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Ahmad Naimzada and Marina Pireddu

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Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

The opposite effect of rational expectations and differentiated information costs for heterogeneous fundamentalists on the stability of an evolutive Muthian cobweb model

Ahmad Naimzada [a*], Marina Pireddu [b†]

[a] Dept. of Economics, Management and Statistics, University of Milano - Bicocca,

U6 Building, Piazza dell'Ateneo Nuovo 1, 20126 Milano, Italy.

[b] Dept. of Mathematics and its Applications, University of Milano - Bicocca,

U5 Building, Via Cozzi 55, 20125 Milano, Italy.

We extend the evolutionary cobweb setting proposed in Hommes and Wagener (2010), in which the share updating mechanism is based on a comparison among the profits realized by the various kinds of agents, by assuming that the market is populated by rational producers, endowed with perfect foresight expectations about prices, in addition to biased and unbiased fundamentalists. Moreover, we suppose that agents face heterogeneous information costs, that are proportional to their rationality degree. Since introducing rational agents enlarges the stability region of the steady state, while considering diversified information costs for fundamentalists shrinks it, we analyze whether one of the two aspects always prevails over the other one when they are jointly taken into account. We also investigate if the chaotic phenomena emerging when enriching the original framework in Hommes and Wagener (2010) with rational agents persist or are inhibited by the introduction of information costs for all agent types. We complete our analysis by studying the network of the relationships among the four settings obtained possibly considering information costs for biased and unbiased fundamentalists and possibly introducing rational agents.

*Tel.: +39 0264485813. E-mail address: ahmad.naimzada@unimib.it

†Tel.: +39 0264485767. E-mail address: marina.pireddu@unimib.it

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1 Introduction

This contribution belongs to the research strand started by Hommes and Wagener (2010) and further developed in Naimzada and Pireddu (2020a,b), proposing here a more general framework, which encompasses the previous three settings.

In Hommes and Wagener (2010) a Muthian cobweb model[1] is considered in which producers can choose between being biased or unbiased fundamentalists.[2] In more detail, biased fundamentalists may be optimists (resp. pessimists) if they predict that the price of the good will always be above (resp. below) the fundamental price, while unbiased fundamentalists predict that prices will always be at their fundamental value. Hommes and Wagener (2010) assume that population shares evolve according to the updating rule adopted in Brock and Hommes (1997) for the case without memory, being based just on a comparison among the profits realized in the previous period by the various kinds of agents. In Hommes and Wagener (2010) it is supposed that all agents face a common zero information cost and focus on the case in which the model is globally eductively stable in the sense of Guesnerie (2002), being stable under naive expectations. Hommes and Wagener (2010) show that the unique steady state, which coincides with the fundamental, is always stable and may coexist with a locally stable period-two cycle.

We extended the framework proposed in Hommes and Wagener (2010) in two different directions in Naimzada and Pireddu (2020a,b).

Namely, in Naimzada and Pireddu (2020a) we assumed that biased and unbiased fundamentalists face heterogeneous information costs that are inversely proportional to the degree of their bias, finding that the equilibrium, when globally eductively stable, may be unstable under evolutionary learning. Therefore, we discovered that the introduction of differentiated infor-

[1]The cobweb model has been widely considered in the literature. See for instance Chapters 4 and 5 in Hommes (2013) and Chapters 2, 3 and 5 in Onozaki (2018), as well as Hommes (2018) for a recent survey on cobweb dynamics.

[2]Other forms of heterogeneity have been studied, in similar contexts, e.g. in Levi et al. (2018) and in Onozaki et al. (2000, 2003).

mation costs has, in the considered setting, a destabilizing effect.
On the other hand, in Naimzada and Pireddu (2020b) we extended the model in Hommes and Wagener (2010) by introducing rational producers, with perfect foresight expectations about prices, which face an information cost like in Brock and Hommes (1997). We found that, locally, adding rational agents enlarges the stability region of the steady state, but globally, with their introduction, the map governing the dynamics, differently from the original setting in Hommes and Wagener (2010) and also from the framework in Naimzada and Pireddu (2020a), is no more monotonically decreasing, and interesting dynamic phenomena can arise, although the unique steady state still coincides with the fundamental value. Hence, in Naimzada and Pireddu (2020b) we discovered that considering rational agents allows for complex dynamic outcomes, characterized by chaotic attractors and multistability phenomena.
Therefore, focusing at first on the equilibrium local stability issue, since heterogeneous information costs for fundamentalists have a destabilizing effect on the steady state, while rational agents play a stabilizing role, in the present work we consider both ingredients, in order to understand if one of the two effects prevails over the other one when they are jointly taken into account. The results that we obtain through our local analysis show that there is not a unique answer, since the overall effect on the size of the local stability region of the steady state depends on the relative values of information costs. More precisely, dealing with rational agents and nonnull information costs for all agent types, the stability region increases with respect to the setting in Hommes and Wagener (2010) e.g. when biased fundamentalists face high information costs, so that their destabilizing role is dampened by the fall in their share, due to their reduced profits. Vice versa, the stability region may decrease with respect to the framework in Hommes and Wagener (2010) e.g. when unbiased fundamentalists and rational agents face high information costs, so that their stabilizing effect is reduced by the decreased share, because of the lower profits they realized. In fact, we shall see that also the comparison between the stability regions with and without information costs for fundamentalists, when rational agents are taken into account, does not give a unique answer, differently from the case in which rational agents were not considered, where the introduction of information costs was always destabilizing. Moreover, with respect to the framework analyzed in Naimzada and Pireddu (2020a), which encompassed heterogeneous information costs for biased and unbiased fundamentalists but without rational agents, we find that the introduction of the latter class of agents enlarges

the stability region. Hence, we obtain a confirmation of the stabilizing effect produced by rational agents (see Hommes 2013), even when information costs for fundamentalists are taken into account, similar to what discovered in Naimzada and Pireddu (2020b), in the absence of information costs.
In order to better understand the complete network of the relationships among the four settings obtained possibly considering information costs for biased and unbiased fundamentalists and possibly introducing rational agents, we conclude our analysis by performing a detailed comparison among the various stability conditions.
In addition, we also investigate if the complex dynamic outcomes emerging when enriching the original framework in Hommes and Wagener (2010) with rational agents, detected in Naimzada and Pireddu (2020b), persist or are inhibited by the introduction of information costs for all agent types. In this respect we find that, although the overall dynamic complexity of the system seems not altered by the introduction of information costs for fundamentalists, in the sense that the routes to chaos and the chaotic attractors look analogous with and without costs for them, we usually witness a reduced complexity degree with nonnull information costs for all agent types. Nonetheless, it is sometimes possible to observe reversed scenarios, in which longer-lasting attractors are obtained when fundamentalists face nonnull information costs, too.
The remainder of the paper is organized as follows. In Section 2 we introduce the framework encompassing rational agents and information costs for all agent types. Section 3 provides a summary of the aspects about the settings without rational agents or information costs for fundamentalists, needed in view of the analysis to be performed in Section 4. In Section 5 we discuss the obtained results and describe the mutual relationships among the four considered settings. In Section 6 we conclude, illustrating possible modeling extensions. Appendix A contains the proof of our main result, while Appendix B encompasses a detailed investigation of the role played by information costs.

2 Introducing the setting with rational agents and information costs

In Naimzada and Pireddu (2020a) we extended the discrete-time evolutionary cobweb setting proposed in Hommes and Wagener (2010) by introducing information costs for biased and unbiased fundamentalists, while in Naimzada and Pireddu (2020b) we enriched the original setting in Hommes and Wagener (2010) by dealing also with rational agents, that are assumed to be the only kind of agents facing information costs, similar to what happened in Brock and Hommes (1997), where rational and naive agents were encompassed. We now jointly consider rational agents and information costs for all agent types, dealing with the simplest case in which, in addition to unbiased fundamentalists, there are two symmetric groups of optimists and pessimists, that share the same bias, but that respectively overestimate and underestimate the price of the good they produce. Also in Naimzada and Pireddu (2020b), where just rational agents faced an information cost, we confined ourselves to the case in which the economy is populated by one couple of groups of symmetrically biased agents, because we therein aimed at showing the dynamic phenomena arising when introducing rational agents in the original setting by Hommes and Wagener (2010). In the same vein, since in the present contribution we will investigate the dynamic outcomes generated by the joint introduction in the framework by Hommes and Wagener (2010) of rational agents and of information costs for all agent types, we assume that the economy is populated by two symmetric groups of optimists and pessimists, like done in Naimzada and Pireddu (2020a), too, where information costs for fundamentalists were taken into account, but rational agents were not considered.

As concerns the Muthian cobweb model, agents have to choose the quantity q of a certain good to produce in the next period in order to maximize their expected profits. Like in Hommes and Wagener (2010), we assume that the supply curve is given by

$$S(p^e) = sp^e, \tag{2.1}$$

where p^e is the expected price and $s > 0$ describes its slope, deriving from the quadratic cost function

$$\gamma(q) = \frac{q^2}{2s}\,. \tag{2.2}$$

The demand function is supposed to be given by

$$D(p) = A - dp.$$

It is positive for large enough values of the parameter A, measuring the market size, whereas $d > 0$ represents the slope of the demand function.
Agents' expectations about the price of the good they have to produce are heterogeneous.[3] In particular, characterizing unbiased fundamentalists, pessimists and optimists by subscripts 0, 1, 2, respectively, their expectations at time t are given by

$$p^e_{i,t} = p^* + b_i, \ i \in \{0, 1, 2\}, \text{ with } b_0 = 0, \, b_1 = -b, \, b_2 = b,$$

where $b > 0$ describes the bias degree of pessimists and optimists, who are symmetrically disposed. The fundamental price

$$p^* = \frac{A}{d + s}$$

can be found equating demand and supply in the case of perfect foresight. Focusing on bias values $b \in (0, p^*)$ prevents the possibility of negative expectations for pessimists.
As done in Naimzada and Pireddu (2020b), we assume that agents may also be rational, correctly predicting the next period price, so that, denoting them by subscript -1, their expectation at time t is given by

$$p^e_{-1,t} = p_t.$$

We suppose that all agents face an information cost $c_i \geq 0$, so that their net profits are described by

$$\pi_{i,t} = p_t S(p^e_{i,t}) - \gamma(S(p^e_{i,t})) - c_i$$

for $i \in \{-1, 0, 1, 2\}$, with S and γ introduced respectively in (2.1) and (2.2). Taking into account the rationality degree of the various kinds of agents, we assume the following ordering for information costs

$$0 \leq c_1 = c_2 = c \leq c_0 < c_{-1} \,. \tag{2.3}$$

[3]Models with agents heterogeneous in their expectations or in the kind of information they use have been proposed and analyzed e.g. in Naimzada and Ricchiuti (2008, 2009) and in Matsumoto and Szidarovszky (2015), respectively.

Namely, optimists and pessimists make symmetric errors in estimating the economic fundamentals, while unbiased fundamentalists know the exact formulation of the demand and supply functions and, behaving as if all other agents were endowed with perfect, they can correctly compute the fundamental value. However, they do not perceive agents' heterogeneity and, differently from rational agents, they cannot predict the next period price.
The share updating rule[4] is based on a comparison among the net profits realized in the previous period by the various kinds of agents. In particular, following Hommes and Wagener (2010), we deal with the discrete choice model in Brock and Hommes (1997) for the case without memory, whose formulation is given by

$$\omega_{i,t} = \frac{\exp(\beta \pi_{i,t-1})}{\sum_{j=-1}^{2} \exp(\beta \pi_{j,t-1})}, \; i \in \{-1, 0, 1, 2\},$$

where $\beta > 0$ is the intensity of choice parameter and $\omega_{i,t}$ denotes the share of agents choosing the forecasting rule $i \in \{-1, 0, 1, 2\}$ at time t.
Starting from the market equilibrium condition at time t

$$A - dp_t = \sum_{i=-1}^{2} \omega_{i,t} S(p^e_{i,t})$$

and introducing the variable $x_t = p_t - p^* = p_t - \frac{A}{d+s}$, the model dynamic equation can be written in deviation from the fundamental as

$$x_t = k(x_{t-1}) = \frac{bs}{d + \omega_{-1,t} s}(\omega_{1,t} - \omega_{2,t}). \tag{2.4}$$

In more explicit terms, the function $k : (-p^*, +\infty) \to \mathbb{R}$ is defined as

$$\begin{aligned} k(x) &= \frac{bs\Big(\exp\Big(-\frac{\beta s}{2}(x+b)^2-\beta c\Big)-\exp\Big(-\frac{\beta s}{2}(x-b)^2-\beta c\Big)\Big)}{d\Big(\exp\Big(-\frac{\beta s}{2}(x+b)^2-\beta c\Big)+\exp\Big(-\frac{\beta s}{2}(x-b)^2-\beta c\Big)+\exp\Big(-\frac{\beta s}{2}x^2-\beta c_0\Big)\Big)+(d+s)\exp(-\beta c_{-1})} \\ &= \frac{bs\Big(\exp\Big(-\frac{\beta s}{2}(x+b)^2\Big)-\exp\Big(-\frac{\beta s}{2}(x-b)^2\Big)\Big)}{d\Big(\exp\Big(-\frac{\beta s}{2}(x+b)^2\Big)+\exp\Big(-\frac{\beta s}{2}(x-b)^2\Big)+\exp\Big(-\frac{\beta s}{2}x^2-\beta(c_0-c)\Big)\Big)+(d+s)\exp(-\beta(c_{-1}-c))}. \end{aligned} \tag{2.5}$$

[4] See Anufriev et al. (2013) for an evolutive macroeconomic framework encompassing biased and unbiased agents. We recall that evolutive mechanisms can be found in continuous-time settings, too. Cf. e.g. Antoci et al. (2019a,b) for recent contributions in this sense.

We will prove in Proposition 2.1 that, for the model formulation in terms of x_t in (2.4), the unique steady state is represented by $x^* = 0$, like it was in the original framework considered in Hommes and Wagener (2010), as well as in the extended settings in Naimzada and Pireddu (2020a,b). We notice that k in (2.5) is differentiable and, when extending its domain to $\mathbb{R}$ by considering larger and larger values of A, it is odd and admits the x-axis as horizontal asymptote for $x \to \pm\infty$. Hence, k is not monotone and this allows for chaotic phenomena.
Before illustrating some possible scenarios for (2.4) in Fig. 3, we state the following result, whose proof is given in Appendix A:

Proposition 2.1 *The only steady state for equation* (2.4) *is given by* $x = 0$. *The equilibrium* $x = 0$ *is locally asymptotically stable for map* k *in* (2.5) *if*

$$b^2 < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right) + (d+s)\exp\left(\frac{\beta b^2 s}{2} - \beta(c_{-1} - c)\right)}{2\beta s^2}. \quad (2.6)$$

Thus, $x = 0$ *may be stable for any* $b > 0$ *or, depending on the considered parameter values, there may exist* $0 < b'_{CR} \leq b''_{CR}$ *such that* $x = 0$ *is stable for each* $b \in (0, b'_{CR}) \cup (b''_{CR}, +\infty)$.

We stress that, if instead of b, in Proposition 2.1 we dealt with β, then we would also find the destabilizing scenario, like it happened in Proposition 3.1 in the supplementary material of Naimzada and Pireddu (2020a), where the framework with information costs for fundamentalists, in the absence of rational agents, was considered.[5] As done in Naimzada and Pireddu (2020b), we chose the bias as bifurcation parameter, both in Proposition 2.1 and in the numerical investigations that we shall perform in Section 4, rather than dealing with the intensity of choice parameter like in Hommes and Wagener

[5] On the contrary, we recall that on increasing b or β in the absence of information costs, just the unconditionally stable and the mixed scenarios described in Proposition 2.1 could arise, where we name a scenario unconditionally stable if the steady state is (globally or locally) stable for every value of the considered parameter, while the mixed scenario is characterized by the stability just for small and for large values of the parameter, with an intermediate interval of instability. Cf. Proposition 2.1 and Corollary 2.1 in Naimzada and Pireddu (2020b) for the framework without rational agents, and Proposition 3.1 and Corollary 3.1 still in Naimzada and Pireddu (2020b) for the framework encompassing rational agents.

(2010),[6] because we are interested in studying the agents' asymptotic heterogeneity, and b just measures their degree of optimism and pessimism.
As previously mentioned, the above presented framework encompasses the setting analyzed in Hommes and Wagener (2010), as well as those in Naimzada and Pireddu (2020a,b). Namely, in the first one, neither information costs nor rational agents were considered, while in Naimzada and Pireddu (2020a) we dealt with information costs for biased and unbiased fundamentalists, but the economy was not populated by rational agents, and finally in Naimzada and Pireddu (2020b) rational agents have been introduced assuming for them a positive information cost like in Brock and Hommes (1997), not faced by biased and unbiased fundamentalists. Despite the just described modeling differences, the settings considered in Hommes and Wagener (2010) and in Naimzada and Pireddu (2020a,b) share a few common features in the outcomes, in addition to some crucial differences. The main similarity concerns the fact that the unique model steady state is given by the fundamental one. However, whereas in Hommes and Wagener (2010) and in Naimzada and Pireddu (2020a) the map governing the dynamics is decreasing, and thus no interesting dynamics can arise, like it happened in Naimzada and Pireddu (2020b) the function k in (2.5) is not monotone, being odd and admitting the x-axis as horizontal asymptote.
Before illustrating in Section 4 the global dynamic scenarios for the new framework, comparing their complexity degree with the one detected in Naimzada and Pireddu (2020b), in order to understand whether the chaotic dynamics and multistability phenomena emerging when enriching the original framework in Hommes and Wagener (2010) with rational agents persist or are inhibited by the introduction of information costs for fundamentalists, we need to briefly recall in Section 3 some aspects of the settings analyzed in Hommes and Wagener (2010) and in Naimzada and Pireddu (2020a,b) that are essential in view of that comparison, as well as in view of comparing the local stability regions in the various frameworks. Namely, the size of the local stability region is affected by the introduction of information costs, as well as of rational agents. In particular, since in Naimzada and Pireddu (2020a) we found that considering diversified information costs in the original setting in Hommes and Wagener (2010) produces a destabilizing effect on the equilibrium, while in Naimzada and Pireddu (2020b) we discovered that adding

[6] We recall that in Naimzada and Pireddu (2020a) we considered both b and β as bifurcation parameters.

rational agents enlarges the stability region of the steady state, in Section 4 we will investigate whether one of the two aspects always prevails over the other one when they are jointly taken into account.

3 Some details about the frameworks without information costs or rational agents

We recall that the dynamic equation considered in Hommes and Wagener (2010), written in deviation from the fundamental, is given by

$$\begin{aligned} x_t &= f(x_{t-1}) \\ &= \frac{bs}{d}\frac{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)+\exp\left(-\frac{\beta s}{2}x_{t-1}^2\right)}. \end{aligned} \tag{3.1}$$

Hommes and Wagener (2010) show in their Theorem A that the map f is always decreasing, having $x = 0$ as unique steady state. The monotonicity of f prevents the emergence of interesting dynamic phenomena and indeed at most period-two cycles can occur.
According to Corollary 2.1 in Naimzada and Pireddu (2020b),[7] the equilibrium $x = 0$ is locally asymptotically stable for (3.1) if

$$b^2 < \frac{d\left(2+\exp\left(\frac{\beta b^2 s}{2}\right)\right)}{2\beta s^2}. \tag{3.2}$$

As a consequence of (3.2), the three scenarios which can occur for increasing values of the bias are those illustrated in Fig. 1 in Naimzada and Pireddu (2020b), where we fixed the other parameters as follows: $A = 18$, $\beta = 15$, $d = 1$, considering $s = 0.5$ in (a), $s = 1.04$ in (b) and $s = 1.6$ in (c). We do not report them here for brevity's sake, since we shall find exactly the same three scenarios in Fig. 1 below, for lower values of s, when recalling the framework encompassing information costs for fundamentalists, in the

[7] The stability condition in (3.2) has not been derived in Hommes and Wagener (2010), because in that work the focus was on the case $s/d < 1$, i.e., on the case in which the Muthian model is globally eductively stable in the sense of Guesnerie (2002), being stable under naive expectations, and, like proven in Theorem A in Hommes and Wagener (2010), when the model is globally eductively stable, it is also evolutionary stable. As shown in Fig. 1 (b) in Naimzada and Pireddu (2020b), the condition $s/d < 1$, although being sufficient, is not necessary for the unconditional stability of $x = 0$.

absence of rational agents.
Namely, the dynamic equation proposed in Naimzada and Pireddu (2020a) for that setting, written in deviation from the fundamental, reads as

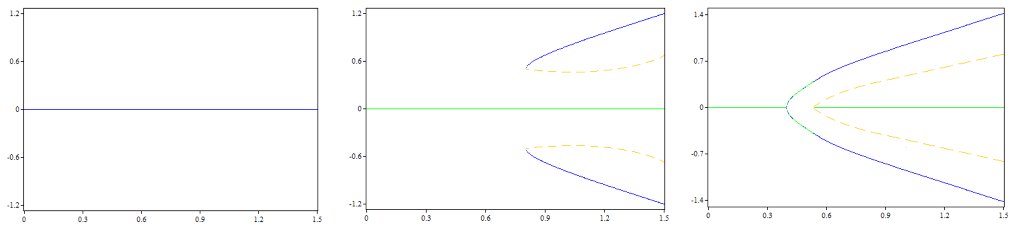
$$\begin{aligned} x_t &= g(x_{t-1}) \\ &= \frac{bs}{d} \frac{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)}{\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)+\exp\left(-\frac{\beta s}{2}x_{t-1}^2-\beta(c_0-c)\right)}. \end{aligned} \tag{3.3}$$

Like f in (3.1), also the map g is always decreasing, admitting $x = 0$ as unique fixed point, and no interesting dynamic phenomena can arise.
According to Corollary 3.1 in Naimzada and Pireddu (2020a), the equilibrium $x = 0$ is locally asymptotically stable for (3.3) if

$$b^2 < \frac{d\left(2+\exp\left(\frac{\beta b^2 s}{2}-\beta(c_0-c)\right)\right)}{2\beta s^2}. \tag{3.4}$$

A comparison between (3.2) and (3.4) immediately suggests that the stability region is reduced by the introduction of information costs so that, differently from the setting in Hommes and Wagener (2010), the steady state $x = 0$ may be unstable even when the Muthian model is eductively stable. This is indeed what happens in Fig. 1 (c). Namely, in Fig. 1 we depict the three scenarios which can occur with (3.3) for increasing values of the bias. In particular, we fix the other parameters as follows: $A = 18$, $\beta = 15$, $d = 1$, $c_1 = c_2 = c = 0.1$, $c_0 = 0.12$, considering $s = 0.5$ in (a), $s = 0.8$ in (b) and $s = 0.95$ in (c). As initial conditions in (a) we have $x_0 = 1$; in (b) and (c) we have $x_0 = 0.01$ for the green points and $x_0 = 1$ for the blue points. We refer the interested reader to Naimzada and Pireddu (2020a) for a deeper investigation of the occurring bifurcations, at which e.g. the unstable period-two cycle (represented in orange, dashed line) emerges, and for the economic interpretation of the various scenarios.
Finally, the dynamic equation proposed in Naimzada and Pireddu (2020b) to describe the framework in which only rational agents face an information cost, written in deviation from the fundamental, reads as

$$\begin{aligned} x_t &= h(x_{t-1}) \\ &= \frac{bs\left(\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)-\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)\right)}{d\left(\exp\left(-\frac{\beta s}{2}(x_{t-1}+b)^2\right)+\exp\left(-\frac{\beta s}{2}(x_{t-1}-b)^2\right)+\exp\left(-\frac{\beta s}{2}x_{t-1}^2\right)\right)+(d+s)\exp(-\beta c_{-1})}. \end{aligned} \tag{3.5}$$



(a) (b) (c)

Figure 1: The bifurcation diagram of g for $b \in (0, 1.5)$ and different initial conditions, for $A = 18$, $\beta = 15$, $d = 1$, $c = 0.1$, $c_0 = 0.12$, and $s = 0.5$ in (a), $s = 0.8$ in (b) and $s = 0.95$ in (c)

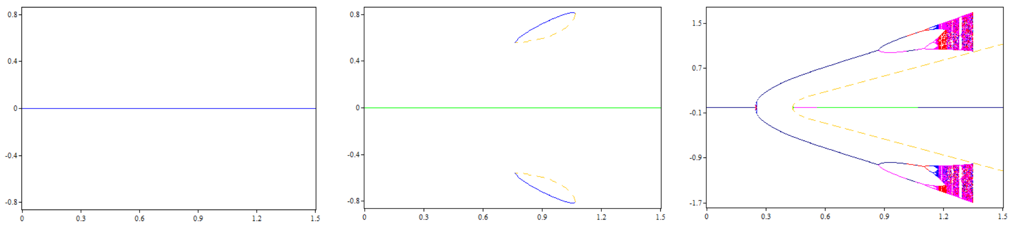
Like k in (2.5), the function h, although being odd and having $x = 0$ as unique fixed point, is not monotone, admitting the x-axis as horizontal asymptote, and thus it may generate interesting dynamic scenarios.
In order to explain when they can occur, we recall that, according to Corollary 3.1 in Naimzada and Pireddu (2020b), the equilibrium $x = 0$ is locally asymptotically stable for (3.5) if

$$b^2 < \frac{d\left(2 + \exp\left(\frac{\beta b^2 s}{2}\right)\right) + (d+s)\exp\left(\beta\left(\frac{b^2 s}{2} - c_{-1}\right)\right)}{2\beta s^2}. \tag{3.6}$$

As observed in Naimzada and Pireddu (2020b), the right-hand side in (3.6) is larger than that in (3.2) for any $c_{-1} > 0$, confirming that rational agents have a positive effect on the system stability.[8] This implies that chaotic phenomena can emerge only when the eductive stability condition for the Muthian model is not fulfilled, since, like it happens in the framework in Hommes and Wagener (2010), when $s/d < 1$ the steady state $x = 0$ is always (globally or locally) stable and at most it coexists with a period-two cycle. On the other hand, for $s/d > 1$ interesting dynamic outcomes may arise. We illustrate the possible scenarios in Fig. 2, where b varies in $(0, 1.5)$ and we fix the other parameters as follows: $A = 18$, $\beta = 15$, $d = 1$, $c_{-1} = 0.15$, taking $s = 0.9$ in (a), $s = 1.04$ in (b) and $s = 1.6$ in (c). As initial conditions in (a)

[8]Nonetheless, the term on the right-hand side in (3.6) is decreasing with c_{-1} and in the limit $c_{-1} \to +\infty$ it coincides with the term on the right-hand side in (3.2), meaning that the stabilizing effect of rational agents tends to disappear when their information cost c_{-1} is too high.

we have $x_0 = 1$; in (b) we have $x_0 = 0.01$ for the green points and $x_0 = 0.9$ for the blue points; in (c) we have $x_0 = 0.01$ for the green points, $x_0 = 0.7$ for the dark blue points, $x_0 = 1$ for the magenta points, $x_0 = 1.3$ for the red points and $x_0 = 1.43$ for the light blue points.



(a) (b) (c)

Figure 2: The bifurcation diagram of h for $b \in (0, 1.5)$ and different initial conditions, for $A = 18$, $\beta = 15$, $d = 1$, $c_{-1} = 0.15$, and $s = 0.9$ in (a), $s = 1.04$ in (b) and $s = 1.6$ in (c)

In Fig. 2 (a) the steady state is globally stable for all bias values. We stress that for the same parameter configuration we would find for the map f in (3.1) a scenario analogous to that depicted in Fig. 1 (b) in Naimzada and Pireddu (2020b), which in turn bears a resemblance to Fig. 2 (b). However, while in Fig. 1 (b) in Naimzada and Pireddu (2020b) the stable (in blue) and unstable (in orange, dashed line) period-two cycles persist for increasing values of the bias, this does not occur in Fig. 2 (b), due to a double reverse fold bifurcation of the second iterate of the map h in (3.5) occurring for $b \approx 1.052$ after which $x = 0$ recovers its global stability.[9] Finally, for the same parameter values used in Fig. 1 (c) in Naimzada and Pireddu (2020b), we observe in Fig. 2 (c) the presence of chaotic dynamics, which can not arise in the framework without rational agents, both without and with information costs, because of the monotonicity of the maps f in (3.1) and g in (3.3). Moreover, with respect to Fig. 1 (c) in Naimzada and Pireddu (2020b), the

[9]We remark that in Naimzada and Pireddu (2020b) we did not find for h the scenario depicted in Fig. 2 (b), in which the steady state, although remaining locally asymptotically stable for increasing values of the bias, loses its global stability through a double fold bifurcation of the second iterate of the map governing the dynamics. The difference in the outcomes is due to the lower value considered in Naimzada and Pireddu (2020b) for the information cost of rational agents, that was $c_{-1} = 0.1$.

local stability region of the steady state is enlarged in Fig. 2 (c).[10] Hence, Fig. 2 confirms both the stabilizing effect produced on the steady state by rational agents and the possibility of witnessing chaotic dynamics when they are taken into account. Indeed, differently from the contexts described by f in (3.1) and by g in (3.3), in which rational agents were not encompassed, the basin of attraction of the steady state may be unconnected due to the presence of the horizontal asymptote for h in (3.5), with its non-immediate components lying outside the basin of attraction of the chaotic attractor(s), when they coexist. In fact, we stress that, on increasing b in Fig. 2 (c), $x = 0$ loses stability for $b \approx 0.245$ through a supercritical flip bifurcation, which gives rise to a stable period-two cycle, followed by two coexisting period-two cycles, born for $b \approx 0.864$ via a double pitchfork bifurcation of the second iterate of h. Raising b further triggers a period-doubling cascade leading to chaos. More precisely, for $b \in (1.155, 1.192)$, we notice the coexistence between two chaotic attractors in two pieces, which merge for $b \approx 1.192$ to give rise to a unique chaotic attractor in two pieces, that disappears for $b \approx 1.328$ by its contact with the unstable period-two cycle born in correspondence to the subcritical flip bifurcation occurring at $x = 0$ when $b \approx 0.437$. We will observe analogous transitions in the frameworks depicted in Figs. 3, 5 and 7, in regard to k in (2.5), too.
We refer the interested reader to Naimzada and Pireddu (2020b) for further details about the occurring bifurcations and for the economic interpretation of the various scenarios, as well as for a more detailed investigation of the possible dynamic outcomes.

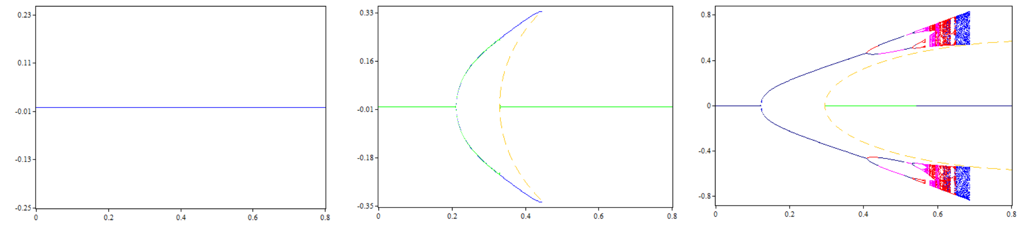
4 Analysis of the setting with rational agents and information costs

We are now in position to investigate whether one of the two local effects recalled at the end of Section 2, i.e., the destabilizing role played by information costs (cf. (3.4) and the comments following it) or the stabilizing role played by rational agents (see (3.6) and the subsequent remarks) on the equilibrium $x = 0$ always prevails over the other one, when dealing with the

[10]Namely, in the framework without rational agents, according to (3.2), $x = 0$ is locally asymptotically stable just for $b \in (0, 0.223) \cup (0.478, +\infty)$, while in the setting encompassing rational agents, according to (3.6), $x = 0$ is locally asymptotically stable for $b \in (0, 0.245) \cup (0.437, +\infty)$.

framework presented in Section 2, which encompasses both rational agents and information costs for all agent types. The answer is not unique and follows by a comparison between the stability condition in (3.2), referring to the framework in Hommes and Wagener (2010), and the stability condition in (2.6), derived in Proposition 2.1 for the new proposed setting. Namely, (3.2) may be stronger or weaker than (2.6) according to which kinds of agents are more penalized by information costs, still maintaining the ordering in (2.3). In particular, the stability region increases with respect to Hommes and Wagener (2010) e.g. when $0 << c \approx c_0 < c_{-1}$, i.e., when biased fundamentalists face high information costs, so that their destabilizing role is dampened by the fall in their share, due to their reduced profits. Vice versa, the stability region may decrease with respect to the framework in Hommes and Wagener (2010) e.g. when $0 \approx c << c_0 < c_{-1}$, that is, when unbiased fundamentalists and rational agents face high information costs, so that their stabilizing effect is reduced by their decreased share, because of the lower realized profits. This shows that, according to the chosen parameter configuration, the local stability region of $x = 0$ in the setting proposed in Section 2 may be larger or smaller than in the setting without rational agents and information costs considered in Hommes and Wagener (2010). When it is smaller, contrarily to what happened in that work, the equilibrium $x = 0$ may be unstable when $s/d < 1$ for the setting introduced in Section 2. However, when the eductive stability condition for the Muthian model is fulfilled we find at most a period-two cycle, while we may witness interesting dynamics for $s/d > 1$ (cf. for instance Fig. 3 (c)).
As we shall see in Section 5, also the comparison between the stability regions with and without information costs for fundamentalists, when rational agents are taken into account, does not give a unique answer, differently from the framework in which rational agents were not considered, where introducing information costs was always destabilizing, as we concluded comparing (3.2) and (3.4).
We further stress that, with respect to the setting analyzed in Naimzada and Pireddu (2020a), in which we considered information costs for biased and unbiased fundamentalists, without encompassing rational agents, the stability region has increased due to the additional introduction of rational agents, as a comparison of the terms on right-hand side of (2.6) and (3.4) immediately shows. Hence, we find a confirmation of the stabilizing effect produced by rational agents (see Hommes 2013), even when information costs for fundamentalists are taken into account.

A more detailed comparison among the stability conditions in (2.6), (3.2), (3.4) and (3.6) can be found in Section 5.

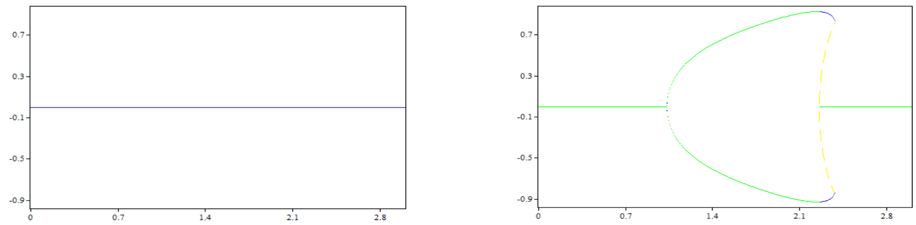


(a) (b) (c)

Figure 3: The bifurcation diagram of k for $b \in (0, 0.8)$ and different initial conditions, for $A = 18$, $\beta = 20$, $d = 1$, $c = 0.1$, $c_0 = 0.12$, $c_{-1} = 0.15$, and $s = 1.3$ in (a), $s = 1.8$ in (b) and $s = 3$ in (c)

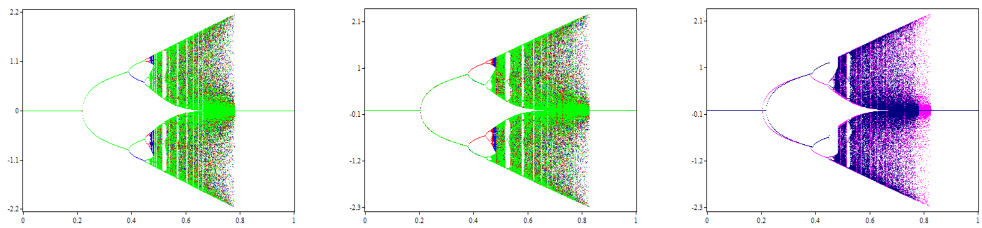
Proceeding with our discussion, which will concern also the second issue raised in Section 2, i.e., the comparison between the complexity degree of the global dynamics arising with and without information costs for fundamentalists in the presence of rational agents, let us report the three main scenarios compatible with Proposition 2.1 in Fig. 3, where b varies in $(0, 0.8)$ and we fix the other parameters as follows: $A = 18$, $\beta = 20$, $d = 1$, $c_1 = c_2 = c = 0.1$, $c_0 = 0.12$, $c_{-1} = 0.15$, considering $s = 1.3$ in (a), $s = 1.8$ in (b) and $s = 3$ in (c). As initial conditions in (a) we have $x_0 = 1$; in (b) we have $x_0 = 0.01$ for the green points and $x_0 = 0.4$ for the blue points; in (c) we have $x_0 = 0.01$ for the green points, $x_0 = 0.38$ for the dark blue points, $x_0 = 0.45$ for the magenta points, $x_0 = 0.50$ for the red points and $x_0 = 0.77$ for the light blue points.
Although in Fig. 3 (a) it holds that $s/d > 1$, and thus the eductive stability condition for the Muthian model is not fulfilled, the steady state is always globally stable. Namely, for the chosen information cost values, the stability region is enlarged with respect to the framework in Hommes and Wagener (2010) when fixing the other parameters as in Fig. 3. Indeed, the bifurcation diagram in that setting looks like that in Fig. 1 (c) in Naimzada and Pireddu (2020b), characterized by a double stability threshold. This shows that the stability of the steady state in the presence of rational agents and information costs for all agent types in general does not imply the stability in the framework considered in Hommes and Wagener (2010).
In Fig. 3 (b), we have again $s/d > 1$, but this time, due to the increased

value of s, we find a scenario similar to Fig. 1 (c) in Naimzada and Pireddu (2020b), in which $x = 0$ is unstable for intermediate values of the bias. However, thanks to the presence of rational agents, the period-two cycle, emerged through a supercritical flip bifurcation, may not persist for excessively large values of the bias. For an explanation, cf. Fig. 8 in Naimzada and Pireddu (2020b) and the corresponding discussion, even if therein information costs for biased and unbiased fundamentalists were not considered.
Passing now to Fig. 3 (c), for a still larger value of $s/d > 1$, thanks again to the presence of rational agents, which allow for the nonmonotonicity of k, we observe the emergence of chaotic dynamics. Making a comparison with Fig. 2 (c), where just rational agents face positive information costs, we can say that the overall complexity degree seems to be lower now. Namely, the routes to chaos and the complex attractors are analogous in the two figures, but in Fig. 3 (c) we deal with larger values for both the destabilizing parameters β and s. Indeed, drawing the bifurcation diagram of map k in (2.5) for $\beta = 15$ and $s = 1.6$, like it was in Fig. 2 (c), we would find the globally stable scenario.
Summarizing, we can then say that, for the parameter configuration considered in Fig. 3, the stability region has increased with respect to the corresponding framework without rational agents and information costs, and that the complexity level of the global dynamics has decreased with respect to the corresponding framework with rational agents, in which fundamentalists do not face information costs. Nonetheless, as observed above, the joint consideration of rational agents and information costs may also reduce the stability region with respect to the setting in Hommes and Wagener (2010), described by (3.1). We illustrate a corresponding example in Fig. 4, where $A = 18$, $\beta = 4$, $s = 0.5$, $d = 1$, $c_1 = c_2 = c = 0.01$, $c_0 = 0.99$, $c_{-1} = 1$, while b varies in $(0, 3)$. In particular, in (a) we consider the framework in Hommes and Wagener (2010), where $x = 0$ is globally stable for every value of b, while in (b) we focus on the framework with rational agents and information costs, where we find a double stability threshold for $x = 0$. Namely, in agreement with Proposition 2.1, in (b) the steady state is stable just for $b \in (0, 1.035) \cup (2.268, +\infty)$, since when $b \approx 1.035$ at $x = 0$ a supercritical flip bifurcation occurs, while for $b \approx 2.268$ a subcritical flip bifurcation takes place at $x = 0$. As initial conditions in (a) we have $x_0 = 0.9$, while in (b) we have $x_0 = 0.01$ for the green points and $x_0 = 0.9$ for the blue points.
Concerning global dynamics, we stress that for the information cost values $c_1 = c_2 = c = 0.01$, $c_0 = 0.1$, $c_{-1} = 1$, in which unbiased agents are



(a) (b)

Figure 4: The bifurcation diagram for $b \in (0, 3)$ and $A = 18$, $\beta = 4$, $s = 0.5$, $d = 1$ of f in (a) and of k in (b) with $c = 0.01$, $c_0 = 0.99$, $c_{-1} = 1$



(a) (b) (c)

Figure 5: The bifurcation diagram for $b \in (0, 1)$ and $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $c_{-1} = 1$ of h in (a) and of k in (b) with $c = 0.01$, $c_0 = 0.1$, obtained for various initial conditions. In (c) we draw them together using just the initial condition $x(0) = 0.01$ and the dark blue (resp. magenta) color for h (resp. k)

much less penalized than in the parameter configuration considered in Fig. 4, when raising the slope of the supply curve to e.g. $s = 3$ we find some interesting dynamic behavior, whose complexity level is this time slightly higher than in the case in which only rational agents face the information cost $c_{-1} = 1$. We report the corresponding bifurcation diagrams in Fig. 5, where $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, while b varies in $(0, 1)$. In particular, in (a) we focus on the framework with rational agents, described by (3.5), in which $c_1 = c_2 = c_0 = 0$, while in (b) we deal with the framework with rational agents and nonnull information costs for all agent types. In Fig. 5 (a) and (b) as initial conditions we have $x_0 = 0.01$ for the green points, $x_0 = 1$ for the red points and $x_0 = 1.2$ for the blue points. In Fig. 5 (c) we draw the bifurcation diagrams of h (in dark blue) and of k (in magenta) using just the initial condition $x(0) = 0.01$, in order to better compare them. Despite the overall similarity, the bifurcation diagram of k, in which all agents face nonnull information costs, displays higher instability and complexity degrees than the bifurcation diagram of h, due to the reduced size of the stability region[11] and the longer persistence of the chaotic attractor. Such differences between Fig. 5 (a) and Fig. 5 (b) can be explained in terms of the information cost values in the two frameworks, focusing in particular on the variation in the cost faced by unbiased fundamentalists, as we shall do in Appendix B. Recalling that unbiased fundamentalists and rational agents play a stabilizing role (see Hommes 2013), from an interpretative viewpoint we can say that, for sufficiently small values of $c_0 \geq 0$ the steady state is stable because, even if the costs faced by rational agents are high ($c_{-1} = 1$) and thus they are not attractive from an evolutive perspective, the information costs faced by unbiased fundamentalists are low enough to make their share significant, due to the net profits they realize, for guaranteeing the system stability; on the other hand, still considering $c_{-1} = 1$ and not varying any other parameter value, when c_0 raises e.g. to 0.1, as it is in Fig. 5 (b), it happens that the decreased attractiveness exerted by unbiased fundamentalists, as a consequence of their reduced net profits, is no more sufficient to stabilize the system and complex dynamics arise.

5 Discussion

Before concluding in Section 6, let us try to summarize the network of relationships among the stability regions for the equilibrium in the new setting proposed in Section 2 and in those described by (3.1), (3.3) and (3.5), since we recall that the last three frameworks are subcases of the more general one introduced in Section 2.

The only implications that always hold true are those reported in the following diagram

$$\begin{array}{ll} \text{HWC} \Rightarrow \text{HWRC} \\ \Downarrow \\ \text{HW} \Rightarrow \text{HWR} \end{array} \tag{5.1}$$

[11]Namely, from conditions (3.6) and (2.6) it respectively follows that $x = 0$ is stable in Fig. 5 (a) for $b \in (0, 0.218) \cup (0.782, +\infty)$, while $x = 0$ is stable in Fig. 5 (b) for $b \in (0, 0.204) \cup (0.828, +\infty)$.

where we denote by HW the original setting considered in Hommes and Wagener (2010), by HWC the setting analyzed in Naimzada and Pireddu (2020a), in which we introduced information costs for biased and unbiased fundamentalists, by HWR the setting studied in Naimzada and Pireddu (2020b), in which we added rational agents to the framework in Hommes and Wagener (2010), and by HWRC the richer framework proposed in the present contribution, in which both rational agents and information costs for all agent types are considered. For instance HWC $\Rightarrow$ HW means that if $x = 0$ is locally asymptotically stable in the setting with information costs, then it is locally asymptotically stable also in the original framework in Hommes and Wagener (2010). Of course, we consider an implication between two different frameworks true when it is satisfied keeping fixed the value of all the parameters that are present in both frameworks and for all feasible values of the parameters that are present in just one of the two frameworks. Due to the nature of the analyzed frameworks, those parameters will be information costs for rational agents or for biased and unbiased fundamentalists, satisfying (2.3).

An alternative way of illustrating diagram (5.1) is provided in Fig. 6.

Checking all the implications in diagram (5.1) is straightforward.

Namely, as observed in Section 4, the validity of HWC $\Rightarrow$ HWRC immediately follows by a comparison of the terms on the right-hand side of (2.6) and (3.4). Similarly, the proof of HW $\Rightarrow$ HWR follows by a comparison of the terms on the right-hand side of (3.2) and (3.6). We stress that the implications HWC $\Rightarrow$ HWRC and HW $\Rightarrow$ HWR confirm the stabilizing role played by rational agents (see Hommes 2013), both when information costs for biased and unbiased fundamentalists are taken into account and when they are missing.

The validity of the implication HWC $\Rightarrow$ HW, which describes the destabilizing effect produced by the introduction of information costs for fundamentalists when the economy is not populated by rational agents, follows by a comparison of the terms on the right-hand side of (3.2) and (3.4), as noticed in Section 3.[12]

[12]We stress that the validity of the implication HW $\Rightarrow$ HWR has been highlighted in Naimzada and Pireddu (2020b), while the implication HWC $\Rightarrow$ HW has not been noticed before, since in Naimzada and Pireddu (2020a) we focused just on the case in which the model is globally eductively stable, and thus the stability condition in (3.2) was always fulfilled. Cf. also Footnote 5. Of course, the implication HWC $\Rightarrow$ HWRC has not been discussed elsewhere, since the setting HWRC had not been considered in the literature

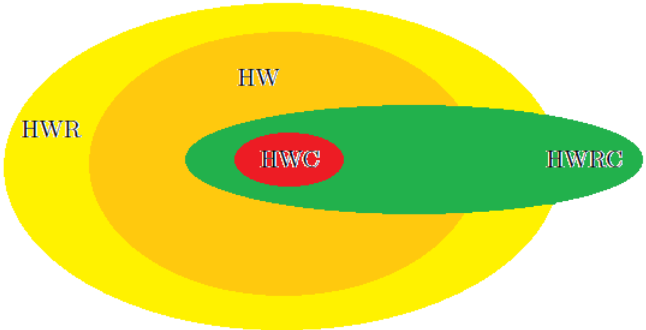


Figure 6: A sketch of the stability regions for the new proposed setting and for the three frameworks that it encompasses. In particular, we depict in orange the stability region corresponding to the framework in Hommes and Wagener (2010), in yellow (red) the stability region corresponding to that same framework in which rational agents (information costs for fundamentalists) are introduced and in green the stability region corresponding to the setting encompassing both rational agents and information costs for all agent types

Observe that the above mentioned comparisons among the terms on the right-hand side of the various stability conditions also show that the implications in diagram (5.1) are strict, since the opposite implications do not hold true for suitable values e.g. of the slope of the supply function, of the intensity of choice parameter and of the bias, while fixing $A = 18$ and $d = 1$, as done along the manuscript. In particular, from this remark it follows that the implication HWR $\Rightarrow$ HWC is not true, because if it were the case that HWR $\Rightarrow$ HWC, then, together with HWC $\Rightarrow$ HW, this would imply that HWR $\Rightarrow$ HW, which is not true in general.
In fact, it is possible to prove by means of counterexamples that all the missing implications in diagram (5.1) are not fulfilled, with the only exception of the implication HWC $\Rightarrow$ HWR, which is true, as it follows from HWC $\Rightarrow$ HW and HW $\Rightarrow$ HWR. Indeed, we have already shown towards the end of Section 4 that HWRC $\not\Rightarrow$ HW and HW $\not\Rightarrow$ HWRC by finding two parameter configurations for which exactly one between (2.6) and (3.2) holds true (cf. Fig. 3 and Fig. 4, with the corresponding comments). Moreover,

yet.

from the discussion in Section 4 about the frameworks in Fig. 2 (c) and in Fig. 3 (c), we also discovered that HWRC $\nRightarrow$ HWR, since drawing the bifurcation diagram of k for $\beta = 15$ and $s = 1.6$, like it was in Fig. 2 (c), we would find the globally stable scenario, while in Fig. 2 (c) we witness a double stability threshold for $x = 0$. Finally, since HW $\nRightarrow$ HWRC then it cannot hold that HWR $\Rightarrow$ HWRC because such implication, together with HW $\Rightarrow$ HWR would imply HW $\Rightarrow$ HWRC, which is not true. Hence HWR $\nRightarrow$ HWRC, as confirmed by Fig. 5, where the stability region is larger for h than for k (see also Footnote 9).

In regard to the global dynamics, as we discussed in Section 4 and in agreement with the findings in Naimzada and Pireddu (2020b), recalled in Section 3, it is the introduction of rational agents which allows for the emergence of chaotic attractors, since the map governing the dynamics is no more monotonically decreasing due to the presence of the same horizontal asymptote for $x \to \pm\infty$, both when information costs for fundamentalists are considered and when they are missing. Indeed, the overall dynamic complexity of the system is not altered by the introduction of information costs for biased and unbiased fundamentalists, like it is confirmed by a comparison between Fig. 1 in Naimzada and Pireddu (2020b) and Fig. 1 in Section 3 in the absence of rational agents, since in both frameworks at most a period-two cycle can arise, as well as by a comparison between Fig. 2 and Fig. 3 when rational agents are taken into account, too. However, in the presence of rational agents, considering the same values for the parameters common to the frameworks with and without information costs for fundamentalists, we usually witness a reduced complexity degree with nonnull information costs for all agent types (see e.g. Fig. 3 (c) and the corresponding comments), even if it is sometimes possible to observe reversed scenarios, in which longer-lasting attractors are obtained when fundamentalists face nonnull information costs, too (cf. Fig. 5).

6 Conclusion

In the present contribution at first we investigated which one between the stabilizing effect produced on the (fundamental) steady state by rational agents and the destabilizing effect produced by the introduction of information costs for fundamentalists in the original evolutionary cobweb framework in Hommes and Wagener (2010) prevails, discovering that the overall result

on the size of the local stability region of the steady state depends on the relative values of information costs. Namely, dealing with rational agents and nonnull information costs for all agent types, we obtained that the stability region increases with respect to the framework considered in Hommes and Wagener (2010) e.g. when biased fundamentalists face high information costs, so that their destabilizing role is dampened by the fall in their share, due to their reduced profits. Vice versa, we found that the stability region may decrease with respect to the framework in Hommes and Wagener (2010) e.g. when unbiased fundamentalists and rational agents face high information costs, so that their stabilizing effect is reduced by their decreased share, because of the lower realized profits. In fact, we discovered that also the comparison between the stability regions with and without information costs for fundamentalists, when rational agents are taken into account, does not give a unique answer, differently from the case studied in Naimzada and Pireddu (2020a) in which rational agents were not considered, where the introduction of information costs was always destabilizing.
We concluded our local analysis by performing a detailed comparison among the stability conditions for the equilibrium in the four settings, obtained possibly considering information costs for biased and unbiased fundamentalists and possibly introducing rational agents. In particular, we found evidence of the stabilizing effect produced by rational agents (see Hommes 2013), even when information costs for fundamentalists are taken into account, similar to what discovered in Naimzada and Pireddu (2020b) in the absence of information costs.
In addition, along the manuscript we investigated if the complex dynamic outcomes emerging when enriching the original framework in Hommes and Wagener (2010) with rational agents, analyzed in Naimzada and Pireddu (2020b), persist or are inhibited by the introduction of information costs for fundamentalists. In this respect we found that, although the overall dynamic complexity of the system seems not altered by the introduction of information costs for biased and unbiased fundamentalists, the routes to chaos and the chaotic attractors being analogous with and without costs for them, we usually witness a reduced complexity degree with nonnull information costs for all agent types, even if it is sometimes possible to observe reversed scenarios, in which longer-lasting attractors are obtained when fundamentalists face nonnull information costs, too.
As concerns possible extensions, we deem that the main weakness of the three settings here recalled and more deeply analyzed, as well as of the new pro-

posed one, concerns the agents' inability to learn from past events. Namely, even when prices undergo a period-two cycle, agents make the same production choice in any time period, without learning from their past mistakes. In view of fixing such issue, the model could be modified endogenizing the beliefs' bias through an adaptive learning mechanism. Such investigation will be performed in a future work.

Appendix A: Proof of Proposition 2.1

The proof of Proposition 2.1 follows steps that are similar to those used to check Proposition 3.1 in Naimzada and Pireddu (2020b). Nonetheless, we present it for sake of completeness.
Proof of Proposition 2.1: It is easy to check that $x = 0$ is a fixed point of k in (2.5). In order to verify that no other steady states may exist, it suffices to notice that $k > 0$ if and only if $x < 0$.
For the derivative of k in correspondence to $x = 0$ it holds that

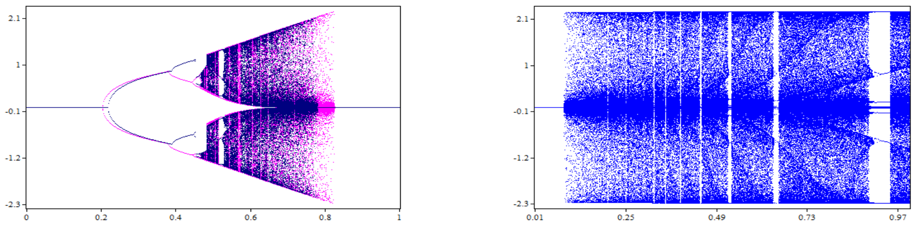
$$k'(0) = \frac{-2b^2\beta s^2 \exp\left(-\beta\left(\frac{b^2 s}{2} + c\right)\right)}{d\left(2\exp\left(-\beta\left(\frac{b^2 s}{2} + c\right)\right) + \exp\left(-\beta c_0\right)\right) + (d+s)\exp\left(-\beta c_{-1}\right)} < 0,$$

so that $x = 0$ is stable when $k'(0) > -1$, i.e., when (2.6) is fulfilled. Introducing $\varphi_1(b) = b^2$ and $\varphi_2(b) = \frac{d\left(2+\exp\left(\frac{\beta b^2 s}{2} - \beta(c_0 - c)\right)\right) + (d+s)\exp\left(\frac{\beta b^2 s}{2} - \beta(c_{-1} - c)\right)}{2\beta s^2}$, for $b \geq 0$ both φ_1 and φ_2 are convex functions with $0 = \varphi_1(0) < \varphi_2(0)$. Because of the exponential function, φ_2 tends to $+\infty$ more rapidly than φ_1 for $b \to +\infty$, and thus there may be two or no intersections between the graphs of φ_1 and φ_2. They intersect once just when their are tangent. □

Appendix B: The role of information costs for unbiased fundamentalists

Before explaining why, comparing Fig. 5 (a) and Fig. 5 (b), we find in the latter framework, characterized by nonnull information costs for all agent types, a longer persisting chaotic attractor, let us more carefully analyze the bifurcation diagrams in Fig. 5. Indeed, some crucial transitions that we observe therein are strictly connected with the shape of the graph of h and k, as well as with the behavior of the forward iterates of suitable points through them - aspects which so far have not been given much attention. In Fig. 5 (a) and Fig. 5 (b), $x = 0$ loses stability through a supercritical flip bifurcation, which gives rise to a stable period-two cycle, followed by two coexisting period-two cycles born via a double pitchfork bifurcation of the second iterate of h and k, which occurs for $b \approx 0.40$ in Fig. 5 (a) and for $b \approx 0.38$ in Fig. 5 (b). Increasing b further triggers a period-doubling cascade leading to chaos. In a first phase, corresponding approximately to $b \in (0.471, 0.483)$ for both h and k, we notice the coexistence between two chaotic attractors in two pieces, which then merge to give rise to a unique chaotic attractor in two pieces. Namely, it approaches $x = 0$, but without encompassing it because the maps h and k vanish just for $x = 0$, that is unstable. The chaotic attractor disappears for $b \approx 0.782$ in the framework in which just rational agents face information costs and for $b \approx 0.828$ in the framework in which the information costs of biased and unbiased fundamentalists are nonnull, too. In the two settings the disappearance of the chaotic attractor is caused by its contact with the unstable period-two cycle born in correspondence to the subcritical flip bifurcation occurring at $x = 0$, for h when $b \approx 0.782$ and for k when $b \approx 0.828$. More precisely, when $x = 0$ becomes locally stable through the subcritical flip bifurcation of h and k an unstable period-two cycle emerges, which, for slightly larger values of b, touches the chaotic attractor, due to its closeness to the origin. Then, like it happens e.g. in Fig. 2 (c), the chaotic attractor disappears and $x = 0$ becomes globally stable since orbits, from the extremal (seemingly) horizontal tracts of the graphs of h and k, reach a neighborhood of the origin and they are now trapped therein, limiting to $x = 0$ (see Fig. 8 (b) for k). Namely, both when the steady state is stable and when it is unstable, the iterate of the maximum and of the minimum of h and k falls on the extremal tracts of the two graphs (cf. Fig. 8 (a) and Fig. 8 (c) for k) and in the next step they are mapped very close to the origin. As long as the steady state was unstable, the iterates remained for several steps close to the origin but then escaped (see Fig. 8 (d) for k), going towards the maximum and minimum values of the two functions, being subsequently mapped on the extremal tracts of the graphs of h and k, repeating similar patterns. In fact, as clearly shown by the bifurcation diagrams in Fig. 5, the most visited regions are neighborhoods of the origin and of the maximum and minimum values for the two functions.

In order to understand why for e.g. $b = 0.82$ in Fig. 5 (b) we still observe the chaotic attractor, while in Fig. 5 (a) it has already disappeared, we focus on

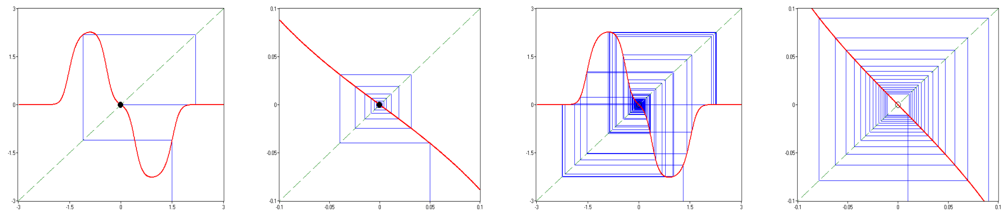


(a) (b)

Figure 7: In (a) the bifurcation diagrams for $b \in (0,1)$ of k with $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $c = 0.01$, $c_{-1} = 1$ and $c_0 = 0.01$ (resp. $c_0 = 0.1$) in dark blue (resp. in magenta), using just the initial condition $x(0) = 0.01$. In (b) the bifurcation diagram for $c_0 \in (0.01, 1)$ of k with $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $b = 0.82$, $c = 0.01$, $c_{-1} = 1$

the role played by information costs. Namely, as we shall see, the explanation is based on the different information cost values in the two frameworks, and in particular on the variation in the cost of unbiased fundamentalists. To better illustrate such point, let us start observing that no sensible differences are produced in the dynamics observed in Fig. 5 (a) by considering $c_1 = c_2 = c = c_0 = 0.01$, rather than null information costs for biased and unbiased fundamentalists. For this purpose, similar to what done in Fig. 5 (c), we report in Fig. 7 (a) the bifurcation diagrams of k obtained for $c = c_0 = 0.01$ and $c_{-1} = 1$ in blue, and for $c = 0.01$, $c_0 = 0.1$ and $c_{-1} = 1$ in magenta, using just the initial condition $x(0) = 0.01$. A comparison with Fig. 5 (c) immediately suggests that no big changes are generated in the local and global dynamic outcomes by raising the information costs for biased and unbiased fundamentalists from 0 to 0.01. In particular, like it happened with null information costs for biased and unbiased fundamentalists in Fig. 5 (a), when $c_1 = c_2 = c = c_0 = 0.01$ it is still true that for $b = 0.82$ the chaotic attractor has already disappeared, while, as we know from Fig. 5 (b), it persists with $c = 0.01$ and $c_0 = 0.1$. Nonetheless, recalling the ordering in (2.3), such small change in the value of $c_1 = c_2 = c$ allows us to consider the effect produced by letting c_0 vary between the extremal values that it may assume, i.e., $c = 0.01$ and $c_{-1} = 1$.[13] Namely, we shall obtain that lower or

[13]We stress that such step may be formalized by writing c_0 as a convex combination of $c = 0.01$ and $c_{-1} = 1$, i.e., as $c_0 = (1-\alpha)c + \alpha c_{-1}$, with $\alpha \in [0,1]$ measuring the penalization degree for unbiased fundamentalists. Indeed, for $\alpha = 0$ unbiased fundamentalists face the



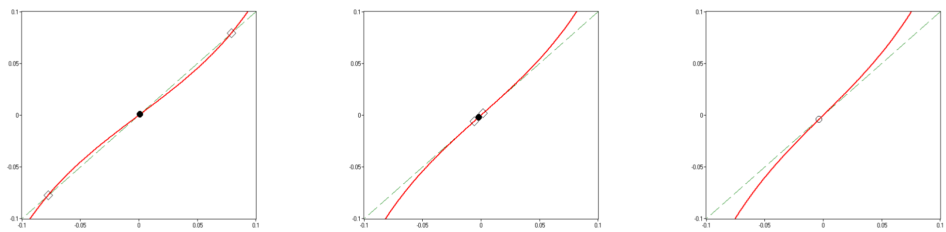
(a) (b) (c) (d)

Figure 8: In (a) the graph of k for $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $c = 0.01$, $c_{-1} = 1$ and $c_0 = 0.01$; in (b) an enlargement of a neighborhood of $x = 0$. In (c) the graph of k for $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $c = 0.01$, $c_{-1} = 1$ and $c_0 = 0.1$; in (d) an enlargement of a neighborhood of $x = 0$

higher values for the information cost of the unbiased fundamentalists yield to different scenarios. Accordingly, we draw in Fig. 7 (b) the bifurcation diagram of k for $c_0 \in (0.01, 1)$ and $b = 0.82$, $c = 0.01$ and $c_{-1} = 1$, which confirms, in agreement with Fig. 7 (a), that for $c_0 = 0.01$ the steady state is stable, while for $c_0 = 0.1$ a chaotic attractor has appeared. Such findings are corroborated by the pictures of the graph of k in Fig. 8, obtained for $b = 0.82$, $c = 0.01$, and $c_0 = 0.01$ in (a) and (b), while in (c) and (d) we set $c_0 = 0.1$. In particular, we illustrate the complete orbits of k in (a) and (b), while we focus on a neighborhood of $x = 0$ in (b) and (d). As explained above when dealing with b as bifurcation parameter, both when the steady state is stable and when it is unstable, the iterate of the maximum and of the minimum of k falls on the extremal (seemingly) horizontal tracts of the graph (see Fig. 8 (a) and Fig. 8 (c)) and in the next step they are mapped very close to the origin. As long as $x = 0$ is stable, it attracts all orbits, as they, starting from the extremal tracts of the graph of k, reach a neighborhood of the origin and they are then trapped therein, limiting to $x = 0$ (see Fig. 8 (b)). When instead the steady state is unstable, the iterates remain for several steps close to the origin but afterward escape (see Fig. 8 (d)), going next to the maximum and minimum values of k, being subsequently mapped on the extremal tracts of the graph, repeating alike patterns. Similar to Fig. 5, the bifurcation diagram in Fig. 7 (b) highlights that the most visited regions of the chaotic attractor, which emerges when $x = 0$ becomes unstable,

lowest possible information cost c they can incur, while for $\alpha = 1$ unbiased fundamentalists face their highest possible information cost c_{-1}.

are neighborhoods of the origin and of the maximum and minimum values, even when dealing with c_0 as bifurcation parameter. In order to investigate the birth of the chaotic attractor, which occurs for $c_0 \approx 0.085$, we report in Fig. 9 the graph of the second iterate of k in a neighborhood of the origin for values of c_0 close to 0.085. In more detail, in Fig. 9 (a) for $c_0 = 0.06$ the steady state is still stable (in this case, it is denoted through a black dot) and its immediate basin of attraction is bounded by the depicted unstable period-two cycle (denoted through empty squares), which however plays no role since, due to the shape of the map, $x = 0$ is actually globally stable; it loses stability through a reverse subcritical flip bifurcation of k for $c_0 \approx 0.085$ (see Fig. 9 (b)) at which the unstable period-two cycle collides with the steady state, which is unstable for larger values of c_0 (see Fig. 9 (c) where $c_0 \approx 0.1$), and it is then denoted through an empty dot.[14]



(a) (b) (c)

Figure 9: The graph of k^2 in a neighborhood of $x = 0$ for $A = 18$, $\beta = 4$, $s = 3$, $d = 1$, $c = 0.01$, $c_{-1} = 1$, and $c_0 = 0.06$ in (a), $c_0 = 0.085$ in (b) and $c_0 = 0.1$ in (c), respectively

As argued in Section 4, e.g. for $c_0 = 0.01$ or $c_0 = 0.06$ the steady state is stable because of the low value of the information cost faced by unbiased fundamentalists - who play a stabilizing role (see Hommes 2013) - despite the high value of the information cost of rational agents, who are stabilizing, too. The steady state becomes unstable for e.g. $c_0 = 0.1$, like it is in Fig. 5 (b), in consequence of the decreased evolutive attractiveness exerted by unbiased fundamentalists, due to their reduced net profits.

[14]The same rule about the symbol to be used to denote the stable/unstable steady state has been applied in Fig. 8, as well.

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