DEMS Department of Economics, Management and Statistics

DEMS WORKING PAPER SERIES

Oligopoly model with interdependent preferences: existence and uniqueness of Nash equilibrium

Marco F. Boretto, Fausto Cavalli and Ahmad K. Naimzada

No. 462 – March 2021

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

Oligopoly model with interdependent preferences: existence and uniqueness of Nash equilibrium[1]

Marco F. Boretto, Fausto Cavalli and Ahmad K. Naimzada

Abstract

We propose a model to describe and study the effect of social interdependent preferences in a Cournot oligopoly based on a game in which the utility functions of firms depend on a combination of weighted profits of their competitors. If social interaction is neglected, the model reduces to the classic Cournot game, diverting from it as the role of social interaction becomes more and more relevant. Several synthetic measures are proposed to summarize the overall behavior of the agents and some configurations characterized by particular interactional structures are presented. Finally, the study of the well-posedness of the proposed framework is investigated, in terms of the existence and uniqueness of Nash equilibria. To this end, we generalize the conditions under which the existence and/or uniqueness of Nash equilibrium in classic game is guaranteed for particular Cournotian oligopoly models without interdependent preferences. In particular, we focus on two families of oligopolies, respectively consisting of "concave" oligopolies and oligopolies with isoelastic demand function.

Keywords: Cournot Game, Preference interdependence, Network, Nash Equilibrium, existence and uniqueness
JEL: D43, C62, C70

1. Introduction

Competition among firms in an oligopolistic market is inherently a setting of strategic interaction. For this reason, the appropriate tool for its analysis is game theory. In Cournot games, the players are the firms, each firm's set of simultaneous actions is the set of its possible outputs and each firm's utility is represented by its profit, namely players compete to achieve the maximum possible profits [23, 14]. However, the economic literature also presents theoretical, empirical and experimental contributions that lead to reconsider the classic Cournot model, mainly for two reasons. First, classic game theory, and

[1]The authors are indebted to Prof. Paolo Bertoletti for his invaluable comments and suggestions, and to the Professors in the Evaluation Committee for their comments during the dissertation of the PhD Thesis of Dr. Marco Boretto. Both contributions helped to improve the quality of the present contribution, which belongs to a research strand we are pursuing on oligopoly modeling with interdependent preferences. Subsequent research topics on the subject will deal with the characterization of the properties of the Nash equilibrium and its comparative statics.

so Cournot games, generally sets on the fundamental hypothesis that agents are self-interested, i.e. aim to maximize their own material payoff and do not take other's state into consideration. According to this assumption, the theory predicts that selection forces favor absolute optimization (or rational) agents [12] and that a different behavior from selfishness is doomed to extinction. Many authors have argued these assumptions are not realistic and poorly descriptive of an oligopolistic market, introducing instead the idea that it is interdependent rather than absolute performance pivotal in the long run survival [1] and that reality seems to suggest that agents acting in an oligopolistic market may not just act in a selfish manner. Second, the classic game theory analysis predicts an equilibrium outcome that rarely emerges in the experimental literature where agents, instead, agree on equilibrium outputs that lie in an interval ranging from the competitive (Walrasian) to the collusive (monopolistic) equilibrium. Earlier experiments suggested that the strategic choices fall in the interval between the Nash and the competitive equilibrium. Among the different experiments on Cournot oligopoly that obtain this result, it's worth mentioning those from Apesteguia *et al.* [3, 4] and Offerman *et al.* [20]. However, in subsequent experiments in which the number of stages of the game was significantly increased, it was shown that the strategic production outputs gradually decrease towards a collusive equilibrium, without even stop at the Cournot-Nash equilibrium. For example, Friedman *et al.* [11] offer an interpretation of how groups of subjects can learn their way out of dysfunctional heuristics and suggest elements for a new perspective on the emergence of cooperation.

The literature on industrial and management economics offers also examples of strong intra-group competitions in which one group has the objective to maximize relative profits, that is the difference between its own profits and the material profits obtained by the competing group[15, 19, 13, 2, 16], with the effect of production choices that are above the Nash equilibrium level of the classic Cournot competition.

The previous pieces of evidence deserve a reconsideration of the modeling of Cournotian competitions, in order to provide possible theoretical explanations of the emergence of equilibria diverting from that in the classic Cournot model. Without any intention of providing an exhaustive review of the theoretical existing literature on the subject, we mention two seminal contributions that provide modeling approaches to account for the emergence of monopolistic or Walrasian equilibria from Cournotian competitions. For the former case, Cyert and De Groot [8] proved that the existence of positive interests (spillovers) in the individual utility deriving from the performances (profits) of the other agents can provide the explanation for the emergence of a learning path driving firms from non-cooperative equilibrium choices to cooperation. A similar approach has been more recently carried out in [17] in which a game with partial cooperation for resource exploitation is studied. On the contrary, Vega-Redondo [25] showed that if firms pursue the maximization of the relative performances considering the negative effects on the utility of the single player caused by the profits of the competitors, Walrasian equilibrium emerges. In addition to these contributions, we mention the cases of partial ownership, partial equity interests [7] as examples of positive interdependence where the fortunes of potential competitors are linked by a positive correlation among material profits, the case of a Cournot duopoly in the presence or in the absence of cross-holdings, pointing the attention to the externalities generated by the two firms [9].

However, the above-mentioned literature lacks a unitary theoretical approach that is able to encompass the range of outcomes that arise in the experimental/empirical literature about oligopolistic Cournotian competitions. Our research aims at proposing a framework that provides an alternative explanation for the convergence towards equilibria ranging from those Walrasian to the collusive one, included the classic Cournot-Nash. The approach we adopt is inspired by the literature on interdependent preferences, in particular from the contribution of Levine [18], in which a theoretical model is proposed with the goal to explain the evidence from the data coming from economic surrogates such as the ultimatum and contribution games among others. Levine's model also represents a novelty with respect to previous literature since it accounts for both positive and negative spillovers of the opponents' monetary payoffs on the utility of each agent. He refers to each of these two scenarios as "altruism" and "spitefulness", respectively. Such an approach has been subsequently refined by Sethi and Somanathan in [22], in which Levine's model is refined by taking into account the possibility of reciprocal behaviors for the agents.

In the present contribution, we introduce a family of Cournotian games in which the utility of a given firm does not necessarily coincide with its profits, but can depend on the profits of the other firms. We allow for both positive and negative spillovers of the material profits of the competitors on the utility of a given firm, represented by interactional weights. The description of the market side is kept general, while homogeneous constant marginal costs are taken into account for all firms. After studying under which conditions the distribution of weights describes economically relevant interactional structures and proposing aggregated measures that portray particular interactional structures, we identify some significant, simple examples of interdependent preferences distributions. With the help of these, we show that the results from Friedman and Apesteguia can be explained with particular interdependent preference structures, without relying on coordination or learning arguments. Depending on how the preference interdependence will be structured, this model can result in different equilibrium choices covering all the spectrum that goes from the Walrasian to the monopolistic equilibrium. Finally, we show that classic general conditions on price functions for the existence and uniqueness of Nash equilibrium can be adapted to take into account preference interdependence, so that the proposed framework is well-posed and allows for considering markets characterized by popular demand functions, as those linear or isoelastic.

The present paper is meant to be the first step in the description and investigation of the effect of a structure of social interdependent preferences in a Cournot oligopoly based on a game in which the utility functions of the firms depend on a combination of weighted profits of their competitors. The following steps are the characterization of the Nash equilibrium, the analysis of the effects of interdependent preferences and the comparative statics of centrality measures at the equilibrium.

The remainder of the paper is organized as follows: in Section 2 we present the model, in Section 3 we introduce some particular structures of interaction to provide a setting to study the problem of existence and uniqueness of the Nash equilibrium, in Section 4 we present existence and uniqueness results. Section 5 bears conclusions and in Appendix we collect proofs.

2. The model

We consider an oligopolistic market in which N firms, identified by an index $i \in \{1, 2, \ldots, N\}$, produce a homogeneous good and compete in choosing the output level $q_i \geq 0$. Each firm faces linear cost function with identical constant marginal cost $c > 0$. Prices are determined by the inverse demand function $p : I \to [0, +\infty)$, $Q \mapsto p(Q)$, where I is a suitable domain. We assume that p is continuous on I, twice-differentiable and strictly decreasing on $I \cap [0, b)$ and null on $I \cap [b, +\infty)$, for some $b \in \mathbb{R} \cup \{+\infty\}$. We collect output levels in a vector $\boldsymbol{q} \in [0, +\infty)^N$.

Each firm realizes a profit given by $\pi_i(q_i, Q_{-i}) = q_i(p(Q) - c)$, where $Q_{-i} = \sum_{j=1, j\neq i}^{N} q_j$ is the aggregate quantity produced by all firms but the i-th one and $Q = Q_{-i} + q_i$ is the aggregate output level of the industry. According to [22], in what follows we refer to π_i as the material payoff of firm i and we assume that each firm has interdependent preferences that are described by the utility function

$$v_i = \pi_i(q_i, \boldsymbol{q}_{-i}) + \sum_{j=1, i\neq j}^{N} \beta_{ij}\pi_j(q_i, \boldsymbol{q}_{-i}) \tag{1}$$

where $\boldsymbol{q}_{-i} \in [0, +\infty)^{N-1}$ is the vector collecting the output levels of all firms but the i-th one and β_{ij} are constant coefficients representing the network of dependences among the agents' preferences. Coefficient β_{ij} weights to what extent preferences of firm i depends on the material payoff of firm j.

Introducing coefficients $\beta_{ii} = 0$ for $i = 1, \ldots, N$, weights β_{ij} can be collected in a hollow matrix B

$$B = \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \cdots & \beta_{1N} \\ \beta_{21} & 0 & \beta_{23} & \cdots & \beta_{2N} \\ \beta_{31} & \beta_{32} & 0 & \cdots & \beta_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{N1} & \beta_{N2} & \beta_{N3} & \cdots & 0 \end{bmatrix},$$

which represents the adjacency matrix of a (possibly negatively) weighted directed graph, in which the coefficient related to the edge connecting vertex i to vertex j represents the (positive or negative) weight through which the utility of firm i depends on the profits of firm j. We note that setting $\beta_{ii} = 0$ allows dropping condition $i \neq j$ in (1). The utility of each firm i is then affected by its own material payoff and by a linear combination of the material payoffs of some of or all the other firms. We denote with $E_i \subset \{1, 2, \ldots, N\} \setminus \{i\}$ the set of all firms whose material payoff affects the utility of firm i (i.e. $\beta_{ij} \neq 0$ if and only if $j \in E_i$). Set E_i corresponds to the (first degree) neighborhood of node i in the graph described by matrix B. To explicitly show that utility function depends on coefficients β_{ij}, in what follows we write $v_i(q_i, \boldsymbol{q}_{-i}, B)$.

The first evident consequence of preference interdependence is that, depending on the sign of β_{ij}, firm i can achieve the same utility by having smaller own profits if the other firms with which it has interaction have larger (when $\beta_{ij} > 0$) or smaller (when $\beta_{ij} < 0$) profits, since a part of the reduced utility coming from own profits can be compensated by the utility coming from the material payoff of other players, as a consequence of the interdependence of preferences.

Since p is a decreasing function, we have that weights β_{ij} have an opposite effect on the marginal utility $\partial v_i / \partial q_i$ with respect to the effect on the utility v_i. As β_{ij} increases, ceteris paribus, the marginal utility of a firm decreases, while the opposite occurs as β_{ij} decreases.

Accordingly to (1), the preferences of each firm are influenced by two levels of interactions in which firms are involved. If we neglect interdependence among preferences, the utility function is affected by the market interaction among firms through profits (actually, in this case v_i exactly corresponds to the profits): at this level, firms are not individually involved, but each of them influences the final price just depending on the quantity they decide to produce, and not based on the firm's identity. If firm i and firm $j \neq i$ produce the same amount q of good, the influence they exert on the price determination is exactly the same. The network of interdependent preferences introduces an additional level of interaction, in which each firm is possibly involved in a way that is different from that of the other firms. At this level, we can say that firms are involved in a network of social interactions, through which each firm has its own neighborhood of firms with which it interacts and to which it is linked, with the neighborhood set possibly ranging from an empty set to the whole industry. Similarly, for each firm i, we have a set of firms whose social preferences depend on the material payoff of firm i. The configurations of outgoing and ingoing links due to social preferences can be, in principle, asymmetric. In an extreme case, the preferences of a firm can be affected by the material payoff of all the other firms and, at the same time, its material payoff may not influence the utility function of any of the remaining firms. And, indeed, vice-versa.

Moreover, each firm can behave in a completely heterogeneous way with respect to each firm with which it interacts. Such heterogeneity is described by the size and the sign of each weight β_{ij}, whose absolute value then describes the degree of social interaction of firm i toward firm j. The sign of β_{ij} identifies the kind of social interaction firm i has toward firm j. To this end, according to the way the agents' behavior is identified in the literature about interdependent preferences, we say that firm i is respectively altruistic, selfish and spiteful toward firm j if $\beta_{ij} > 0, \beta_{ij} = 0$ and $\beta_{ij} < 0$, respectively. We stress that such expressions are not intended to connote a moral or psychological involvement of firms, but they are simply borrowed from the literature about interdependent preferences

([18, 22]). In what follows, when we say that firm i is, for instance, altruistic toward firm j we mean that the preferences of firm i are socially linked to the material payoff of firm j and that the spillover of the material payoff of firm j on the utility of firm i is positive, without entering into details of the reasons for which such spillover is positive.

The distribution of weights β_{ij}, $j = 1, \ldots, N$ characterizes the social interaction of firm i toward the whole industry, as well as the distribution of weights β_{ij}, $i = 1, \ldots, N$ characterizes the social interaction that the industry has toward a given firm j. In some cases, it can be useful to summarize these two sets by means of a couple of synthetic measures. To this end, we identify each element of vector[2] $B\boldsymbol{u}$ as the *overall outgoing degree* of social interaction. Element $(B\boldsymbol{u})_i$ corresponds to the i-th row summation of elements of the weight matrix B, i.e. it aggregates all the weights that firm i places on the material payoff of its competitors. Similarly, we identify each element of vector $\boldsymbol{u}^T B$ as the overall *ingoing degree* of social interaction. In this case, element $(\boldsymbol{u}^T B)_j$ provides the j-th column summation of elements of weight matrix B, i.e. it aggregates all the weights that all the firms in the industry place on the material payoff of a given firm j. We stress that identical synthetic measures can correspond to completely different weights' distributions, so in most cases they just allow capturing the average, outgoing or ingoing, degree of social interaction.
The following example shows the above-mentioned elements.

Example 1. (A general network of social interactions)
We consider the 7×7 weighted matrix B

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0.5 & 0.2 & 0 & -0.5 & 0 \\ 0.2 & 0.3 & 0 & 0.7 & 0.5 & 0.9 & 0 \\ -0.5 & 0.4 & 0.2 & 0 & -0.3 & 0.7 & 0 \\ -0.1 & -0.15 & -0.19 & 0 & 0 & -0.1 & 0 \\ -0.02 & -0.18 & -0.12 & -0.13 & -0.09 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2}$$

which gives rise to the network shown in Figure 1, where the number (in red) on each node corresponds to a firm. A directed/oriented link or edge between two nodes $i \neq j$ depicts the dependence of the utility of firm i on the material payoff of firm $j \neq i$, represented by $\beta_{ij} \neq 0$. The network is oriented in the sense that each arrow indicates the direction along which such dependence realizes. The interdependence between two firms can realize in two ways: it can be unidirectional (in this case the number of links connecting two firms is unique and directed from a node to another one, e.g. as in the case of firms 4 and 5)

[2]With $\boldsymbol{u} \in \mathbb{R}^N$ we denote the vector whose components are $u_i = 1, i = 1, \ldots, N$.

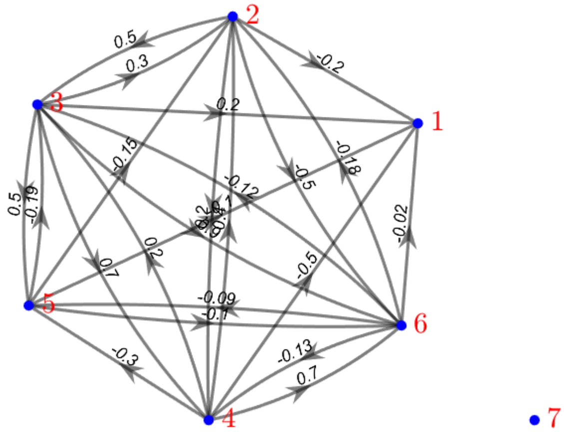

Figure 1: Graphical representation of the network described by weight matrix B in 2

or not (in this case we have a couple of links, e.g. as in the case of firms 3 and 5). In the network of Figure 1, the weight that quantifies the extent of the link between firm i and j is reported above the edge connecting node i to node j, and corresponds to coefficient β_{ij} which represents the magnitude of the "social interest" of player i towards player j.
Each row of matrix B collects the weights that the corresponding firm gives to the opponents' profits. For example, if we focus on the first row (firm 1) we can see that it represents a self-interested firm which opponents profits' weights are all equal to zero, so that the utility of such firm coincides with its own material profits.
A different situation is depicted by the second row where firm $i = 2$ considers in its own utility also the material payoff of the competitors, placing on them both positive and negative weights. For example, firm 2 weights negatively the profits of firm 1 ($\beta_{21} = -0.2$) while evaluates positively the profits of firm 3 ($\beta_{23} = 0.5$). In this case the utility of firm 2 is diminished when firm 1's profits increase, and increased when firm 3's profits grow up.

As we already noted, the graph is oriented. For example, the first node ($N = 1$) has no outgoing but five ingoing links with different signs and weights. This situation perfectly depicts the case, we mentioned above, of a self-interested firm, which nonetheless contributes to some of the opponents' utility.
Moreover, if we compare node 1, node 3 and node 7 we can note that the number of outgoing and/or ingoing edges is very different and it can be used as a first

indication of the level and kind of social interaction of the firm. Firm 7 has no social interactions with the other firms, as its preferences do not take into account the profits of the other firms and its profits are not taken into account in the utility of any of the other firms. The unique channel of interaction of firm 7 with the other firms is the market. Conversely, firm 1 is involved in the network of social interaction. Even if its preferences do not take into account the profits of the other firms, its profits do affect the preferences of all the other firms. Finally, firm 3 is completely involved in the network of social interactions, with both outgoing and ingoing links with the other firms. Its utility derives from a combination of its own profits and a fraction of the profits of every other its opponent. If we look at the column vector $B\mathbf{u} = [0, 0, 2.6, 0.5, -0.54, -0.54, 0]^T$ resulting from each row summation, we can interpret each element as indicating the overall outgoing degree of social interaction of the corresponding firm. A positive value represents a firm as being on average altruistic, a negative value as being on average spiteful and a value equal to zero as being on average self-interested.

A clarification over the terminology used might be useful. For socially altruistic on average, we mean a firm that positively binds its own utility to the material payoff of others; however, a firm may evaluate differently two different firms, in the sense it can be altruistic towards the first one and spiteful towards the other. Let consider firm 1 and firm 2 which can be defined as generally self-interested. Although their social degree coincides the two firms are indeed very different, on average. Looking at the solely outgoing degree on average might be misleading. In this case firm 1 is actually self-interested towards each competitor, while firm 2 is self-interested only on average.

The same line of reasoning can be applied to the comparison between firm 3 and 4. They both act altruistically on average, but the former is always acting altruistically with all its opponents while the latter only on average. Another interesting insight of the composition of matrix B is given by the comparison between firm 5 and firm 6, which both have the same social degree of spitefulness and both act as spitefully towards each competitor, weighting however in different ways the profits of each of their competitors. For this reason, it makes sense to identify the general attitude (or degree) of a firm towards its opponents.

Let consider the row vector $\mathbf{u}^T B = [-0.62, 0.37, 0.39, 0.77, 0.11, 1]$ resulting from each column summation. Each of its values indicates the ingoing social degree of interaction, i.e. how each firm in the network is taken into account on average in the opponents' utilities. For example, the first element indicates that firm 1 is on average negatively considered in the opponents' utility. On the opposite side, the second element indicates that firm 2 positively contributes, on average, to its opponents' utilities. Finally, the last element identifies a firm that, on average, does not contribute to its opponents' utilities.

Each firm tries to maximize its own utility by choosing the quantity to produce. Such setting can be described by a game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, where $\mathcal{N} = \{1, 2, \ldots, N\}$ is the set of players, $S_i \subset [0, +\infty)$ is the set of strategies of each player i and function v_i defined in (1) is the utility function for the i-th firm, for $i \in \mathcal{N}$. A particular case among the games belonging to Γ is the classic Cournot game, namely game $\Gamma_0 = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, O)) = (\mathcal{N}, S_i, \pi_i(q_i, \boldsymbol{q}_{-i}))$ obtained setting B equal to the null matrix O. In Γ_0 firms choose the quantity to produce in order to maximize material payoff, i.e. profits.

As already discussed, the first straightforward effect of considering interdependent preferences is that we can identify an additional channel of interaction among firms, along with the usual market interaction. Such latter channel is the unique one that is present in the classic Cournot game Γ_0 and establishes a "global", market-related, form of interaction among all firms, mediated by the common inverse price function through the aggregate output level. Interdependent preferences establish another, possibly local or even one-to-one form of interaction, described by the distribution of coefficients β_{ij}. As such coefficients divert from 0, game Γ diverts from Γ_0, with the role of social interaction that becomes more and more relevant as $|\beta_{ij}|$ increase.

3. Relevant structures of social interaction

As it will become evident in what follows, the general framework described by game Γ allows for the description of a wide range of situations. In order to simplify and improve the economic interpretation of the analytical results that will be provided in the remainder of the thesis, it is convenient to introduce some simplified scenarios, which are characterized by networks of social interactions with particular structures.

The first and simplest structure we consider consists of "homogeneous" weight distribution. This corresponds to an economic scenario in which firms are assumed to be identical with respect to their information about their competitors and identical with respect to the way such knowledge influence their preferences. In this setting, the preferences of all firms are affected by the profits of any one of their competitors by the same extent β. The matrix describing the network of social interactions is then $B = \beta(U - I)$ where U is the $N \times N$ matrix whose elements are equal to 1, and I is the $N \times N$ identity matrix. For example, if $N = 3$ we have the following 3×3 matrix

$$B = \begin{bmatrix} 0 & \beta & \beta \\ \beta & 0 & \beta \\ \beta & \beta & 0 \end{bmatrix} \tag{3}$$

to which corresponds the network described in Figure 2.

The present scenario is very unsophisticated and it is even inappropriate to speak about a "structure" of social interaction, because of the very regular distribution of weights. However, it deserves some investigation as it allows casting a first glance at the possible equilibrium configurations described by game Γ.

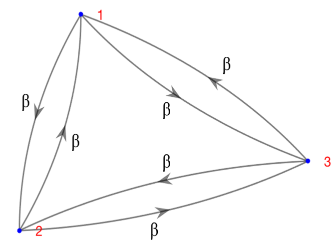

Figure 2: Graphical representation of the network described by weight matrix B in 3

Proposition 1. *Let us consider an oligopoly for which the network of social interaction is described by a matrix B in which $\beta_{ij} = \beta$ for any $i \neq j$, $1 \leq i, j \leq N$. Let p be an inverse demand function for which game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$ has a unique internal equilibrium $\boldsymbol{q}^*(\beta)$ for any β in $(-1/(N-1), 1)$.*

We have that as $\beta \to 1^-$ the aggregate equilibrium output level $Q^*(\beta)$ converges to the equilibrium output level Q^*_M of a monopoly in which p is the inverse demand function.

We have that as $\beta \to \left(-\frac{1}{N-1}\right)^+$ the equilibrium aggregate output level $Q^*(\beta)$ converges to the aggregate equilibrium output level Q^*_C of a competitive market in which p is the inverse demand function.

Moreover, on increasing β in $(-1/(N-1), 1)$ we have that the aggregate equilibrium output level $Q^*(\beta)$ is a continuous function that monotonically varies from Q^*_M to Q^*_C.

An oligopoly is usually described as a market structure dominated by a few firms and characterized by an intermediate degree of competition, lying between monopoly (just one firm, minimum competition degree) and perfect competition (many firms, maximum competition degree). The family of games Γ considered in Proposition 1 describe oligopolies that provide a continuum of outcomes (identified by the industry output levels) that range between such extremal outcome levels. As weights β approach 1, the setting with interdependent preferences tends to describe the setting in which a social planner coordinates the agents' production in order to maximize the aggregate industry profits and just looking at the aggregate output level at the equilibrium corresponds to that of a monopoly market. Similarly, as weights β approach $-1/(N-1)$, the aggregate output level at the equilibrium corresponds to that of a competitive market.

In game Γ, the transition between the monopolistic and competitive markets (aggregate) equilibria do not (only) occurs on increasing the number of suppliers populating the market, but it takes place, for any given number of firms, as the distribution of weights describing interaction among firms decrease from the uniform distribution $\beta = 1$ to the uniform distribution $\beta = -\frac{1}{N-1}$. We stress that even such a very simplified setting is able to represent all the possible situations, in terms of aggregate equilibrium outcomes, ranging from the monopolistic limit to the competitive limit scenarios.

The previous proposition also provides two intrinsic bounds on weights β_{ij}, leading to the following assumption

Assumption 1. $-\frac{1}{N-1} < \beta_{ij} < 1$,

so that, with coefficients in such range, we can compare the aggregate Nash equilibrium of any game Γ to the (aggregate) output levels of a monopoly and of a competitive market, as it guarantees the following result, in which Q_M^* and Q_C^* are defined in Proposition 1.

Proposition 2. *Let us consider an oligopoly with a unique, internal Nash equilibrium and for which the network of social interaction is described by a matrix B that satisfies Assumption 1. Then $Q_M^* < Q^* < Q_C^*$, where Q^* is the unique aggregate equilibrium output level.*

In terms of aggregate output levels (and hence of the degree of competitiveness), games Γ allow for a continuous transition between two extreme market situations. Accordingly, we can address such two extreme situations as the "monopolistic limit" and the "competitive limit" of sequences of games Γ.

We highlight that, in the case of a duopoly, Assumption 1 provides $-1 < \beta_{ij} < 1$ and we find the same symmetric bound on weights that are used in [22] and in the literature strand about oligopolies ([7]). In general situations, the bound provided by Assumption 1 is asymmetric, with potentially larger positive than negative weights in absolute value.

Without any intention of providing an exhaustive review of all the possible particular structures, we mention three alternative configurations of interaction between firms. The second configuration of the network we consider is obtained setting $\beta_{ij} = \beta_i \in (-1/(N-1), 1)$ for $i, j = 1, \ldots, N$ and $i \neq j$. In such setting we have that each firm has a homogeneous behavior with respect to all the other firms in the industry, but the behavior of each firm can be different with respect to that of the other firms. This corresponds to an economic scenario in which each firm, due to the complexity of the framework and/or since it lacks of distinctive information concerning each (or at least some) of its competitors, is not able to develop an articulated and heterogeneous network of social interactions. So it consequently behaves in the same way with respect to all its competitors (we can say that it interacts with the remainder of the industry). However, firms are assumed to be possibly heterogeneous, so that they can be either uniformly altruistic, selfish or spiteful toward any other firm j, with a constant degree of social interaction. Summarizing, such a configuration is characterized by firms that are heterogeneous but each of them homogeneously takes into account all its competitors.

The third configuration we consider is described in terms of the transposed matrix of the previous case. This structure is obtained setting $\beta_{ij} = \beta_j \in (-1/(N-1), 1)$ for $i, j = 1, \ldots, N$ and $i \neq j$. In such setting the information endowment and knowledge about each firm is very elevated and shared among all firms, so that all firms consider a given firm in the same way, and they consequently behave in the same way with respect to it, but the behavior of each firm can be different with respect to that of the other firms. All firms j have the same social preferences toward a given firm i, being either uniformly altruistic, selfish or spiteful toward the firm i, with a constant degree of social interaction. However, preferences toward firm i can be different from those toward firm k.

In the last possible configuration of the network the overall outgoing degree of social interaction is the same for all firms, i.e. vector $B\boldsymbol{u}$ has identical elements. In this case no restriction is imposed on each weight $\beta_{ij} \in (-1/(N-1), 1)$, the summation of each row just has to provide the same value. All firms have, on average, the same overall degree of altruism or spitefulness (or they can even be selfish), but the way the social preferences of firm i are influenced by material profits of firm j can be different on varying i and j.

4. Existence and uniqueness of Nash equilibria with interdependent preferences

We want to generalize the conditions under which the existence and/or uniqueness of Nash equilibrium in classic game Γ_0 is guaranteed for particular Cournotian oligopoly models without interdependent preferences. To this end, following e.g.[21, 26, 6] we consider two settings, respectively consisting of "concave" oligopolies (i.e. for which assumptions on the inverse demand function and network of interdependent preferences guarantee the concavity of the utility function) and oligopolies with isoelastic demand function, as an economically relevant crucial example of a setting that provides a game in which the best response functions are not monotonic.

We start considering the family of oligopolies for which the payoff function is concave. In such setting, to guarantee the uniqueness of the equilibrium, it is necessary to introduce a bound on the maximum possible strategy chosen by the agents. On the contrary, without such assumption, it is possible to see that multiple equilibria can occur even without interdependent preferences (see e.g. [21]). To this end, we introduce the capacity limit $L_i > 0$ for each firm $i \in \mathcal{N}$, which represents the maximum output level that each firm is able to supply. For more details about such aspects we refer to [21, 6].

To provide a suitably rich family of oligopolies that both include relevant examples and at the same time for which existence and uniqueness of the Nash equilibrium is guaranteed, we introduce some assumptions on the inverse demand function p and on coefficients β_{ij}. In what follows, the set of the oligopolies that fulfill the following Assumptions will be identified with $\mathcal{O}$.

Assumption 2. *For any $q_i \in [0, L_i], i \in \mathcal{N}$ and for $Q \in \left[0, \sum_{k=1}^N L_i\right]$ we have $p'(Q) < 0$ and for any $z \in \left[0, \sum_{k=1}^N L_i\right]$ we have*

$$\begin{cases} p''(Q)z + p'(Q) < 0, \\ -p''(Q)\frac{z}{N-1} + p'(Q) < 0. \end{cases} \tag{4}$$

The previous condition is the generalization to the case of interdependent preferences of decreasing marginal revenue condition $p''(Q)q_i + p'(Q) < 0$, which is given for concave oligopolies without preferences' interdependence (see [26]), corrected to account for the contribution of interdependent preferences. We remark that the former condition in (4) is always fulfilled for a concave function, while the latter one is always fulfilled for a convex function. Finally, Assumption 2 is fulfilled for a linear demand function and it's worth noting that the former condition in (4) is increasingly less restrictive as N increases.

Concerning the admissible distributions of weights, Assumption 1 just provides a first restriction on the economically relevant values of β_{ij}. However, the resulting set of weights is still too wide to guarantee the existence and/or uniqueness of the Nash equilibrium of Γ. If we applied to game Γ the assumption that in the literature of games on networks allows obtaining existence and uniqueness of the Nash equilibrium (see [5]), we would impose $\rho(B) < 1$. However, the family of oligopolies described by games obtained adopting such condition would be too restricted. For instance, it would be not possible to consider a sequence of games Γ approaching the monopolistic limit, as in such case we must necessarily have $\rho(B) > 1$ in a neighborhood of the limit. Moreover, we stress that the above-mentioned condition is applied in the literature to a situation in which $\beta_{ij} \leq 0$.

As it is evident in the proofs of the following propositions and accordingly to the literature, the problem of studying the existence and uniqueness of the Nash equilibrium of Γ can be rephrased into a linear complementarity problem (from now one, LCP) (see e.g. [21, 24]). For an LCP, well-posedness is guaranteed if the matrix associated with the corresponding problem is a P-matrix, i.e. a matrix in which all the principal minors are strictly positive (for a survey about P-matrices we refer to [24]). As we can see from the proofs of the following propositions, the matrix associated to the linear complementarity problem arising from the optimization problem related to the Nash equilibrium of game Γ is the matrix $I + B$. This leads to the assumption

Assumption 3. *Matrix $I + B$ is a P-matrix.*

The previous Assumption can be seen as a generalization of assumption $\rho(B) < 1$ (as such condition, when $-\frac{1}{N-1} < \beta_{ij} \leq 0$, guarantees that $I+B$ is a P-matrix) and has basically the same economic interpretation: local complementaries have to be small enough to avoid the emergence of a non-finite equilibrium solution. The first relevant consequence of Assumption 3 is that $I + B$ is an invertible matrix, which will play a key role in the characterization of the internal Nash equilibrium in terms of the inverse of $I + B$. In addition, it bears several interesting properties that allow studying (and characterizing) the family of oligopolies in $\mathcal{O}$. For example, starting from a game in $\mathcal{O}$ it is possible to vary coefficients with continuity to obtain Γ_0, which indeed belongs to $\mathcal{O}$. This guarantees that any oligopoly in $\mathcal{O}$ can be studied by considering a continuous family of oligopolies in $\mathcal{O}$ with progressively larger coefficients, always starting from the purely selfish scenario. In particular, if an oligopoly of N firms belongs to $\mathcal{O}$, all the oligopolies obtained rescaling the coefficients of some (possibly one or even all) firms by any coefficient $\beta \in [0,1]$ will describe oligopolies belonging to $\mathcal{O}$. Moreover, if an oligopoly of N firms belongs to $\mathcal{O}$, also the oligopolies obtained removing one firm has to belong to $\mathcal{O}$.

Now we consider the existence and uniqueness of Nash equilibrium in the case of concave oligopoly.

Proposition 3. *Under Assumptions 1-3, game Γ has at least a Nash equilibrium $\boldsymbol{q}^*$ with $q_i^* \in [0, L_i]$. If $q_i^* < L_i$ for each $i \in \mathcal{N}$, then the equilibrium is unique. Moreover, in the particular case of the linear demand function, game Γ always has a unique Nash equilibrium.*

The previous Assumptions guarantee a setting for which the Nash equilibrium exists, and if it belongs to $[0, L_i)^N$, it is also unique (this is the case in which the capacity limit of no firms coincides with its equilibrium output level).

Assumption 3 is a suitable setting also for "non-concave" oligopolies, as for example in the relevant case of the isoelastic demand function.

Proposition 4. *Under Assumptions 3, if $p(Q) = 1/Q$, game Γ has a unique Nash equilibrium $\boldsymbol{q}^*$.*

We stress that the equilibrium provided by the previous proposition can be also a boundary equilibrium, and this just depends on the network of social interactions among firms.

5. Conclusions

We introduced an oligopolistic market in which N firms produce a homogeneous good and compete in choosing the output quantity given the individual interdependent preferences structure described by a utility function that depends both on the individual profits and on a linear combination of the profits of some of or all the other firms. The introduction of an interdependent preferences structure provides a framework that is able to, simultaneously deal, in the individual utility function, with both positive and negative effects due to the material payoffs of the other players. This provides a generalized setting that allowed us to encompass in a unified setting all the effects evidenced by the experimental literature, exposed in the introduction of the paper, in terms of the outcome of the game. In fact, even considering a very prototypical and simplified scenario, this setting proved to be capable to describe a wide range of situations. Considering a homogeneous weight distribution (i.e matrix $B = \beta(U - I)$) we characterized families of games both in the case of uniform positive interdependence (altruistic preferences) and the case of uniform negative interdependence (spiteful preferences) for which volumes of production are coherent with different market forms, ranging fluidly from the monopolistic (as the uniform distribution of weights converges to the 1) to the competitive limit (as it converges to the $-\frac{1}{N-1}$), passing through the classic Cournot oligopoly.
We considered a network in which the weights the single firm places on opponents' material payoffs are the same in order to represent a scenario in which

the single firm does not discriminate between its opponents but treats each competitor the same way. We also considered the case of a network in which is the industry that homogeneously behaves toward each single firm, namely the weights the whole industry places on the single firm's material payoffs are the same. Finally, we considered a preference structure in which the firms behave on average the same way, namely the summation of their social weights are the same.

Moreover, we showed how the proposed approach provides a reliable framework to work with whose behavior, with respect to the existence and uniqueness of Nash equilibrium, is in line with the classic oligopoly modeling without interdependence of preferences. Assumption 2, on the inverse demand function, together with Assumptions 1,3, on the coefficients of interdependence, allowed us to prove the existence and uniqueness of the Nash equilibrium for families of oligopolies (i.e. oligopolies described by a game in which the utility function is concave) that include classic and relevant examples, such as the case of concave oligopolies and the case of isoelastic demand functions. Concerning the admissible distributions of weights, Assumption 1 provides the first restriction on the economically relevant values of coefficients β_{ij} that is not sufficient to guarantee the existence and/or uniqueness of the Nash equilibrium of the game with interdependence of preferences. Therefore, we rephrased the problem of studying the existence and uniqueness of the Nash equilibrium into a linear complementarity problem which guarantees the well-posedness of the matrix, associated with the corresponding problem, as it satisfies the condition to be a P-matrix. We showed that if $I + B$ is a P-matrix it bears several interesting properties that allow studying (and characterizing) the family of oligopolies by considering a continuous family of oligopolies with progressively larger but suitable coefficients. In particular, P-matrix assumption is actually a generalization of the $\rho(B) < 1$ condition that it is often imposed in game theory on networks for the existence and uniqueness of the Nash equilibrium.

Appendix

Proof of Proposition 1. The utility function is

$$v_i = q_i(p(q_i + Q_{-i}) - c) + \beta \sum_{j=1, i \neq j}^{N} (q_j(p(q_i + Q_{-i}) - c))$$

from which we have that marginal utility is

$$\frac{\partial v_i}{\partial q_i} = p(Q) - c + q_i p'(Q) + \beta Q_{-i} p'(Q) \tag{5}$$

Since we are considering the aggregated output level at an internal equilibrium $\boldsymbol{q}^*$, first order condition necessarily requires $\partial_{q_i} v_i = 0$, Thanks to this and summing the right hand side of (5) for $i = 1$ to N we obtain

$$Qp'(Q) - \frac{N}{(N-1)\beta + 1} \cdot (c - p(Q)) = 0, \tag{6}$$

By assumption, if $\beta > 0$ the previous equality is solved by a unique $Q^*(\beta) > 0$ for any $\beta > 0$. We have that $\lim_{\beta \to 1^-} Q^*(\beta)$ is then the solution to $Qp'(Q) = c - p(Q)$, which is exactly the output level of a monopoly with inverse demand function $p(Q)$. Conversely, if $\beta < 0$, we have that $\lim_{\beta \to -\frac{1}{N-1}} Q^*(\beta)$ is then the solution to $c = p(Q)$, which is exactly the output level of a competitive market in which the inverse function is $p(Q)$. Concerning the monotonicity of $Q^*(\beta)$, let us introduce function $f_\beta(Q)$ defined by the left hand side of (6). Since $Q^*(\beta)$ is a maximum point, we have $f_\beta(Q^*(\beta)) > 0$ for $Q < Q^*(\beta)$ and $f_\beta(Q^*(\beta)) < 0$ for $Q > Q^*(\beta)$. In particular, since p is strictly decreasing, we have $p(Q^*(\beta)) - c > 0$ for $Q < Q^*(\beta)$ and $p(Q^*(\beta)) - c < 0$ for $Q > Q^*(\beta)$. This means that if $\beta_1 > \beta_2$,noting that $N/((n-1)\beta + 1)$ decreases as β increases, $f_{\beta_2}(Q) > f_{\beta_1}(Q) > 0$ for $Q < Q^*(\beta_1)$, which implies that the solution to $f_{\beta_2}(\beta) = 0$ must fulfill $Q^*(\beta_1) < Q^*(\beta_2)$. □

Lemma 1. *Let B be an invertible matrix that fulfills Assumption 1 and assume that $\boldsymbol{\xi} = (I + B)^{-1}\boldsymbol{u}$ is componentwise nonnegative. Then $\boldsymbol{u}^T(I + B)^{-1}\boldsymbol{u} > 1$.*

Proof. Let N be the size of B. Thanks to Assumption 1, we have that $\rho(I+B) = K < N$, so $\rho\left(\frac{I+B}{N}\right) < 1$ and we can use series expansion

$$(I + B)^{-1} = \sum_{n=0}^{+\infty} \left(I - \frac{I + B}{N}\right)^n \frac{1}{N}$$

We then have

$$\begin{aligned}
\boldsymbol{u}^T(I + B)^{-1}\boldsymbol{u} &= 1 + \boldsymbol{u}^T \left(I - \frac{(I + B)}{N}\right)(I + B)^{-1}\boldsymbol{u} \\
&= 1 + \boldsymbol{u}^T \left(I - \frac{(I + B)}{N}\right)\boldsymbol{\xi} = 1 + \boldsymbol{u}^T \left(I\left(1 - \frac{1}{N}\right) - \frac{B}{N}\right)\boldsymbol{\xi}
\end{aligned}$$

Elements in $\boldsymbol{u}^T\left(I\left(1 - \frac{1}{N}\right) - \frac{B}{N}\right)$ are given by

$$1 - \frac{1}{N} - \sum_{i=1, j\neq i}^{N} \frac{\beta_{ij}}{N} > 1 - \frac{1}{N} - \sum_{i=1, j\neq i}^{N} \frac{1}{N} > 0$$

so $\boldsymbol{u}^T\left(I\left(1 - \frac{1}{N}\right) - \frac{B}{N}\right)\boldsymbol{\xi} > 0$. This allows concluding. □

Proof of Proposition 2. Since by assumption we know that a unique internal equilibrium $\boldsymbol{q}^*$ exists, marginal utility must vanish at it, having

$$\frac{\partial v_i}{\partial q_i} = p'(Q^*)q_i^* + p(Q^*) - c + \sum_{j\neq i}^{N} \beta_{ij} p'(Q^*)q_j^* = 0,$$

or, in vector form,

$$p'(Q^*)\boldsymbol{q}^* + (p(Q^*) - c)\boldsymbol{u} + p'(Q^*)B\boldsymbol{q}^* = 0 \Leftrightarrow (I + B)\boldsymbol{q}^* = \frac{c - p(Q^*)}{p'(Q^*)}\boldsymbol{u}.$$

Since the solution to the previous system is unique, we must have that matrix $I+B$ is invertible, so we can write

$$\boldsymbol{q}^* = \frac{c - p(Q^*)}{p'(Q^*)}(I+B)^{-1}\boldsymbol{u},$$

and, aggregating, we obtain

$$Qp'(Q) - \boldsymbol{u}^T(I+B)^{-1}\boldsymbol{u} \cdot (c - p(Q)) = 0.$$

Thanks to Lemma 1, we have that $\boldsymbol{u}^T(I+B)^{-1}\boldsymbol{u} > 1$, and hence from simple geometrical considerations similar to those concluding the proof of Proposition 1 we have that the solution to the last equation must lie between Q_M^* to Q_C^*. $\square$

Proof of Proposition 3. We start noting that Assumption 2 guarantees the concavity of the utility function of each player. The existence of a Nash equilibrium is then a consequence of Nikaido-Isoda Theorem (see e.g. [10]) for more details). Now assume that for equilibria there hold $q_i < L_i$ for all $i \in \mathcal{N}$.

We find the best response function of the i-th firm, for a given vector of strategies $\boldsymbol{q}_{-i}$. In principle, we have to distinguish three cases:

a) $\partial_{q_i} v_i(0) \leq 0$: since v_i is strictly concave on $[0, L]$, in this case it is also strictly decreasing it attains its maximum at $q_i = 0$;

b) $\partial_{q_i} v_i(L_i) \geq 0$: in this case the concavity of v_i guarantees that v_i is strictly increasing and hence it attains its maximum at $q_i = L_i$;

c) in the remaining situations Assumption 2 guarantees the existence and uniqueness of a solution to equation $p'(z_i+Q_{-i})z_i+p(z_i+Q_{-i})-c+\sum_{j=1}^N \beta_{ij}p'(z_i+Q_{-i})z_j = 0$, since the right-hand side is strictly decreasing on $(0, L)$, positive for $z_i \to 0^+$ and negative for $z_i \to L^-$.

We then have

$$\mathrm{BR}_i(\boldsymbol{q}_{-i}) = \begin{cases} 0 & \text{if} \quad \partial_{q_i} v_i(0) \leq 0 \\ L_i & \text{if} \quad \partial_{q_i} v_i(L_i) \geq 0 \\ z_i & \quad\text{otherwise} \end{cases} \tag{7}$$

Any equilibrium with $q_i < L_i$ for all $i \in \mathcal{N}$ must fulfill

$$q_i \frac{\partial v_i}{\partial q_i} = q_i \left(p'(q_i + Q_{-i})q_i + p(q_i + Q_{-i}) - c + \sum_{j \neq i}^N \beta_{ij} p'(Q) q_j \right) = 0 \tag{8}$$

and

$$\frac{\partial v_i}{\partial q_i} = p'(Q)q_i + p(Q) - c + \sum_{j \neq i}^N \beta_{ij} p'(Q) q_j \leq 0 \tag{9}$$

Conditions (8) and (9) can be equivalently rewritten as

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T \left(\boldsymbol{q} + \frac{p(Q)-c}{p'(Q)} \boldsymbol{u} + B\boldsymbol{q} \right) = 0 \\ \boldsymbol{q} + \frac{p(Q)-c}{p'(Q)} \boldsymbol{u} + B\boldsymbol{q} \geq 0 \end{cases}$$

Let us introduce $y = -\frac{p(Q)-c}{p'(Q)}$, so the previous system is

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T(\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0 \\ \boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q} \geq 0 \\ y = -\frac{p(Q)-c}{p'(Q)} \end{cases}$$

Note that the first three conditions describe a linear complementarity problem, in which the pattern of solution $\boldsymbol{q}$ (i.e. the position of null vs. non-null components) is independent of y. Thanks to Assumption 3, for each $y > 0$, there exists a unique solution q (different from the null vector) to such problem in which we have either $q_i > 0$ or $q_i = 0$ for each $i = 1, \ldots, N$.

Let $\tilde{B}$ be the matrix in which the ith row and column are made by null elements if $q_i = 0$ while the remaining elements are those of B. Note that $I + \tilde{B}$ is a P matrix and hence it is invertible. Let $\tilde{\boldsymbol{u}}$ be a vector in which the ith element is null if $q_i = 0$ while the remaining elements are equal to 1. We stress that $\tilde{B}$ just depends on the distribution of degrees of interaction, and not on demand function and marginal costs. Note that the last problem is now equivalent to

$$\begin{cases} \boldsymbol{q} - y\tilde{\boldsymbol{u}} + \tilde{B}\boldsymbol{q} = 0 \\ y = -\frac{p(Q)-c}{p'(Q)} \end{cases}$$

so we can write $\boldsymbol{q} = -\frac{p(Q)-c}{p'(Q)}(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}}$. Summing we obtain

$$Qp'(Q) = (c - p(Q))\boldsymbol{u}^T(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}}$$

in which the right-hand side is a strictly increasing function, while the left-hand side is decreasing, since its derivative is $p'(Q)+Qp''(Q)$, which is indeed negative if p is concave, but it is negative as well when p is convex thank to Assumption 2. This guarantees the uniqueness of Q. If in addition $q_i > 0$, we have that the solution can be written as $\boldsymbol{q} = -\frac{p(Q)-c}{p'(Q)}(I + B)^{-1}\boldsymbol{u} = -\frac{p(Q)-c}{p'(Q)}\boldsymbol{\xi}$. Multiplying both sides by $\boldsymbol{u}^T$ we have $Q = -\frac{p(Q)-c}{p'(Q)}\sum_{j=1}^{N}\xi_i$, which, combined with the relation for $\boldsymbol{q}$, allows concluding. $\square$

Proof of Proposition 4. Utility function is

$$v_i = q_i\left(\frac{1}{Q} - c\right) + \sum_{i=1}^{N}\beta_{ij}q_j\left(\frac{1}{Q} - c\right) = \frac{1}{Q}\left(q_i - cQ + \sum_{i=1}^{N}\beta_{ij}q_j(1 - cQ)\right)$$

The null vector can not be the Nash equilibrium, as p is not defined for $Q = 0$. It is easy to see that a Nash equilibrium can not have more than $N - 2$ null components. In fact, by contradiction, without loss of generality, let us assume that $q_i = 0$ for $i > 2$, so that we have

$$v_i = q_i\left(\frac{1}{q_1 + q_2} - c\right) + \beta_{i,-i}q_{-i}\left(\frac{1}{q_1 + q_2} - c\right),\ i = 1, 2$$

and

$$\frac{\partial v_1}{\partial q_1} = \frac{-cq_1^2 - 2cq_1q_2 - \beta_{12}q_2 + q_2 - cq_2^2}{(q_1+q_2)^2}, \qquad \frac{\partial v_2}{\partial q_2} = -\frac{\beta_{21}q_1 - q_1 + cq_1^2 + cq_2^2 + 2cq_1q_2}{(q_1+q_2)^2}$$

If $q_2 > 0$, we have two possibilities: v_1 is strictly decreasing if $1 - \beta_{12} - cq_2 < 0$ or it is concave and unimodal. In the first case, the best response is $q_1 = 0$, but the best response to $q_1 = 0$ can not be $q_2 > 0$ (utility function v_2 is strictly decreasing in this case). So we necessarily need that q_1 and q_2 are strictly positive at the equilibrium.

In the general case, marginal utility is

$$\frac{\partial v_i}{\partial q_i} = -\frac{q_i}{Q^2} + \frac{1}{Q} - c - \sum_{j=1}^{N} \beta_{ij}\frac{q_j}{Q^2} = \frac{1}{Q^2}\left(-cQ^2 + Q - q_i - \sum_{j=1}^{N}\beta_{ij}q_j\right)$$

so its sign is determined by the sign of the second degree polynomial $\partial_{q_i} v_i(q_i) = -cq_i^2 - 2cq_iQ_{-i} - cQ_{-i}^2 + \sum_{j=1}^{N}(1-\beta_{ij})q_j$, which represents a concave parabola, strictly decreasing for $q_i \geq 0$. We then have two possibilities for the best response

- $\mathrm{BR}_i(q_{-i}) = 0$, in which case we necessarily have $\partial_{q_i} v_i(0) \leq 0$
- $\mathrm{BR}_i(q_{-i}) > 0$

so at a Nash equilibrium q we must have a couple of relations similar to (8) and (9), so we can again write the equilibrium condition as

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T\left(\boldsymbol{q} + \frac{p(Q)-c}{p'(Q)}\boldsymbol{u} + B\boldsymbol{q}\right) = 0 \\ \boldsymbol{q} + \frac{p(Q)-c}{p'(Q)}\boldsymbol{u} + B\boldsymbol{q} \geq 0 \end{cases}$$

Let us introduce $y = -\frac{p(Q)-c}{p'(Q)} = Q - cQ^2$, so the previous system is

$$\begin{cases} \boldsymbol{q} \geq 0 \\ \boldsymbol{q}^T(\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0 \\ \boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q} \geq 0 \\ y = Q - cQ^2 \end{cases}$$

Note that the first three conditions describe a linear complementarity problem, in which the pattern of solution q (i.e. the position of null vs. non-null components) is independent of y. Thanks to Assumption 3, for each y, there exists a unique solution $\boldsymbol{q}$ (different from the null vector) to such problem in which we have either $q_i > 0$ or $q_i = 0$ for each $i = 1, \ldots, N$. However, $y > 0$, as otherwise $\boldsymbol{q}^T(\boldsymbol{q} - y\boldsymbol{u} + B\boldsymbol{q}) = 0$ would have the unique null solution, which is not consistent with $y > 0$ and would provide $Q = 0$, which is impossible as p is not defined at $Q = 0$.

Let $\tilde{B}$ be a matrix in which the ith row and column are made by null elements if $q_i = 0$ while the remaining elements are those of B. Note that $I + B$ is a P matrix and hence it is invertible. Let $\tilde{\boldsymbol{u}}$ be a vector in which the ith element is null if $q_i = 0$ while the remaining elements are equal to 1.

We indeed have

$$\begin{cases} \boldsymbol{u} - y\tilde{\boldsymbol{u}} + \tilde{B}\boldsymbol{u} = 0 \\ y = Q - cQ^2 \end{cases} \Leftrightarrow \begin{cases} \boldsymbol{q} = y(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}} \\ y = Q - cQ^2 \end{cases}$$

from which $Q = y\boldsymbol{u}^T(I + \tilde{B})\tilde{\boldsymbol{u}} = y\mu$. The last equation can be rewritten as

$$\frac{Q}{\mu} = Q - cQ^2 \Leftrightarrow \mu = \frac{1}{1 - cQ}$$

since $Q \neq 0$. The previous equation has a unique solution since $\mu > 1$. This follows from Lemma 1 noting that $\mu = \boldsymbol{u}^T(I + \tilde{B})^{-1}\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}}^T(I + \hat{B})^{-1}\hat{\boldsymbol{u}}$ in which $\hat{B}$ is the submatrix obtained from $\tilde{B}$ by removing all the rows/columns for which $q_i = 0$ and $\hat{\boldsymbol{u}}$ is a constant unitary vector with as many elements as the non-null components in $\boldsymbol{q}$. $\square$

References

[1] A. Alchian. Uncertainty, evolution and economic theory. *Journal of Political Economy*, (58):211–221, 1950.

[2] R. Antle and A. Smith. An Empirical Investigation of the Relative Performance Evaluation of Corporate Executives. *Journal of Accounting Research*, 24:1–39, 1986.

[3] J. Apesteguia, S. Huck, and J. Oechssler. Imitation-theory and experimental evidence. *Journal of Economic Theory*, 136:217–235, 2007.

[4] J. Apesteguia, S. Huck, J. Oechssler, and S. Weidenholzer. Imitation and the evolution of Walrasian behavior: Theoretically fragile but behaviorally robust. *Journal of Economic Theory*, 145:1603–1617, 2010.

[5] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's Who in Networks. Wanted: The Key Player. *Econometrica*, 74(5):1403–1417, 2006.

[6] G.I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky. *Nonlinear Oligopolies Stability and Bifurcations*. Springer, 2010.

[7] M.J. Clayton and B.N. Jorgensen. Optimal Cross Holding with Externalities and Strategic Interactions. *The Journal of Business*, 78(4):1505–1522, 2005.

[8] R.M. Cyert and M.H. DeGroot. An Analysis of Cooperation and Learning in a Duopoly Context. *The American Economic Review*, 63(1):24–37, 1973.

[9] J. Farrell and C. Shapiro. Asset ownership and market structure in oligopoly. *Journal of Economics*, 21:275–292, 1990.

[10] F. Forgo, J. Szep, and F. Szidarovszky. *Introduction to the Theory of Games: Concepts, Methods, Applications*. Springer, 1999.

[11] D. Friedman, S. Huck, R. Oprea, and S. Weidenholzer. From imitation to collusion: Long-run learning in a low-information environment. *Journal of Economic Theory*, 155:185–205, 2015.

[12] M. Friedman. Essays in Positive Economics. *University of Chicago Press*, 1953.

[13] R. Gibbons and K.J. Murphy. Relative performance evaluation for chief executive officers. *Industrial and Labor Relations Review*, 43, 1990.

[14] J.H. Hamilton and S.M. Slutsky. Endogenous timing in duopoly games: Stackelberg or Cournot equilibria. *Games and Economic Behavior*, 2:29–46, 1990.

[15] B. Holmstrom. Moral Hazard in Teams. *The Bell Journal of Economics*, 13(2):324–340, 1982.

[16] S.N. Janakiraman, R.A. Lambert, and D.F. Larcker. An Empirical Investigation of the Relative Performance Evaluation Hypothesis. *Journal of Accounting Research*, 30(1):53–69, 1992.

[17] M. Kopel and F. Szidarovszky. Resource Dynamics under Partial Cooperation in an Oligopoly. *Journal of Optimization Theory and Applications*, 128:393–410, 2006.

[18] D.K. Levine. Modeling altruism and spitefulness in experiments. *Rev. Econ. Dynam.*, 1:593–622, 1998.

[19] K.J. Murphy. Executive Compensation. *Marshall School of Business, University of Southern California*, 1999.

[20] T. Offerman, J. Potters, and J. Sonnemans. Imitation and Belief Learning in an Oligopoly Experiment. *Review of Economic Studies*, 69:973–997, 2002.

[21] K. Okuguchi and F. Szidarovszky. *The Theory of Oligopoly with Multi-Product Firms*. Springer-Verlag Berlin Heidelberg, 1999.

[22] R. Sethi and E. Somanathan. Preference Evolution and Reciprocity. *Journal of Economic Theory*, 97:273–297, 2001.

[23] N. Singh and X. Vives. Price and quantity competition in a differentiated duopoly. *The RAND Journal of Economics*, pages 546–554, 1984.

[24] M. Tsatsomeros. Lecture notes on matrices with positive principal minors: Theory and applications, 2017. Available on Internet.

[25] F. Vega-Redondo. The Evolution of Walrasian Behavior. *Econometrica*, 65(2):375–384, 1997.

[26] X. Vives. *Oligopoly Pricing, Old Ideas and New Tools*. MIT Press, 2010.