big(\sum_{n>k} f_n\Big) \\ &\leq \sum_n m(A_n) + [1 + \mathfrak{n}_c(\rho)] \lim_k \sum_{n>k} \rho(f_n) \\ &= \sum_n m(A_n). \end{aligned}$$

Thus, $\mathscr{M}_0(\rho) \subset \mathbb{P}_{ca}(\Omega)$. If, $\mathfrak{m}_c(\rho) < 1$ then $\mathfrak{m}(\rho) < 1$ as well so that claim (*b*) follows from the preceding remarks and Theorem 2. ☐

The conditions for the existence of a countably additive pricing measure listed under (*a*) and (*b*) are perhaps deceptively simple. In fact the inequality $\mathfrak{n}_c(\rho) > -\infty$ implies that, if $\langle f_n \rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of negligible functions, then necessarily $\sup_n f_n$ has to be negligible as well. Thus, e.g., $\mathscr{N}_*$ has to be closed with respect to countable unions. This property requires a rather deep reformulation of the axioms (4) that characterize economic rationality, $\geq_*$, and there may well be cases in which such additional conditions are simply contradictory. If, e.g., Ω is a separable metric space and $\mathscr{N}_*$ consists of sets of first category then, as is well known, $\mathbb{P}_{ca}(\Omega, \mathscr{N}_*) = \varnothing$, see [27, Théorème 1].

In the classical setting of continuous time finance, the order $\geq_*$ is generated by some countably additive probability and each $X \in \mathscr{X}$ is of the form $X = W_0^X + \int \theta^X dS$ where S is, e.g., a locally bounded semimartingale with respect to the given probability measure, θ is a predictable process integrable with respect to S and its price, in the absence of frictions, is $\rho(X) = W_0^X$. Although the condition $\mathfrak{m}_c(\rho) = 0$ is immediate the inequality $\mathfrak{n}_c(\rho) > -\infty$ is not at all obvious. In fact, even when $0 \leq X_n \leq X_{n+1}$ and $X_0 = \lim_n X_n \in \mathfrak{B}(\Omega)$ it is not easy to show that there is θ^0 such that

$$\sum_n \int \theta^{X_n} dS = \int \theta^0 dS. \tag{57}$$

7. Monotonicity

Let us return to the common order $\geq_*$ with which we started. Given the discussion in the introduction, it is natural to ask whether $\geq_*$ arises from some probability or some family of probabilities i.e whether $\geq_*$ is of the form (6) or (7). This may be regarded as a decision theoretic issue, completely independent from financial theory. The answer to this question is, in the general case, in the negative.

First we notice that $\mathscr{N}_*$ is closed with respect to subsets and to union, i.e. it is an ideal (of sets). For fixed $A \notin \mathscr{N}_*$ the family $\{A \setminus N : N \in \mathscr{N}_*\}$ is a filter and it is thus possible to construct an ultrafilter probability P_A assigning unit mass to each element in it. Thus $P_A \in \mathbb{P}(\Omega, \mathscr{N}_*)$ and $P_A(A) = 1$ and so $\mathscr{N}_*$ coincides with the intersection of the null sets of the collection $\mathscr{P} = \{P_A : A \notin \mathscr{N}_*\}$. Needless to say, it may well be that none of the elements of $\mathscr{P}$ is countably additive (or even has a non trivial countably additive part).

Second, even letting $\mathscr{P}$ be a family of probability functions with the property that $A \in \mathscr{N}_*$ if and only if $\sup_{P \in \mathscr{P}} P(A) = 0$, this would not imply that $\geq_*$ coincides with the ranking $\geq_{\mathscr{P}}$ as defined in (7) – unless the elements of $\mathscr{P}$ are countably additive. To this end we need an additional condition:

$$\text{if} \quad f + \varepsilon >_* 0 \quad \text{for all} \quad \varepsilon > 0 \quad \text{then} \quad f \geq_* 0 \qquad f \in \mathfrak{F}(\Omega). \tag{4d}$$

Indeed it is easily seen that upon adding (4d) to (4a) – (4c), $f \geq_* 0$ is equivalent to $\{f < -\varepsilon\} \in \mathscr{N}_*$ for all $\varepsilon > 0$ so that $\geq_*$ coincides with $\geq_{\mathscr{P}}$ for any $\mathscr{P} \subset \mathbb{P}(\Omega)$ with the above property.

Theorem 6. *Let the partial order $\geq_*$ satisfy properties (4a)–(4c). Then,*

(a). $\geq_*$ may be represented by some $P \in \mathbb{P}(\Omega)$ if and only if (i) $\geq_*$ satisfies (4d) and (ii) there exists $\rho \in \Pi(\mathfrak{B}(\Omega))$ with $\mathfrak{m}(\rho) < 1$;

(b). $\geq_*$ may be represented by some $P \in \mathbb{P}_{ca}(\Omega)$ if and only if, in addition to (i) and (ii) above, ρ satisfies (iii) $\mathfrak{n}_c(\rho) < +\infty$.

Proof. If $P \in \mathbb{P}(\Omega)$ represents $\geq_*$ then (4d) is necessarily true. Moreover, identifying P with an element of $\Pi(\mathfrak{B}(\Omega))$, condition (*ii*) holds. If in addition P is countably additive, then (*iii*) is also true. This proves necessity for both claims. Conversely, assume that (*i*) and (*ii*) hold. By Theorem 2 there exists $P \in \mathscr{M}(\rho)$.

If (*iii*) is also true, then we deduce from Theorem 5 that such P is countably additive. The proof is complete upon noting that in either case $\mathscr{N}_*$ coincides with P null sets and that, as remarked above, $f \geq_* 0$ if and only if $\{f \leq -\varepsilon\} \in \mathscr{N}_*$. □

We derive from this simple result the surprising conclusion that, assuming (4d), then in order for the existence of a cash additive extension of the price system it is necessary that economic rationality is defined in probabilistic terms. It was noted in [13, p. 546] that condition (4d) should be viewed as a robustness criterion of the statement $f \geq_* 0$ in the face of measurement errors of arbitrarily small magnitude.

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