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No. 463 – March 2021

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http://dems.unimib.it/

Characterization of Nash equilibria in Cournotian oligopolies with interdependent preferences

Marco F. Boretto, Fausto Cavalli and Ahmad K. Naimzada[1]

Abstract

We study the effects on the Nash equilibrium of the presence of a structure of social interdependent preferences in a Cournot oligopoly, described in terms of a game in which the network of interactions reflects on the utility functions of firms through a combination of weighted profits of their competitors as in [7]. Taking into account the channels of social and market interactions, we detail the consequence of preference interdependence on the best response of a firm, focusing on both direct and high degree of interdependence effects between two given firms. We characterize the Nash equilibrium in terms of social and market interactions among firms, through a Bonacich-like centrality measure and a scalar index describing the degree of competitiveness that characterizes an oligopoly with interdependent preferences. Finally, we study the equilibrium of some scenarios described by regular structures of interaction.

Keywords: Cournot Game, Preference interdependence, Nash Equilibrium
JEL: D43, C62, C70

1. Introduction

There are both experimental and empirical evidence that in an economic setting corresponding to an oligopoly market, the equilibrium outcome can significantly differ from the Nash equilibrium arising in the classic Cournot game used to model oligopolistic competitions. Laboratory experiments allowed us to point out that agents often choose strategies that overestimate the Nash equilibrium, moving close to the competitive one (see e.g. Offerman *et al.* [19] and Apesteguia *et al.* [3, 4]) and this in particular concerns the initial stages of experiments. If the number of stages is increased, new scenarios can emerge, with a progressive adjustment of choices toward reduced output levels, lower than those corresponding to the Nash equilibrium and in agreement with those of a collusive equilibrium, as shown in the experiment by Friedman *et al.* [13]. Similarly,

[1]The authors are indebted to Prof. Paolo Bertoletti for his invaluable comments and suggestions, and to the Professors in the Evaluation Committee for their comments during the dissertation of the PhD Thesis of Dr. Marco Boretto. Both contributions helped to improve the quality of the present contribution, which belongs to a research strand we are pursuing on oligopoly modeling with interdependent preferences and follows the contribution in [7]. Next research topic on the subject will deal with comparative statics of the Nash equilibrium.

several researches on industrial and management economics ([15, 18, 14, 2, 16]) showed that overproduction, with respect to the Nash equilibrium quantities, can occur in the presence of strong intra-group competitions.

Even if some theoretical attempts to provide an explanation of such phenomena have been proposed (see [10], [17], [22], [9], [11]), a unitary approach that is able to encompass all the observed scenarios is missing. In particular, Friedman *et al.* [13] conjectured that the emergence of agents' choices resembling those of a cooperative setting could be explained in terms of a "learning-to-cooperate" behavior of the players, who through time can learn how to make choices of collective interest, looking for high collective payoffs after experiencing that deviating from a cooperative-like behavior results in a disadvantage for everybody.

In [7] we introduced an interdependent preferences structure in a Cournot oligopoly that provided a framework that incorporated, in the individual utility function, the effects due to the profits (material payoffs) of the other players, i.e. firms are involved in a network of social interactions. The resulting family of games proved to be able to provide a first evidence of encompassing all the effects evidenced by the experimental literature in terms of the aggregated outcome of the game. As we showed in [7], the developed framework is robust in terms of the existence and uniqueness of Nash equilibria, and its well-posedness is in line with that of classic Cournot oligopoly without interdependent preferences. In the present contribution we focus on the characterization of the Nash equilibrium, going deeper into the understanding of the effect of altruistic (i.e. individual utility has a positive spillover from the profits of a competitor), selfish (i.e. individual utility has no spillover from the profits of a competitor) or spiteful (i.e. individual utility has negative spillover from the profits of a competitor) behaviors of an agent with respect to his opponents. Moreover, in addition to that due to social interaction we aim at understanding the effects on the equilibrium of the market interaction. Firstly, we characterize the effect of preference interdependence in games in terms of strategic substitutability/complementarity, in order to understand how the network of social interaction alter the degree of strategic interaction between two interdependent firms, and consequently altering the way a given firm optimally responds to a change in the strategy of one of its opponents. In order to completely understand how best response mechanism of a player affects the equilibrium in the presence of interdependence of preferences we extend the analysis to the n possible degrees of interdependence effects between two given firms, taking into account also n-th order feedback effects on each firm.
We then characterize the Nash equilibrium through the two channels of interaction among firms, namely the market and the social interaction, and explain the individual role of each of these two channels on the Nash equilibrium. Concerning the social interaction we make use of some of the elements related to the theory on networks, such as the centrality measures, and consider their roles on the Nash equilibrium. Similarly, we introduce a measure of the degree of competitiveness that is able, for every game Γ, to pin the exact situation represented by the game.

We conclude by considering particular structures in light of elements characterizing both the social interaction and the market interaction.

The present paper is the second step in analysis and explanation of the effects of preference interdependence among firms in a Cournot oligopoly game. For a detailed introduction of the problem, an exhaustive description of the model and precise results about its well-posedness , we refer to [7], even if we try to keep as much as possible self-explanatory the present contribution, in particular having in mind its focus and the presented results. The next steps in the research strand will be the comparative statics of centrality measures at the equilibrium.

The remainder of the paper is organized as follows. In Section 2 we recall the main aspects of the model introduced in [7] and some relevant results; in Section 3 we study the effects of preference interdependence on the best response of a player; in Section 4 we study the effects of preference interdependence on the Nash equilibrium, which are then mathematically formalized in the propositions presented in Section 5. In Section 6 we focus on some particular structures of interaction, while in Section 7 we present conclusions and future research aims. Proofs of propositions are collected in the Appendix.

2. The model

The model under investigation is that introduced in [7], to which we refer for a complete and detailed description. For the reader's sake, we summarize it and briefly recall existence and uniqueness results. The economic setting under consideration is characterized by $i = 1, 2, \ldots, N$ firms, producing homogeneous goods, competing in a Cournotian oligopolistic market characterized by a suitably smooth inverse demand function $p : I \to [0, +\infty)$, $Q \mapsto p(Q)$, with Q identifying the aggregate output of the industry. Assuming the same linear cost function $C(q) = cq$ for all the firms, we have that realized profits (material payoffs) correspond to $\pi_i(q_i, Q_{-i}) = q_i(p(Q) - c)$, where q_i is the output level decision for the i-th firm and $Q_{-i} = Q - q_i$. Firms aim at maximizing a utility function that, in addition to their own material payoff, can (positively or negatively) depend on the material payoffs of the other firms, i.e. they are characterized by a network of social interaction and have interdependent preferences (see [21] and [7]). The extent to which the utility function of firm i depends on the material payoff of firm j is described by weights $\beta_{ij}, i \neq j$ whose absolute value describes the degree of social interaction of firm i toward firm j. Coefficients β_{ij} can be both positive, null or negative, identifying the kind of social interaction of firm i toward firm j. In agreement with the literature about interdependent preferences, in the first case we say that i is altruistic toward firm j, while in the second and third one we respectively say that it is selfish and spiteful toward firm j. Setting $\beta_{ii} = 0$ and collecting weights in a $N \times N$

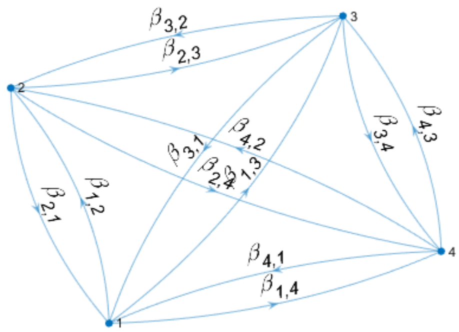


Figure 1: Graphical representation of the network described by weight matrix B in (1)

matrix B (see (1) and Figure 1 for an example in the 4×4 case)

$$B = \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 0 & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & 0 & \beta_{24} \\ \beta_{41} & \beta_{32} & \beta_{43} & 0 \end{bmatrix}, \tag{1}$$

we have that the utility function of firm i can be written as

$$\begin{aligned} v_i(q_i, \boldsymbol{q}_{-i}, B) &= \pi_i(q_i, \boldsymbol{q}_{-i}) + \sum_{j=1, i\neq j}^{N} \beta_{ij} \pi_j(q_i, \boldsymbol{q}_{-i}) \\ &= q_i(p(q_i + Q_{-i}) - c) + \sum_{j=1, i\neq j}^{N} \beta_{ij}(q_j(p(q_i + Q_{-i}) - c)), \end{aligned} \tag{2}$$

where $\boldsymbol{q}_{-i} \in [0, +\infty)^{N-1}$ is the vector collecting the output levels of all firms but the i-th one.

The previous framework can be described by a game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, in which players in set $\mathcal{N} = \{1, 2, \ldots, N\}$ choose strategies in sets $S_i \subset [0, +\infty)$ in order to maximize their utility function v_i defined in (2). Among the games defined by Γ we have the classic Cournot game, namely game $\Gamma_0 = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, O)) = (\mathcal{N}, S_i, \pi_i(q_i, \boldsymbol{q}_{-i}))$ obtained setting B equal to the null matrix O. In Γ_0 firms choose the quantity to produce in order to maximize material payoff, i.e. profits.

We recall that in [7] the well-posedness of game Γ was investigated. In particular, it was shown that a natural bound on weights of interaction is provided by

Assumption 1. $-\frac{1}{N-1} < \beta_{ij} < 1$,

which guarantees that on varying β_{ij}, in terms of aggregate output level, the outcome of game Γ ranges from that corresponding to a monopoly (monopolistic limit) to that corresponding to a competitive market (competitive limit), both indeed with inverse demand function p. The existence and uniqueness of the Nash equilibrium is guaranteed under suitable assumptions on the demand function, e.g. if we consider an isoelastic demand function $p(Q) = 1/Q$ or if we consider an oligopoly with a concave utility function. This last scenario occurs for instance if

Assumption 2. *For any* $q_i \in [0, L_i], i \in \mathcal{N}$ *and for* $Q \in \left[0, \sum_{k=1}^{N} L_i\right]$ *we have* $p'(Q) < 0$ *and for any* $z \in \left[0, \sum_{k=1}^{N} L_i\right]$ *we have*

$$\begin{cases} p''(Q)z + p'(Q) < 0, \\ -p''(Q)\frac{z}{N-1} + p'(Q) < 0, \end{cases} \tag{3}$$

and under suitable assumption on the structure of interaction, i.e. when

Assumption 3. *Matrix* $I + B$ *is a* P*-matrix.*

We stress that condition (3) is fulfilled by a linear demand function.

3. First order effect of interdependent preferences

In this section we study the direct effect of preference interdependence on the best response of a player [2]. The interdependence of preferences has an effect on the strategic interaction among agents. To this end, we recall that the common way to characterize the strategic interaction in games is in terms of strategic substitutability/complementarity. According to [8], we recall that strategy of player j has an effect of strategic substitutability (complementarity) on the strategy of player i if increasing q_j reduces (resp. increases) marginal profits of player i. For regular payoff functions v_i, the kind of strategic interaction (strategic substitutability/complementarity) between the strategies of two firms is then identified by the (negative/positive) sign of the second order cross derivative $\partial^2 v_i / \partial q_i \partial q_j$. In game Γ_0, strategic interaction just depends on the shape of the (inverse) demand function that characterizes the market, while in the general case Γ it is significantly affected by a network of social interactions.

[2] The presented results are referred to the Nash equilibrium of any game Γ, and not just to those fulfilling Assumptions of the previous section, which guarantee the existence and/or uniqueness of the Nash equilibrium. In this sense, Assumption 1 is fundamental of the proposed model while violating Assumption 3 leads to situations in which the matrix $I + B$ is not invertible, a necessary condition for the characterization of the internal equilibrium of the model. Conversely, Assumption 2, which concerns the demand function, is not mandatory, i.e. presented results are referred to each Nash equilibrium of games defined by a demand function that not necessarily satisfies Assumption 2 nor is isoelastic.

The goal of the next propositions is to investigate the effect on strategic substitutability/complementarity when preference interdependence is introduced. In what follows, we refer to this as first order effect. To fix ideas, let us consider a setting for which game Γ_0 is characterized by strategic substitutability (as, for instance, in the case of a linear demand function).

Proposition 1. *Assume that game Γ_0 is characterized by strategic substitutability. Then, $\partial^2 v_i/\partial q_i \partial q_j$ decreases (increases) due to an increase (a decrease) of coefficient β_{ij}.*

According to Proposition 1, a first effect of preference interdependence is to alter the degree of strategic interaction between two interdependent firms. If firm i is altruistic toward firm j, the strategic substitutability characterizing q_i with respect to q_j in Γ_0 is reinforced, while it is weakened if firm i is spiteful toward firm j, and in this latter case the kind of strategic interaction can possibly turn into strategic complementarity.

The economic rationale of this can be understood by looking at the form of the utility function and recalling the subsequent comments. Without interdependent preferences, thanks to the assumption of strategic substitutability, we have that if the strategy of a firm j increases this has the effect of reducing marginal profits of firm i. However, if the utility of firm i depends on profits of firm j, we can observe that the same marginal utility can be achieved with smaller or larger marginal profits if firm i is altruistic or spiteful toward firm j, respectively.

The first consequence of this is on the way firm i optimally responds to a change in the strategy of firm j. If we assume that for each $\boldsymbol{q}_{-i}$ there exists a unique best response $q_i > 0$ (e.g. if v_i is strictly concave in q_i), we have that if firm i is altruistic toward firm j, an increase in the strategy of firm j reduces the strategic response of firm i, while the opposite occurs for a spiteful behavior. We have the understandable consequence that an altruistic behavior induces a less aggressive interaction, while in the presence of a spiteful behavior the resulting interaction is more aggressive.
We exemplify the first order effects for a particular social interaction structure and a linear demand function.

Example 1. (First order effects)
Let consider a market characterized by inverse demand function

$$p(Q) = \max\{a - bQ, 0\}$$

populated by 4 firms, whose interdependent preferences are described by matrix

$$B = \begin{bmatrix} 0 & 0.61 & 0 & -0.32 \\ -0.2 & 0 & 0.73 & -0.17 \\ 0.43 & -0.08 & 0 & -0.23 \\ -0.3 & 0.81 & 0 & 0 \end{bmatrix}, \tag{4}$$

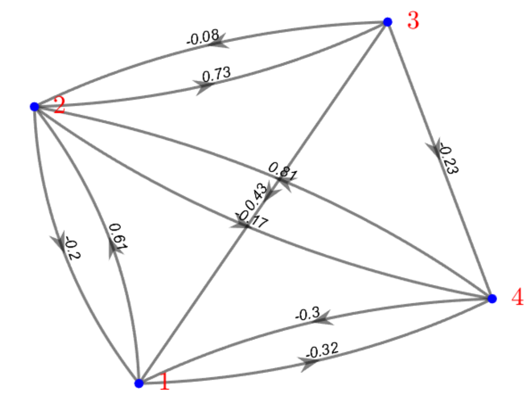


Figure 2: Graphical representation of the network described by weight matrix B in (4)

which generates the network graph in Figure 2. The utility of the generic firm $i = 1, \dots, 4$ has the following form

$$\begin{aligned} v_i(\mathbf{q}) = \pi_i(q_i, \mathbf{q_{-i}}) + \beta_{i1}\pi_1(q_1, \mathbf{q_{-1}}) + \beta_{i2}\pi_2(q_2, \mathbf{q_{-2}}) \\ + \beta_{i3}\pi_3(q_3, \mathbf{q_{-3}}) + \beta_{i4}\pi_4(q_4, \mathbf{q_{-4}}), \end{aligned}$$

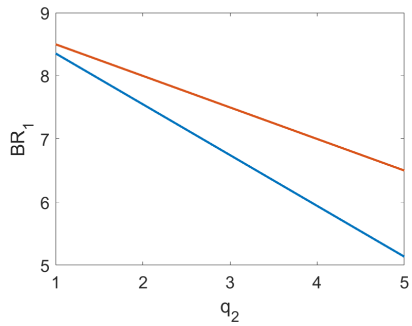
where $\beta_{ii} = 0$. We stress that it is easy to see that matrix B in (4) fulfills Assumptions 1,3, so, since demand function p is linear, the resulting game has a unique internal Nash equilibrium (see Proposition 3 in [7]). In what follows, we assume that we deal with suitable strategies for which the best response is strictly positive. For such reason, we can drop the max function. The utility of firm 1 becomes

$$v_1(\mathbf{q}) = q_1(p(Q) - c) + (0.61q_2 - 0.32q_4)(p(Q) - c).$$

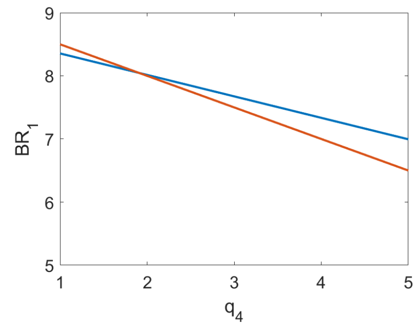
The role of weights β_{ij} on the utility is evident. Increasing β_{12} (i.e. increasing the degree of altruism of firm 1 toward firm 2), keeping constant the output quantities, the utility increases, while, indeed, the opposite occurs if we decrease β_{12}. Instead, considering weight β_{14} we can note that increasing the spitefulness degree, from -0.32 to -0.33, will decrease the value of the utility. The marginal utility of firm 1 with respect to its decision variable q_1 is

$$\frac{\partial v_1(\mathbf{q})}{\partial q_1} = p(Q) - c - bq_1 - 0.61bq_2 + 0.32bq_4.$$

The first remark concerns the role of weights β_{ij}, as they act in an opposite way on the marginal utility with respect to the utility v_i. In fact, looking at



(a)



(b)

Figure 3: Best response with (blue line) and without (red line) preference interdependency: comparing the slope of the blue line and the red line both in 3(a) and in 3(b) is clear how the interdependent weights modify the degree of strategic substitutability.

the marginal utility we can note how an increase (decrease) in the β_{12} weight will decrease (increase) the marginal utility of firm 1. Contrariwise, an increase (decrease) in the β_{14} weight will increase (decrease) the marginal utility of firm 1. In order to determine the optimal strategy that maximizes the utility of a firm, we impose first order condition

$$\frac{\partial v_i(\mathbf{q})}{\partial q_i} = 0 \Longleftrightarrow p(Q) - c - bq_i - \beta_{i2}bq_2 - \beta_{i3}bq_3 - \beta_{i4}bq_4 = 0,$$

which implicitly defines the reaction or best response function of firm i with respect to the choices of its other competitors

$$\mathrm{BR}_i(\mathbf{q}_{-i}) = \frac{a-c}{2b} - \frac{q_2(1+\beta_{i2})}{2} - \frac{q_3(1+\beta_{i3})}{2} - \frac{q_4(1+\beta_{i4})}{2}.$$

So, for firm 1 the best response function is

$$\mathrm{BR}_1(\mathbf{q}_{-1}) = \frac{a-c}{2b} - \frac{q_2(1+0.61)}{2} - \frac{q_3}{2} - \frac{q_4(1-0.32)}{2}.$$

First, looking at the equation of the best response we can note how the quantity of the other firms negatively affects the reaction of the firm under consideration. Increasing the strategic decision of the output quantity of one of the competitors decreases the optimal quantity chosen by the firm considered, accordingly to the strategic substitutability (the stronger an opponent plays the lower the player considered will respond) characterizing a Cournotian game with linear demand function (even in the presence of interdependent preferences). Firms are therefore bound (linked) by strategic substitutability, whose degree is indeed affected by the weights describing the network of social interactions. We want to point out how changing the β_{ij} affects the strategic substitutability of pairs of strategies.

In order to show the effects of preference interdependence on strategic substitutability, we focus on firm 1 and we vary the strategies of its opponents, in particular the strategy of firm 2. This means that the interdependence effect we are looking at is that highlighted in red in Figure 4(a). Since now, we set $a = 21$, $b = 1$, $c = 1$, while we set initially the output quantities $q_i = 1, \forall i \neq 1$.

We stress that firm 1 best response is affected by firm 2 strategy through the two different channels of interaction, namely the market interaction and the social interaction, (see Figure 4(b)). In what follows, we want to put in evidence the fraction of the change in the best response of player 1 due to a change in the strategy of player 2 that is solely due to the dependence of preferences of firm 1 on the material payoff of firm 2. Considering the previous parameters' values we obtain

$$\mathrm{BR}_1(1,1,1) = 8.355.$$

As a comparison it is useful to compute also the best response in the case of game Γ_0 (i.e. when $\beta_{ij} = 0, \forall i, j$), obtaining

$$\mathrm{BR}_1^C(1,1,1) = 8.5.$$

Now we let q_2 vary slightly, setting $q_2 = 1.5$. In this case the best response for the case with interdependence is

$$\mathrm{BR}_1(1.5,1,1) = 7.9525,$$

while for the selfish case we have

$$\mathrm{BR}_1^C(1.5,1,1) = 8.25.$$

Therefore, we can compute the variation of the best response after a change in the value of output quantity q_2

$$\Delta_{BR_1} = \mathrm{BR}_1(1.5,1,1) - \mathrm{BR}_1(1,1,1) = -0.4025,$$

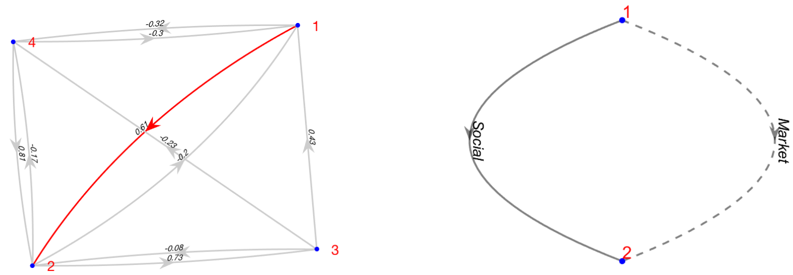
while without interdependence of preferences we have

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(1.5,1,1) - \mathrm{BR}_1^C(1,1,1) = -0.25.$$

We can note how the variation is different in the two cases, given the same variation in the strategic output q_2. In case of altruistic interdependence the strategic substitutability is amplified and the first order effect on the best response of firm 1 due to the dependence of its utility function on the material payoff of firm 2 can be quantified by

$$FOE_{1,2} = \Delta_{BR_1} - \Delta_{BR_1^C} = -0.1525.$$

In what follows we refer to first order market effect to the outcome due to market interaction among firms, while we refer to first order social effect to the outcome due to the network of social interactions among firms. We stress that since we are most interested in effects due to social interdependence of firms, in what



(a) (b)

Figure 4: Figure 4(a) is the graphical representation of the first order social dependence of firm 2 on firm 1. In Figure 4(b) are shown the two components of first order effects: the market (dashed line) and the social (solid line) effect.

follows we use notation $FOE_{i,j}$ (with $i \neq j$) as the first order effect in the best response of player i due to a change in the strategy of player j that is solely due to the dependence of preferences of firm i on the material payoff of firm j. Now we repeat the previous steps letting q_4 vary by the same amount and we compare the variation in the best responses of the interdependent case (BR_1) and selfish case (BR_1^C) (i.e. we focus on the interaction described in Figure 5 by the highlighted portion of the graph), respectively obtaining

$$\Delta_{BR_1} = \mathrm{BR}_1(1, 1, 1.5) - \mathrm{BR}_1(1, 1, 1) = -0.17$$

and

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(1, 1, 1.5) - \mathrm{BR}_1^C(1, 1, 1) = -0.25.$$

We note that this time $\Delta BR_1 < \Delta BR_1^C$. The reason is that firm 1 now acts spitefully towards firm 4. In this case, the first order effect on the best response of firm 1 due to the dependence of its utility function on the material playoff of firm 4 can be quantified by

$$FOE_{1,4} = \Delta_{BR_1} - \Delta_{BR_1^C} = +0.08.$$

To summarize, the first order effect is essentially an alteration of the degree of strategic interaction between the firms, increasing the strategic substitutability in the case in which there is altruism and decreasing it in the case of spitefulness (compared to the selfish case). Since the stronger the strategic substitutability effect is the smaller is the best response, we have that altruism acts in a subtractive way on the best response quantity, while spitefulness acts additively.

4. High order effects

To completely understand how best response mechanism of a player affects the equilibrium under the effect of interdependent preferences we can not limit

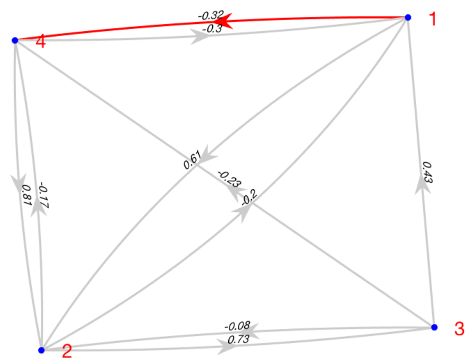


Figure 5: Graphical representation of the first order social dependence of firm 4 on firm 1

to the "direct" effect described in the previous proposition. To show this, we focus on a simple example. Assume that the utility of firm i depends on the profits of firm k and that, in turn, the utility of firm k depends on profits of firm j and that both interdependence effects are of altruistic kind. According to the previous proposition, there is a direct effect on the best response of firm k due to the strategy q_j, which in the particular case leads to a less aggressive reply than without interdependent preferences and the decreasing monotonicity of marginal profits of firm k is bolstered. However, again according to the previous proposition, to such an additional decrease corresponds an additional increase to the marginal profits of firm i. We then have an indirect, second degree interdependence effect between firm i and j, mediated by the interactions involving firm k, which results in a more aggressive response of firm i to the strategy of q_j. The previous reasoning can be repeated considering all the possible couplings of "altruist"-"spiteful" behavior. It easy to see that we have a sort of "rule-of-signs" that allows predicting such second order effect: if we identify "altruism" with "-" (meaning the reduced response to strategies) and "spitefulness" with "+" (meaning the increased response to strategies), two subsequent behaviors lead to a composite behavior that can be identified by the "sign" product of the two starting behaviors. We stress that since $|\beta_{ik}| < 1$ and $|\beta_{kj}| < 1$, the second order effect is reduced with respect to the first order effects induced both by preference interdependence between firm i and firm k and between firm k and firm j, but it can be larger than the first order effect arising from the direct dependence of the utility function of firm i and that of firm j.

Indeed, the previous considerations can be repeated to take into account effects on the response of firm i to the strategy q_j mediated by 2,3,$\ldots,n,\ldots$ firms (i.e. considering a path of length $2,3,\ldots,n,\ldots$ starting in node i and ending in node j), giving rise to second, third,..., n-th,... order effects. This is particularly relevant when we study the equilibrium of the game, as the overall effect of firms interaction consequent to interdependent preferences will require to take into account every k-th order effects, for any $k \geq 1$.

Note that a change in the strategy of firm i has an effect on the marginal utility of firm i itself, which is due to high order effects of interdependence among firms' preferences, in addition to the obvious direct effect on the utility exerted by the change of the marginal material payoff of firm i. To describe this, assume that the utility function of firm i depends on the material payoff of firm j and vice-versa (i.e. $\beta_{ij} \neq 0$ and $\beta_{ji} \neq 0$) and let us assume a sequence of consecutive choices. The strategy of firm i has a first order effect on the best response of firm j, which, in turn, reflects on the best response of firm i, giving rise to a second order effect. At the equilibrium such "consecutive choices" and the consequent effects simultaneously occur, but this suggests that among high order effects we then have to take into account also n-th order feedback effects on each firm.
We reconsider the setting studied in Example 1 to highlight second order effects.

Example 2. (Second order effects)
Let us consider a structure of interdependent preferences as described by matrix 4. The goal of next example is to highlight the effects on the best response of player 1 due to a change in the strategy of player 3, taking into account also the effects that such change has have on the best response of player 2. The situation we are going to consider is highlighted in Figure 6(a). In particular, we can consider the effect of the variation in the best response of firm 1 after a change in the strategic quantity of firm 3, both in the case of the presence of interdependence of preferences

$$\Delta_{BR_1} = \mathrm{BR}_1(q_2, 1.5, q_4) - \mathrm{BR}_1(q_2, 1, q_4) = -0.25$$

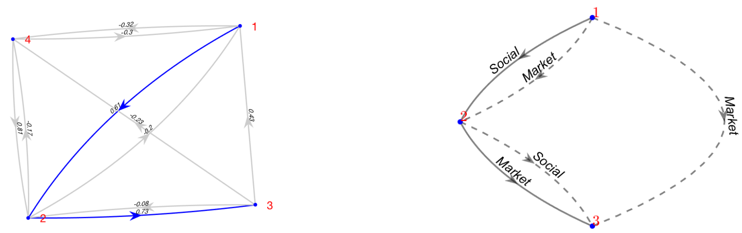
and in the case without interdependence of preferences

$$\Delta_{BR_1^C} = \mathrm{BR}_1^C(q_2, 1.5, q_4) - \mathrm{BR}_1^C(q_2, 1, q_4) = -0.25.$$

As we note, the two quantities coincide since the weight that binds the utility of firm 1 to the material payoff of firm 3 is zero ($\beta_{13} = 0$, hence no edge directly connecting firm 1 to firm 3 is highlighted in red in Figure 6(a)). A change in the strategy of firm 3 causes only a direct first order market effect on the best response of firm 1, while has a null first order social effect on the best response of firm 1, i.e.

$$FOE_{1,3} = \Delta_{BR_1} - \Delta_{BR_1^C} = 0.$$

Furthermore, we can note $\Delta_{BR_1} < 0$.
We recall that, according to example 1, notation $FOE_{i,j}$ is used to identify



(a) (b)

Figure 6: Figure 6(a) is the graphical representation (in blue) of the second order indirect connection of firm 1 to firm 3, mediated by firm 2. In Figure 6(b) are shown the components of the first and second order effects: the market (dashed line) and the social (solid line) effect of interdependence. Note that firm 3 and firm 1 are bound by the market effect only, since $\beta_{13} = 0$ (for this reason, in Figure 6(a), there is no solid line connecting the two firms.)

the first order social effect on the best response of firm i after a change in the strategic choice of firm j.

Then, we calculate the variation of the best response of firm 2 after a change in the value of the strategic quantity q_3, both in the case of interdependence of preferences and without. We respectively obtain

$$\Delta_{BR_2} = \mathrm{BR}_2(q_1, 1.5, q_4) - \mathrm{BR}_2(q_1, 1, q_4) = -0.4325$$

and

$$\Delta_{BR_2^C} = \mathrm{BR}_2^C(q_1, 1.5, q_4) - \mathrm{BR}_2^C(q_1, 1, q_4) = -0.25.$$

In this case, Δ_{BR_2} depends both on the market interaction effect and the first order social effect, since $\beta_{23} = 0.73$, while $\Delta_{BR_2^C}$ only depends on the market interaction effect.

We can note that $\Delta_{BR_2} < \Delta_{BR_2^C}$, so the first order effect on the best response of firm 2 due solely to the interdependence of its utility function on the material payoff of firm 3 is given by

$$FOE_{2,3} = \Delta_{BR_2} - \Delta_{BR_2^C} = -0.1825.$$

Since firm 2 is bound to firm 3 by a positive coefficient, a change in the strategic quantity of the latter will have the first order social effect to decrease the best response of the former. Namely, firm 2 is altruistic towards firm 3, therefore it reacts to an increase in the choice of its opponent by decreasing its strategic quantity.

Now we want to compute, in the model with interdependent preferences, the change in the best response of firm 1, after a change in q_3, also taking into account the best response of firm 2. If we consider in BR_1 the effects on q_2 of a change in q_3 we obtain

$$\Delta_1 = BR_1(BR_2(q_1, 1.5, q_4), 1.5, q_4) - BR_1(BR_2(q_1, 1, q_4), 1, q_4) = 0.098.$$

We can reiterate this logic considering the model without interdependency

$$\Delta_1^C = BR_1^C(BR_2^C(q_1, 1.5, q_4), 1.5, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4) = -0.125,$$

where Δ_1 considers both the effects due to the market interaction and the (direct and indirect) effects due to the interdependence of preferences. To understand what contributes to make Δ_1 differ from Δ_1^C, we must pay attention to the superimposition of the subsequent market and social effects, as shown in Figure 6(b).

If an overlapping of market and social effects was already present in the $FOE_{2,3}$, now we have the superimposition of multiple second order effects of the two kinds, that is

1. the superimposition of two consecutive market effects (dashed subsequent lines connecting firm 2-firm 3 and firm 1-firm 2)
2. a market effect that superimposes to a social effect (solid line connecting firm 2-firm 3 and dashed line connecting firm 1-firm 2)
3. a social effect that superimposes to a market effect (dashed line connecting firm 2-firm 3 and solid line connecting firm 1-firm 2)
4. the superimposition of two social effects (solid subsequent lines connecting firm 2-firm 3 and firm 1-firm 2)

The last contribution is what we will consider as the second order social effect of a change in the choice of firm 3 on the best response of firm 1, mediated by the best response of firm 2, namely $SOE_{1,2,3}$. The difference

$$\Delta_1 - \Delta_1^C = 0.223 \tag{5}$$

allows us to subtract the first and second order effects referred solely to the market interaction. What we are left with consists of the superimposition of several effects that we want to isolate. First, we should take into account the first order effect on the best response of firm 1 due to the change in q_3, which however in our case is equal to zero ($FOE_{1,3} = 0$). We then have to tackle the mixed higher order effects, namely those induced by the change in q_3 on the best response of firm 2, which in turn causes a second order effect on the best response of firm 1. This last effect is partially due to the market interaction between firm 1 and all its opponents and partially due to the interaction firm 1 has with firm 2 through the structure of interdependence of preferences. The former effect is represented by the terms

$$\begin{aligned} BR_1^C(BR_2(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2(q_1, 1, q_4), 1, q_4) - \\ (BR_1^C(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4)) = 0.091 \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
BR_1(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1(BR_2^C(q_1, 1, q_4), 1, q_4) - \\
(BR_1^C(BR_2^C(q_1, 1.5, q_4), 1, q_4) - BR_1^C(BR_2^C(q_1, 1, q_4), 1, q_4)) = 0.076.
\end{aligned}
\tag{7}
$$

Finally, we obtain the solely second order effect of the interdependence of preferences between firm 1 and firm 3, mediated by firm 2. This effect is quantifiable in

$$
\begin{aligned}
BR_1((q_2 + BR_2 - BR_2^C), 1, q_4) - BR_1(q_2, 1, q_4) - \\
(BR_1^C(q_2 + BR_2 - BR_2^C), 1.5, q4)) - BR_1^C(q_2, 1, q_4)) = 0.056.
\end{aligned}
\tag{8}
$$

We can note how the following identity is satisfied

$$
(5) = (6) + (7) + (8)
$$

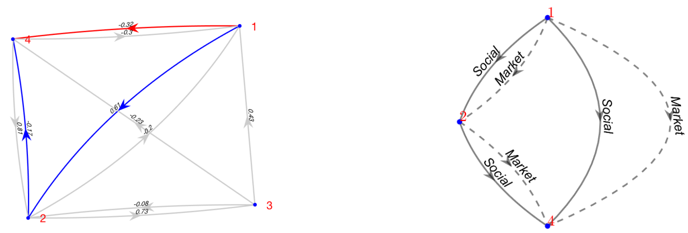
Hence, a firm that is positively linked to a second firm, which in turn is positively bound to a third one causes a positive second order effect of the latter on the former. An increase in q_3 causes a negative first order effect on the best response of firm 2. Firm 1, which is positively linked to firm 2 observing a decrease in the best response of the opponent, increases its best response. If we ignore for a while what happens in between and we focus only on the cause-effect chain of a variation of q_3 on the best response of firm 1 we would see that the increase in q_3 increases the best response of firm 1, namely in this specific setting $FOE_{1,3} = 0$ but $SOE_{1,2,3} > 0$.

We now consider the second order interdependent effect on the strategic choice of firm 1 after a change in q_4, mediated by the interdependency between firm 2 and firm 4 (see Figure 7(a)). Following the previous analysis (all the involved effects are reported in Figure 7(b)) we may skip the intermediate computation and focus only on the sign of $SOE_{1,2,4}$, that is

$$
\begin{aligned}
BR_1((q_2 + BR_2 - BR_2^C), q_3, 1) - BR_1(q_2, q_3, 1) - \\
(BR_1^C(q_2 + BR_2 - BR_2^C), q_3, 1.5)) - BR_1^C(q_2, q_3, 1)) = -0.043
\end{aligned}
$$

In this case, a firm that is positively linked to a second firm, which in turn is negatively bound to a third one causes a negative second order effect of the latter on the former. An increase in q_4 causes a positive first order effect on the best response of firm 2. Firm 1, which is positively linked to firm 2 observing an increase in the best response of the opponent, decreases its best response. If we focus only on the cause-effect chain of a variation of q_4 on the best response of firm 1 we would see that the increase in q_4 decreases the best response of firm 1, namely in this specific setting $FOE_{1,4} > 0$ and $SOE_{1,2,4} < 0$.

Finally, we investigate the feedback effect, that is what happens when a given firm decides to change its strategic choice and this decision causes, initially, a first order effect on the best response of an opponent which, ultimately, causes



(a) (b)

Figure 7: Figure 7(a) is the graphical representation (in blue) of the second order indirect connection of firm 1 to firm 4, mediated by firm 2. The direct connection of firm 1 to firm 4 is highlighted in red. In Figure 7(b) are shown the components of the first and second order effects: the market (dashed line) and the social (solid line) effect of interdependence.

an additional change in the best response of the given firm. For instance, we can compute the first order social effect on q_2 after an increase in q_1

$$FOE_{2,1} = 0.05$$

Therefore, an increase (decrease) in q_1 causes an increase (decrease) in the best response of firm 2, since $\beta_{21} < 0$.
Then, we can compute the second order social effect

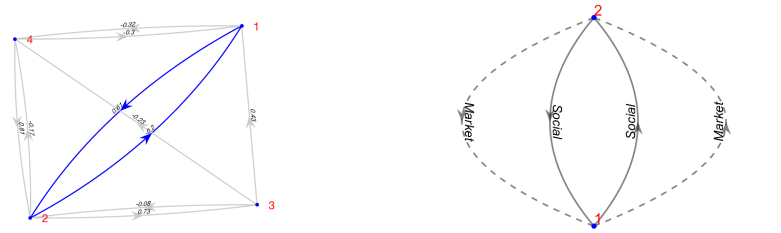
$$SOE_{1,2,1} = -0.015$$

In this case, we can see that an initial increase in q_1 generates a feedback effect that ultimately causes a decrease in the best response of firm 1, accordingly to $\beta_{12}\beta_{21} > 0$.

5. Characterization of Nash equilibria

The previous considerations are crucial to understand the characterization of equilibria in terms of the effects of the network structure of social interactions, as shown in the next proposition. We focus on internal equilibria as for boundary equilibria such effects could be hindered or changed by the fact that some firms are actually not active at the equilibrium, as in the case of those having null equilibrium strategies, or because production levels reached the capacity limit. In any case, we stress that the following results could be suitably modified for boundary equilibria.

Proposition 2. *Let $\boldsymbol{q}^*$ be an internal Nash equilibrium for game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, and let Q^* be the corresponding aggregate equilibrium output of the industry.*



(a) (b)

Figure 8: Figure 8(a) is the graphical representation of the second order feedback connection on firm 1 through firm 2. In Figure 8(b) the market (dashed line) and the social (solid line) effect of interdependence.

Then there exists a vector $\boldsymbol{\xi} \in (0, +\infty)^N$*, which just depends on coefficients* β_{ij}*, such that*

$$\boldsymbol{q}^* = Q^* \boldsymbol{\sigma} = Q^* \frac{\boldsymbol{\xi}}{\mu}, \tag{9}$$

with $\mu = \sum_{i=1}^{N} \xi_i$ *and where the aggregate equilibrium quantity satisfies*

$$Q^* p'(Q^*) = (c - p(Q^*))\mu, \tag{10}$$

while vector $\boldsymbol{\xi}$ *is defined by*

$$\boldsymbol{\xi} = (I + B)^{-1} \boldsymbol{u}, \tag{11}$$

in which the i*-th component represents a measure of the centrality of the* i*-th firm in the network described by matrix* B*. At an internal equilibrium, the utility achieved by each firm is the same, corresponding to* $v_i = \left| \frac{Q^*(p(Q^*)-c)}{\mu} \right|$*. At the equilibrium, each firm realizes profit* $\pi_i^* = \sigma_i Q^*(p(Q^*) - c)$*,* $i = 1, \ldots, N$*.*

The Nash equilibrium is characterized through the two channels of interaction among firms: the market interaction and the social interaction, whose influences can be identified in both relations (9) and (10). The effects related to the latter channel are all encompassed in $\boldsymbol{\xi}$ (and, consequently, in μ), which just depends on the network structure of social interaction. The effects related to the former channel are encompassed in Q^*, which however depends both on the inverse demand function and costs (i.e. on the unique elements characterizing the market interaction) and on μ, which is determined by the distribution of weights β_{ij}. Vector $\boldsymbol{\xi}$ has positive elements, each of which provides a centrality measure of the corresponding firm in the network of social interactions. The centrality of a firm determines its market share, which exactly equals the fraction that the centrality measure of the firm represents with respect to the sum of the centrality measures of all the firms in the industry[3]. Consequently, the centrality measure (and hence the network of social interactions) determines the ordering of firms with respect to realized profits, as firms with larger centrality measures have higher profits. Ceteris paribus, the more a firm's centrality measure is large, the greater will be its market share and the ordering of firms with respect to their centrality measures provides the ordering of firms with respect to their market share. We stress that since firms are homogeneous in all respects but the distribution of social preferences, without a network of interdependence, all firms would produce the same quantity and would realize the same profits at the equilibrium. It's worth noticing that even if firms choose heterogeneous equilibrium strategies, they are homogeneous with respect to the achieved equilibrium utility. This is probably due to a "compensation" effect in the utility. Since firms maximize a linear combination of profits, the equilibrium is realized when a lack in the contribution given to v_i by the own material payoff is symmetrically compensated for the material payoffs of the competitors[4].

Social preferences of firms determine another key element characterizing the equilibrium, i.e. the scalar μ, which is defined as the sum of the centrality measures of all the firms. It is possible to show (see Lemma 1 in the Appendix in [7]) that $\mu \in (1, +\infty)$, where the limit $\mu \to 1$ is realized in the monopolistic limit, while $\mu \to +\infty$ is realized in the competitive limit and in the case of oligopoly described by Γ_0 we have $\mu = N$. To deepen the explanation of the meaning of μ we provide the following result, focused on the most interesting case of a unique equilibrium.

Corollary 1. *Let us consider an inverse demand function for which* (11) *has a unique solution $Q^*(\mu)$ for each $\mu \in (1, +\infty)$. Then $Q^*(\mu)$ is an increasing function.*

As a consequence of the previous result, we have that μ is a scalar index that is related to the degree of competitiveness that characterizes an oligopoly with interdependent preferences. We remark that μ can be (inversely) connected with the Lerner index, a common measure of market power. In fact, recalling that Lerner index can be written as $L(Q^*) = (p(Q^*) - c)/p(Q^*)$ and that $\eta Q^* = -p(Q^*)/(Q^* p'(Q^*))$ is the demand elasticity, we can write $\mu = (L(Q^*)\eta(Q^*))^{-1}$.

Note that, besides the limiting games that correspond to the monopolistic and competitive limits, in general we have an infinite set of games, different with respect to the network of interdependent preferences, that are characterized by the same value of μ, i.e. by the same degree of competitiveness. We give

[3]We stress that if we also take into account firms that are not active at the equilibrium, solving the Nash equilibrium problem we again find a vector $\boldsymbol{\xi}$, in which null elements identify non-active firms.

[4]We thank Prof. Bertoletti for stimulating considerations about such a point. Indeed, that provided is just a tentative insight, and it would deserve further investigation considering different, non-linear dependency of the utility function on the material payoffs.

evidence of this in the next example.

Example 3. (Network of social interaction and competitiveness)
Let consider the following couple of 5×5 matrices

$$B_1 = \begin{bmatrix} 0 & -0.0926 & 0.3880 & -0.0885 & -0.2069 \\ 0.3221 & 0 & -0.0344 & -0.1526 & -0.1352 \\ -0.0985 & 0.0675 & 0 & 0.0936 & -0.0626 \\ 0.1195 & 0.2360 & -0.1278 & 0 & -0.2278 \\ 0.0054 & 0.1173 & -0.0201 & -0.1025 & 0 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0 & -0.1243 & 0.1736 & -0.1958 & 0.1465 \\ 0.0788 & 0 & 0.3368 & -0.2065 & -0.2090 \\ 0.2129 & -0.0690 & 0 & -0.2456 & 0.1017 \\ 0.1678 & -0.1164 & 0.0298 & 0 & -0.0812 \\ 0.3836 & -0.3188 & -0.3224 & 0.2575 & 0 \end{bmatrix}$$

A direct computation provides

$$\tilde{B}_1 = (I + B_1)^{-1} = \begin{bmatrix} 0.9322 & 0.0505 & -0.3374 & 0.1434 & 0.2113 \\ -0.2953 & 0.9242 & 0.1624 & 0.1098 & 0.0990 \\ 0.1155 & -0.0416 & 0.9449 & -0.0785 & 0.0596 \\ -0.0201 & -0.2605 & 0.1261 & 0.9660 & 0.1886 \\ 0.0299 & -0.1362 & 0.0147 & 0.0838 & 1.0078 \end{bmatrix}$$

and

$$\tilde{B}_2 = (I + B_2)^{-1} = \begin{bmatrix} 1.0485 & 0.1060 & -0.2525 & 0.1885 & -0.0905 \\ -0.1457 & 1.0588 & -0.2452 & 0.0597 & 0.2724 \\ -0.2406 & 0.0530 & 1.0051 & 0.2205 & -0.0380 \\ -0.2238 & 0.1267 & 0.0115 & 0.9503 & 0.1353 \\ -0.4686 & 0.2813 & 0.3398 & -0.2269 & 1.0745 \end{bmatrix},$$

for which we can calculate the column vectors of the centrality measures $\boldsymbol{\xi}$, composed by the row summations, for both matrices B_1 and B_2

$$\boldsymbol{\xi}^{\mathbf{1}} = \tilde{B}_1 \cdot \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \boldsymbol{\xi}^{\mathbf{2}} = \tilde{B}_2 \cdot \boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Therefore, the degrees of competitiveness that characterize the two different structures of interdependent preferences coincide

$$\mu_1 = \boldsymbol{u}^T \boldsymbol{\xi}^{\mathbf{1}} = 5 = \mu_2 = \boldsymbol{u}^T \boldsymbol{\xi}^{\mathbf{2}},$$

even if the two interdependent structures described by B_1 and B_2 are very different.

Similarly, if we consider the couple of 5×5 matrices

$$B_1 = \begin{bmatrix} 0 & -0.1617 & -0.1663 & -0.0535 & 0.1315 \\ 0.0507 & 0 & -0.2800 & -0.2307 & 0.2099 \\ -0.1621 & -0.0159 & 0 & -0.1684 & 0.0964 \\ 0.0872 & -0.0643 & -0.0913 & 0 & -0.1815 \\ -0.1744 & 0.2839 & -0.1648 & -0.1947 & 0 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 0 & 0.0857 & 0.0417 & -0.1661 & -0.2112 \\ -0.1410 & 0 & -0.2785 & 0.1886 & -0.0191 \\ -0.2039 & 0.1150 & 0 & -0.0209 & -0.1402 \\ -0.1143 & 0.3121 & -0.2696 & 0 & -0.1782 \\ -0.2271 & 0.1271 & 0.0982 & -0.2482 & 0 \end{bmatrix},$$

direct computation provides

$$\tilde{B}_1 = (I + B_1)^{-1} = \begin{bmatrix} 0.9824 & 0.2188 & 0.2050 & 0.1033 & -0.1761 \\ -0.0651 & 1.0658 & 0.2781 & 0.2510 & -0.1964 \\ 0.1317 & 0.0784 & 1.0436 & 0.1811 & -0.1015 \\ -0.0409 & 0.0114 & 0.1230 & 1.0568 & 0.1829 \\ 0.2036 & -0.2493 & 0.1527 & 0.1824 & 1.0439 \end{bmatrix},$$

and

$$\tilde{B}_2 = (I + B_2)^{-1} = \begin{bmatrix} 1.0515 & -0.2036 & -0.0515 & 0.2766 & 0.2602 \\ 0.1853 & 0.9921 & 0.2256 & -0.1353 & 0.0656 \\ 0.2293 & -0.2011 & 0.9539 & 0.1467 & 0.2045 \\ 0.1659 & -0.4335 & 0.1642 & 1.1772 & 0.2595 \\ 0.2339 & -0.2602 & -0.0933 & 0.3578 & 1.0951 \end{bmatrix},$$

for which we can calculate the column vectors of the centrality measures $\boldsymbol{\xi}$, composed by the row summations, for both matrices B_1 and B_2

$$\boldsymbol{\xi^1} = \tilde{B}_1 \cdot \boldsymbol{u} = \begin{bmatrix} 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \end{bmatrix}, \quad \boldsymbol{\xi^2} = \tilde{B}_2 \cdot \boldsymbol{u} = \begin{bmatrix} 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \\ 1.333 \end{bmatrix}.$$

Therefore, the degrees of competitiveness that characterize the two different structures of interdependent preferences coincide

$$\mu_1 = \boldsymbol{u}^T \boldsymbol{\xi^1} = 6.667 = \mu_2 = \boldsymbol{u}^T \boldsymbol{\xi^2}$$

The two couples of matrices highlight the fact that, in general we may have an infinite family of structures of interdependent preferences characterized by the same value of μ, i.e. by the same degree of competitiveness. In particular, the first couple of matrices, composed by firms that on average are self-interested, realize in an oligopoly described by Γ_0.

When the equilibrium is internal, we have that the market share actually corresponds to the Katz-Bonacich centrality measure $(I - \alpha M)^{-1}\boldsymbol{u}$ associated with the network described by a matrix M (see e.g. [6]). In this case the network is that induced by the structure of social interaction. We stress that the present setting differs from those usually studied in the literature for two aspects. Firstly, in most cases coefficient α is positive, while in the present case we have $\alpha = -1$. Moreover, the network is described by a weighted adjacency matrix in which links can be both positive and negative.

It is relevant to understand how firms' interaction determines the centrality measure. From the mathematical viewpoint, this is encompassed in condition (11), from which we have that matrix $(I + B)^{-1}$ provides the complete characterization of the corresponding internal equilibrium, for a given inverse demand function p. If we write

$$(I+B)^{-1} = \begin{bmatrix} \tilde{\beta}_{11} & \tilde{\beta}_{12} & \tilde{\beta}_{13} & \cdots & \tilde{\beta}_{1N} \\ \tilde{\beta}_{21} & \tilde{\beta}_{22} & \tilde{\beta}_{23} & \cdots & \tilde{\beta}_{2N} \\ \tilde{\beta}_{31} & \tilde{\beta}_{32} & \tilde{\beta}_{33} & \cdots & \tilde{\beta}_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{\beta}_{N1} & \tilde{\beta}_{N2} & \tilde{\beta}_{N3} & \cdots & \tilde{\beta}_{NN} \end{bmatrix},$$

from Proposition 2 we can say that each $\tilde{\beta}_{ij}$ encompasses the aggregate effect due to any order dependence of social preferences of firm i with respect to firm j. Recalling the comments following Proposition 1, there is a strategic influence of firm j on firm i due to social interaction not only if the utility of firm i directly depends on the material payoff of firm j, but also as a consequence of any path of length n of subsequently dependent preferences that starts from firm i and ends on firm j. Without interdependence, we indeed have $\tilde{\beta}_{ij} = 0$ for $i \neq j$ and $\tilde{\beta}_{ii} = 1$. So the more $\tilde{\beta}_{ij}$ for $i \neq j$ differs from 0, the greater is the aggregate effect of any order due to the preference interdependence that links firm i to firm j. Similarly, the more $\tilde{\beta}_{ii}$ differs from 1, the greater is the feedback effect of any order on firm i due to the firms' network of social interactions.

Each component ξ_i of the centrality measure $\boldsymbol{\xi}$ is simply the sum of all $\tilde{\beta}_{ij}$, i.e. ξ_i is the aggregate effect due to any order dependence of social preferences of firm i with respect to the whole industry. Each $\tilde{\beta}_{ij}$ is then the contribution to the centrality measure of firm i of any order social interaction of firm i with firm j. Note that, independently of the altruistic or spiteful behavior of firm i with respect to firm j, the sign of $\tilde{\beta}_{ij}$ can be positive or negative. This means that, it is in general false that if firm i is, for example, altruistic with respect to firm j, then $\tilde{\beta}_{ij}$ will be negative for sure and this will reduce the centrality of firm i. Weights β_{ij} only accounts for a first order effect of preference interdependence, which is indeed the potentially more relevant one, but aggregating all the effects of the $n > 1$ order indirect dependence the resulting effect can be, in principle, of any kind.

To deepen the role of $(I + B)^{-1}$ and $\boldsymbol{\xi}$ we focus on the case in which the network is such that $\rho(B) < 1$, so we can use Neumann series[5] expansion of $(I + B)^{-1}$

$$(I + B)^{-1} = \sum_{n=0}^{\infty} (-B)^n = I + (-1)B + B^2 + \ldots + (-1)B^n + \ldots \tag{12}$$

from which we have

$$\boldsymbol{\xi} = (I + (-1)\, B + B^2 + (-1)\, B^3 + \cdots)\, \boldsymbol{u}, \qquad \xi_i = \sum_{k=1}^{N} \tilde{\beta}_{ik},\ i = 1, \ldots, N. \tag{13}$$

The first addend in both (12) and (13) represents the situation in which we do not have interdependent preferences, i.e. when B is the null matrix. In such case, since firms are identical, we simply have $\boldsymbol{\xi} = \boldsymbol{u}$. All the other terms account for effects of increasing orders. In particular, each element $(-B)^n_{ij}$ of each addend in (12) represents the effect of order n of firm i with respect to firm j, namely the sum of all the effects exerted by firm j on firm i along any possible path of $n+1$ firms with interdependent preferences starting in i and ending in j. The "$-$" sign in front of B accounts for the effect on the best response of interdependent preferences, which, as a consequence of Proposition 1, is quantified by $-B$. A positive (respectively, negative) coefficient has the direct effect to reduce (respectively, increase) the strategic response so this provides a negative contribution to the market share at the equilibrium.

The economic meaning of (alternating) signs in front of each term in the Neumann series in (12) can be understood by recalling the considerations about the role of altruistic and spiteful behaviors on first order and high order effects.

[5] We stress that a series expansion can be performed also when $\rho(B) \geq 1$, but the resulting expression is much more involved and less instructive for the explanation of high order effects, which however holds in this case as well.

Element $(-B)_{ij}$ represents the direct effect of firm i toward j and it indeed corresponds to $-\beta_{ij}$. Sign "$-$" shows that if firms i is altruistic toward firm j (i.e. $\beta_{ij} > 0$), it will provide a negative (direct) contribution to the centrality of firm i, while if firm i is spiteful toward firm j (i.e. $\beta_{ij} < 0$), it will provide a positive (direct) contribution to the centrality of firm i, while no contribution to the centrality of firm i is provided if firm i is selfish toward firm j (i.e. $\beta_{ij} = 0$).

Similarly, matrix B^2 collects the indirect second order effects on couples of firms. Element $(B^2)_{ij}$ is obtained by adding $\beta_{ik} \cdot \beta_{kj}$, namely adding all the effects due to the dependence of firm i preferences on the choices of firm j mediated by all firms $k = 1, \dots, N$. If either firm i is selfish with respect to firm k or firm k is selfish with respect to firm j, the second order effect due to the path ikj is indeed null. If the kind of social interaction of firm i toward firm k and of firm k toward firm j is of the same kind (either both altruistic or both spiteful) the resulting contribution to the centrality of agent i with respect to agent j (mediated by agent k) is actually reinforced due to the synergical result of two consecutive effects of the same kind. Conversely, two consecutive effects of different kinds weaken the contribution to the centrality of firm i.

Each element $\tilde{\beta}_{ij}$ of $(I+B)^{-1}$ is then the result of all the interactions of any order linking firm i to material payoff of firm j. The way $\boldsymbol{\xi}$ changes is completely described in terms of the distribution of $\tilde{\beta}_{ij}$, and this has a strong influence on the way $\boldsymbol{q}^*$ and $\boldsymbol{\pi}^*$ change.
Once more we take into account the scenario considered in Example 1 to show the properties of its Nash equilibrium.

Example 4. (Characterization of Nash equilibrium)
The goal of this example is to show, through a numeric case study, the effects of the structure of preferences interactions on the equilibrium, as analyzed in Proposition 3. We start noting that the spectral radius of matrix B is $\rho(B) = 0.6173 < 1$. and we compute the inverse matrix

$$\tilde{B} = (I+B)^{-1} = \begin{bmatrix} 0.7542 & -0.6184 & 0.4514 & 0.2401 \\ 0.3661 & 0.6432 & -0.4695 & 0.1185 \\ -0.3112 & 0.1549 & 0.8869 & 0.1307 \\ -0.0702 & -0.7065 & 0.5157 & 0.9760 \end{bmatrix},$$

which provides the column vector of the centrality measures

$$\boldsymbol{\xi} = \tilde{B}\boldsymbol{u} = \begin{bmatrix} 0.8273 \\ 0.6582 \\ 0.8614 \\ 0.7150 \end{bmatrix},$$

which is a vector of positive elements that determines the ordering with respect to the output levels and profits at the equilibrium. We then calculate the index that encompasses the degree of competitiveness that characterizes the model with interdependent preferences

$$\mu = \boldsymbol{u}^T \boldsymbol{\xi} = 3.062$$

from which we can calculate the market share σ, corresponding to the relative measure of centrality of each firm

$$\boldsymbol{\sigma} = \frac{\boldsymbol{\xi}}{\mu} = \begin{bmatrix} 0.2702 \\ 0.2150 \\ 0.2813 \\ 0.2335 \end{bmatrix}$$

Solving the system of the N equations $BR_i(\mathbf{q}^*_{-i}) = q^*_i$ we find the unique Nash equilibrium $\boldsymbol{q}^*_i = \begin{bmatrix} 4.073 & 3.241 & 4.241 & 3.521 \end{bmatrix}^T$, to which corresponds the total equilibrium output $Q^* = 15.076$. Note that the following identity is satisfied $Q^* p^{'}(Q^*) = (c - p(Q^*))\mu$

We recall that each element β_{ij} of B encompasses the first order effect of preference interdependence between firm i and firm j. Instead, each element $\tilde{\beta}_{ij}$ of $(I+B)^{-1}$ is the result of all the social interactions of any order linking firm i to firm j.

In general, the signs of elements of matrix $-B$ (i.e. first order effects) are those most relevant for the signs of elements of $\tilde{B}$. However, exceptions are possible. By comparing matrix $\tilde{B}$ and matrix $-B$

$$\tilde{B} - (-B) = \begin{bmatrix} \cdot & -0.0084 & 0.4514 & -0.0799 \\ 0.1661 & \cdot & 0.2605 & -0.0515 \\ 0.1188 & 0.0749 & \cdot & -0.0993 \\ -0.3702 & 0.1035 & 0.5157 & \cdot \end{bmatrix}$$

we are able to highlight the contribution of the higher order effects to the equilibrium. First, we note how it is possible that a null first order effect turns to be positive or negative, such as the case of β_{13}.

Second, we can note that the sign of $\tilde{\beta}_{41}$ is not the opposite of that of β_{41}. We recall from the previous analysis that, if $\beta_{ij} < 0$ then $FOE_{i,j} > 0$ and this effect is generally the predominant one in $\tilde{\beta}_{ij}$. But this is not the case since $|\beta^{(2)}_{41}| + |\beta^{(3)}_{41}| > |\beta_{41}|$.

Entering more into details, weight $\beta_{12} = 0.61$ encompasses an altruistic behavior on behalf of firm 1 towards firm 2 (in the sense that an increase in the strategic choice of firm 2 will lead firm 1 to decrease its best response). In this case, the weight $\tilde{\beta}_{12} = -0.6184$ confirms that the aggregate effect due to any order dependence of social preferences of firm 1 with respect to firm 2 is to reduce the centrality of firm 1. In this case the contribution of the first order effect to the value of $\tilde{\beta}_{21}$ is the dominant one, as $\tilde{\beta}_{21} \neq -\beta_{12}$. The cumulated $n > 1$ effects just contribute by slightly reinforcing the first order effect by 0.0084. Conversely is the situation involving firm 4 and firm 1. The weight $\beta_{41} = -0.3$ encompasses a spiteful behavior on behalf of firm 4 towards firm

1 but the aggregate effect due to any order dependence of social preferences of firm 4 with respect to firm 1 is not to increase the centrality of firm 4 but to decrease ($\tilde{\beta}_{41} = -0.0702$) the centrality measure of firm 4. This means that high order indirect effects on the centrality measure of firm 4 due to firm 1 are stronger than first order effect (on aggregate, they amount to 0.3702).
Finally we also draw attention to the dependence of preference of firm 1 on firm 3 material payoff. Although the utility of firm 1 does not depend on the material payoff of firm 3 ($\beta_{13} = 0$), there exists a positive aggregate effect ($\tilde{\beta}_{13} = 0.4514$) due to higher order dependence of social preferences of firm 1 with respect to firm 3 that increases the centrality of firm 1.

Since $\rho(B) = 0.6173 < 1$, in order to quantify the role of high order social interaction, we can use Neumann series expansion. In particular, each element $(-B)^n_{ij}$ of each addend in Neumann series represents the effect of order n of firm i with respect to firm j, namely the sum of all the effects exerted by firm j on firm i along any possible path of $n+1$ firms with interdependent preferences starting in i and ending in j. For instance, the matrix B^2 collects the indirect second order effects on couples of firms

$$B^2 = \begin{bmatrix} -0.0260 & -0.2592 & 0.4453 & -0.1037 \\ 0.3649 & -0.3181 & 0 & -0.1039 \\ 0.0850 & 0.0760 & -0.0584 & -0.1240 \\ -0.1620 & -0.1830 & 0.5913 & -0.0417 \end{bmatrix} = [\beta^{(2)}_{ij}].$$

One thing catching the eyes of the reader, is the coefficient $\beta^{(2)}_{23} = 0$. In general, a value $\beta^{(n)}_{ij} = 0$ may depend by the fact that the n order effects binding firm i to firm j are all equal to zero or that such effect cancels out. We recall that B^2 collects the indirect second order effects for couples of firms. Element $\beta^{(2)}_{23}$ is obtained

$$\beta^{(2)}_{23} = \sum_{z=1}^{N} \beta_{2z} \cdot \beta_{z3} = (-0.2 \cdot 0) + (0 \cdot 0.73) + (0.73 \cdot 0) + (-0.17 \cdot 0) = 0,$$

i.e. there are length 2 paths starting from firm 2 and ending in firm 3 provides contributions that cancel.

Finally, it is worth noting that if we compare the relative centrality index $\boldsymbol{\sigma}$ and the column vector $B\boldsymbol{u}$ (coming from the row summation of each player's coefficients), there is no correspondence between the outgoing degree of social interaction of a firm and its relative centrality in the network. For instance, a more altruistic, on average, firm can be more central in the network with respect to a less altruistic, on average, firm

$$B\boldsymbol{u} = \begin{bmatrix} 0.29 \\ 0.36 \\ 0.12 \\ 0.51 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.2702 \\ 0.2150 \\ 0.2813 \\ 0.2335 \end{bmatrix}$$

For example, firm 4 is more altruistic, on average, than firm 2 ($(B\boldsymbol{u})_4 > (B\boldsymbol{u})_2$) and its relative centrality index is bigger than the one of firm 2 ($\sigma_4 > \sigma_2$). Therefore, firm 4 obtains more profits than firm 2 even if it exerts a more altruistic behavior on average.

6. Nash equilibria in relevant network structures

In Proposition 2 we have shown that Nash equilibria can be characterized in terms of the Bonacich centrality measure $\boldsymbol{\xi}$, which quantifies the relevance that a firm has from being in the network of social interaction. It is indeed affected by the way the utility function of firm i directly depends on the material payoff of its competitors, but it can be significantly altered by the indirect effects of other firms' preferences structure. The vector of centrality measures has a twofold descriptive power. Firstly, the distribution of centrality measures determines the ordering of firms with respect to their market share, describing how much a firm is dominant inside the market. Moreover, aggregating all the ξ_i we are able to quantify the degree of competitiveness, as it indicates where the equilibrium production of the industry places between the monopolistic and the competitive limit.

However, the point of view adopted to determine the Bonacich centrality measure can be reversed. Instead of considering the overall equilibrium effect arising from the way in which firms take into account their competitors in their preferences, we can focus on the overall equilibrium effect from the way in which firms are taken into account by their competitors. This is described by the Friedkin-Johnsen [12] centrality measure $\boldsymbol{\chi} = (I + B^T)^{-1}u$, which quantifies the influence that a firm has from being part of the network of social interaction. Such measure does not directly determine the equilibrium performance of firms and their ordering with respect to their relevance is in general independent of that with respect to their influence.

In what follows, we reconsider particular structures of interdependence of preferences in light of the results of Proposition 2 and to cast a first glance at the role on the equilibrium of the ordering of firms with respect to their altruistic/spiteful behavior. A systematic analysis is beyond of the scopes of the present contribution and will be the subject of future researches, for now we limit to provide a preliminary picture focusing on some extreme situations in terms of ordering firms with respect to their relevance and influence. We provide results for each of the relevant structures reported in Section 3 of [7], which we briefly summarize. In the first situation we consider an economic scenario in which firms share the same common information about each other and are affected in the same way by such information, so that they are characterized by a uniform and homogeneous distribution of preferences. All the interaction degrees are then identical to β.

The second configuration we consider is characterized by firms that, since the framework is characterized by a high degree of complexity and/or they are not endowed with individual information about any of their competitors, exhibit undifferentiated preferences with respect to all the other players. Their behavior is then homogeneous with respect to each their competitors, but two different firms can have different behaviors. In such scenario we set $\beta_{ij} = \beta_i \in (-1/(N-1), 1)$ for $i, j = 1, \ldots, N$ and $i \neq j$, so that firms have social preferences that are either uniformly altruistic, selfish or spiteful toward any other firm j, with a constant degree of social interaction. The third configuration we consider firms are endowed with an elevated informational degree about each player, so they all have identical behavior toward a given player. However, two different firms can be differently taken into account by all their competitors. The scenario is realized by setting $\beta_{ij} = \beta_j \in (-1/(N-1), 1)$ for $i, j = 1, \ldots, N$ and $i \neq j$, so that all firms j have the same social preferences toward a given firm i, being either uniformly altruistic, selfish or spiteful toward the firm i, with a constant degree of social interaction. Finally, we consider the structure in which the overall outgoing degree of social interaction is the same for all firms, i.e. vector $B\boldsymbol{u}$ has identical elements.

We start studying the case of uniform weights. Thanks to the simplicity of this case, we report the analytical expressions of the elements characterizing the structure of social interaction at the equilibrium and we provide the explicit expression of the equilibrium for two relevant inverse demand functions, namely the (piecewise) linear and the isoelastic ones.

Proposition 3. *Let $B = \beta(U - I)$ where U is the $N \times N$ matrix whose elements are equal to 1, and I is the $N \times N$ identity matrix and let β satisfy Assumption 1. We then have $\tilde{B} = (I + B)^{-1} = \tilde{\beta}_a I + \tilde{\beta}_b U$, where $\tilde{\beta}_a = \frac{(N-2)\beta+1}{-(N-1)\beta^2+(N-2)+1}$ and $\tilde{\beta}_b = \frac{-\beta}{-(N-1)\beta^2+(N-2)+1}$, to which corresponds*

$$\xi_i = \frac{1}{(N-1)\beta + 1}, \sigma_i = \frac{1}{N}, \ i = 1, \ldots, N,$$

and

$$\mu = \frac{N}{(N-1)\beta + 1}.$$

If the inverse demand function is $p(Q) = \max\{a - bQ, 0\}$, for marginal cost we have $c < a$ and the capacity limit is suitably close to a/b we have that the unique Nash equilibrium is internal and has

$$q_i^* = \frac{(a-c)}{b\,(\beta\,(N-1)+1) + Nb}, \ Q^* = \frac{N\,(a-c)}{b\,(\beta\,(N-1)+1) + Nb}, \ i = 1, \ldots, N$$

If the inverse demand function is $p(Q) = 1/Q$, we have the unique internal Nash equilibrium characterized by

$$q_i^* = \frac{(N-1)(1-\beta)}{N^2 c}, \ Q^* = \frac{(N-1)(1-\beta)}{Nc}, \ i = 1, \ldots, N$$

We stress the fact that, if we consider a model without interdependence of preferences ($\beta = 0$) we obtain the exact equilibrium quantities of the model with

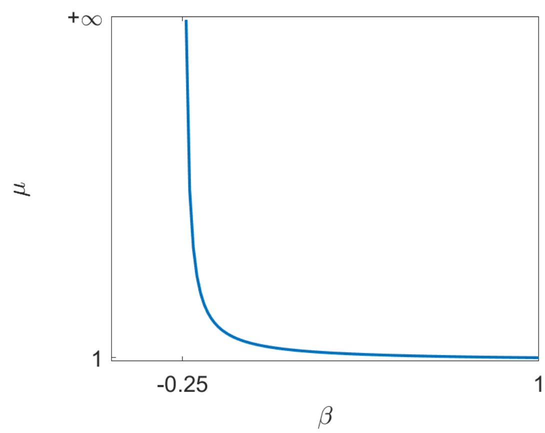


Figure 9: μ is a decreasing function of β and is equal to 1 when $\beta \to 1$ and is equal to $+\infty$ when $\beta \to -\frac{1}{(N-1)}$

homogeneous costs function and isoelastic demand function of Puu [20](in the case of a duopoly) and of Ahmed and Agiza [1](in the case of n competitors).

Indeed, in such a simplified framework, all firms are identical at the equilibrium. However, such a scenario allows us to show that the proposed model can represent all the possible configurations, in terms of competitiveness degree, ranging from the monopolistic limit to the competitive one. We can note that, for each N, μ is a decreasing function of β and is equal to 1 when $\beta \to 1$ and equal to $+\infty$ when $\beta \to -\frac{1}{(N-1)}$, so that, in game Γ, the transition between the monopolistic and competitive markets (aggregate) equilibria do not (only) occurs on increasing the number of firms populating the market, but it takes place, for any given number of firms, as the distribution of weights describing interaction among firms decreases from the uniform distribution $\beta = 1$ to the uniform distribution $\beta = -1/(N-1)$. As an example, we report the plot of function $\mu(\beta)$ in the case of $N = 5$ in Figure 9. The previous example guarantees that there exists at least a network of social interaction for which the (aggregate) equilibrium is characterized by a given $\mu \in (1, +\infty)$. Finally, we remark that, thanks to the symmetric and homogeneous weights' distribution, in the previous example all firms have identical relevance and influence. Now we focus on the case of constant outgoing degrees. The explicit expressions of elements of $(I+B)^{-1}, \boldsymbol{\xi}$ and $\boldsymbol{\chi}$ and of μ are quite involved, so are not reported and can be found in Appendix. What is relevant in the present scenario is the ordering at the equilibrium of such quantities. To this end, without loss of generality, we assume that firms are ordered from the most spiteful/least altruistic to the least spiteful/most altruistic one.

Proposition 4. *Let $\beta_{ij} = \beta_i$ for $1 \leq i, j \leq N$, with $i \neq j$ such that the corresponding matrix B satisfies Assumptions 1 and 3. Moreover, assume that*

$\beta_i \leq \beta_j$ for $1 \leq i < j \leq N$. We then have

$$\xi_r \geq \xi_s,\ \sigma_r \geq \sigma_s,\ \chi_r \leq \chi_s,\ \textit{for } 1 \leq r < s \leq N.$$

Consequently, we indeed have $q_r^* \geq q_s^*$ for $1 \leq r < s \leq N$.

Proposition 4 shows that the more spiteful the firm, the more central in the industry is and therefore the higher is the market share it owns. Moreover, the more spiteful the firm is, the less influence it exerts on the competitors. We stress that, removing the assumption of uniform behavior of firms with respect to their competitors (i.e. $\beta_{ij} = \beta_i$) it is no more true that if firm i has an average outgoing degree of social interaction that is greater than that of firm j, then it will be less relevant in the market. It is easy to see that the particular structure of the scenario in Proposition 4 is such that the higher is the average outgoing degree of social interaction, the lower is its average outgoing degree of social interaction, so the results are a joint effect of both the orderings.

Now we focus on the case of constant ingoing degrees. Also in this case, we leave the explicit expressions of elements of $(I+B)^{-1}, \boldsymbol{\xi}$ and $\boldsymbol{\chi}$ and of μ to the Appendix and we focus on the ordering of such elements at the equilibrium, assuming again that firms are ordered from that with the smallest ingoing degree to that with the largest one.

Proposition 5. *Let $\beta_{ij} = \beta_j$ for $1 \leq i, j \leq N$, with $i \neq j$ such that the corresponding matrix B satisfies Assumptions 1 and 3. Moreover, assume that $\beta_r \leq \beta_s$ for $1 \leq r < s \leq N$. We then have*

$$\xi_r \leq \xi_s,\ \sigma_r \leq \sigma_s,\ \chi_r \geq \chi_s,\ \textit{for } 1 \leq r < s \leq N.$$

Consequently, we indeed have $q_r^* \leq q_s^*$ for $1 \leq r < s \leq N$.

Proposition 5 shows that the more negatively the firm is taken into account on average in the opponents' utilities, the less central in the industry is and therefore the lower is the market share it owns. Moreover, the more negatively the material payoff of the firm influences, on average, the utility of the competitors, the more influence it exerts on the industry. Also in this case, it is worth remarking that the particular structure of the scenario in Proposition 5 is reversed with respect to that in Proposition 4, as in the former case the higher is the average ingoing degree of social interaction, the higher is its average outgoing degree of social interaction, and in fact the ordering is reversed with respect to that in Proposition 4.

Proposition 6. *At the equilibrium firms produce the same amount of good if and only if they have the same centrality index or, equivalently, $\sum_{j=1, i\neq j}^{N} \beta_{ij} = \beta$ for each i.*

Proposition 6 specifies under which condition we have a "homogeneous" equilibrium scenario, which actually corresponds to the equilibrium scenario of game Γ_0. If vector $B\boldsymbol{u}$, which collects the overall outgoing degrees of social interaction, has identical elements, then the behavior of firms at the equilibrium is homogeneous, independently of any possible "local" heterogeneity in the distribution of weights β_{ij}. We stress that the previous proposition can not be generalized, in the sense that, as also shown in Example 4, vectors $B\boldsymbol{u}$ do not provide in the general case sufficient information to draw conclusions about the behavior of firms at the equilibrium.

7. Conclusions

In this paper we studied the role of preference interdependence on the resulting properties of the Nash equilibrium for an oligopoly in which firms are involved in a network of social interactions. The first effect of introducing preference interdependence into the model is to alter the degree of strategic interaction between two firms. In such a way an altruistic firm optimally responds to a change in the strategy of one of its opponents in a less aggressive interaction, while the opposite occurs for a spiteful firm. To completely understand how best response mechanism of a firm affects the equilibrium in the presence of interdependence of preferences the analysis had to be extended to the n possible degrees of interdependence effects between two given firms. For instance, we had to consider also the effect of a change on the best response of any intermediary agents which in turn influences the best response of the reference player. Interestingly enough, we noted that, although the first order effect is generally the most important on the best response of a given firm, it may happen that high order effects are, instead, the main components of a response, that in the same case, can be counterintuitive if looking at the sole structure of interdependence of preferences provided by matrix B. In addition, among the high order effects, we take into account n-th order feedback effects on each firm.
The effect is that the Nash equilibrium is strongly connoted in terms of elements related to the social interaction. With respect to this, a fundamental role is played by the vector $\boldsymbol{\xi}$ and scalar μ. The former one is determined by the distribution of weights β_{ij} and provides the centrality (or relevance) measure of each firm in the network of social interactions and ultimately determines the market share of each firm. The latter one is a key element characterizing the aggregate equilibrium and encompasses the degree of competitiveness that characterizes an oligopoly with interdependent preferences.
Through Proposition 2 we showed how the internal Nash equilibrium can be expressed in a simple way in terms of the Bonacich index ($\boldsymbol{\xi}$) and the degree of competitiveness (μ) of the market.

The next goal of the present research strand is to investigate how the equilibrium changes on varying the structure of social interaction, with particular reference to the relevant quantities characterizing the equilibrium. With this respect, the goal will be to study how the most common centrality indexes in the literature depend on a change of the interactional structure, focusing in addition to the Bonacich index, on Friedkin-Johnsen centrality measure proposed in [12] and on the intercentrality measure introduced in [5].

Appendix

Proof of Prop. 1. Assumption about strategic substitutability in Γ_0 guarantees that $\partial^2\pi_r/\partial q_r q_s = q_r p'(Q) + p''(Q) < 0$ for any r, s. The degree of strategic interaction between i and j is given by

$$\frac{\partial^2 v_i}{\partial q_i q_j} = p'(Q) + q_i p''(Q) + \sum_{r=1, r\neq j}^{N} \beta_{ir} q_r p''(Q) + \beta_{ij}(q_j p'(Q) + p''(Q)),$$

which, as a consequence of strategic substitutability in Γ_0, negatively depends on β_{ij}. This concludes the proof. $\square$

Proof of Prop. 2. At an internal equilibrium q first order condition must hold, so we have

$$p'(Q)q_i + p(Q) - c + \sum_{j=1}^{N} \beta_{ij} p'(Q) q_j = 0,$$

which, in vector form, can be rewritten as $p'(Q)(I+B)\boldsymbol{q} + (p(Q)-c)\boldsymbol{u} = 0$. Setting $y = -(p(Q)-c)/p'(Q)$, the last system becomes

$$\begin{cases} (I+B)\frac{\boldsymbol{q}}{y} = \boldsymbol{u} \\ y = -\frac{p(Q)-c}{p'(Q)} \end{cases}$$

Since the game has solution q, the former vector equation has at least a solution that can be written as $\frac{\boldsymbol{q}}{y} = (I+B)^{+}\boldsymbol{u} + [I - (I+B)^{+}(I+B)]\boldsymbol{z} = \boldsymbol{\xi}$, where A^{+} is the Moore-Penrose inverse and $\boldsymbol{z}$ is an arbitrary vector (in the particular case of an invertible matrix $I+B$ we obtain (11)).

Left multiplying both sides by $\boldsymbol{u}^T$ we immediately obtain (10) and then, using $y = Q/(\boldsymbol{u}^T\boldsymbol{\xi})$, we find (9). Profits immediately follow. $\square$

Proof of Corollary 1. From equation (10) we can define function

$$f_\mu(Q) = Qp'(Q) + (p(Q)-c)\mu.$$

Since $Q^*(\mu)$ is a maximum point, we have $f_\mu(Q^*(\mu)) > 0$ for $Q < Q^*(\mu)$ and $f_\mu(Q^*(\mu)) < 0$ for $Q > Q^*(\mu)$. in particular, since p is strictly decreasing, we have $p(Q^*(\mu)) - c > 0$ for $Q < Q^*(\mu)$ and $p(Q^*(\mu)) - c < 0$ for $Q > Q^*(\mu)$. This means that if $\mu_2 > \mu_1$ $f_{\mu_2}(Q) > f_{\mu_1}(Q) > 0$ for $Q < Q^*(\mu_1)$, which implies that the solution to $f_{\mu_2}(Q) = 0$ must fulfill $Q^*(\mu_1) < Q^*(\mu_2)$. $\square$

Proof of Propositions 3, 4 and 5. We start proving Proposition 4, so let B such that $\beta_{ij} = \beta_i$ for $i \neq j, i, j \in \mathcal{N}$. We can write $B = D + \boldsymbol{b}\boldsymbol{u}^T$, where D and $\boldsymbol{b}$ are respectively a diagonal matrix in which $d_{ii} = 1 - \beta_i$ for $i \in \mathcal{N}$ and a vector with $b_i = \beta_i$ for $i \in \mathcal{N}$. Thanks to the Sherman-Morrison formula we can write $(I+B)^{-1} = D^{-1} - \frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$. It is easy to see that the elements of $D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}$ are given by $a_{ij} = \frac{\beta_i}{(1-\beta_i)(1-\beta_j)}$ while $1+\boldsymbol{u}^T D^{-1}\boldsymbol{b} = 1+\sum_{j=1}^{N} \frac{\beta_j}{1-\beta_j}$.

Note that thanks to Assumption 1 we have

$$1+\sum_{j=1}^{N}\frac{\beta_j}{1-\beta_j} > 1-\sum_{j=1}^{N}\frac{1}{N}=0,$$

so $I+B$ is invertible and the Sherman-Morrison formula can be applied.

The generic elements of $\tilde{B}$ are then

$$\tilde{\beta}_{ii}=\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}},\quad \tilde{\beta}_{ij}=-\frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}, i\neq j \tag{14}$$

We have $\boldsymbol{\xi}=D^{-1}\boldsymbol{u}-\frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^TD^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^TD^{-1}\boldsymbol{b}}$ and $\boldsymbol{u}^T\boldsymbol{\xi}=\boldsymbol{u}^TD^{-1}\boldsymbol{u}-\frac{\boldsymbol{u}^TD^{-1}\boldsymbol{b}\boldsymbol{u}^TD^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^TD^{-1}\boldsymbol{b}}$.

The generic component of the centrality index results

$$\begin{aligned}
\xi_i &= \frac{1}{1-\beta_i}-\frac{\displaystyle\sum_{j=1}^{N}\frac{\beta_i}{(1-\beta_i)(1-\beta_j)}}{\displaystyle 1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\\
&= \frac{1}{1-\beta_i}\left(\frac{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\beta_i\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\\
&= \frac{1}{1-\beta_i}\left(\frac{1-N+(1-\beta_i)\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)
\end{aligned}$$

Noting that

$$
\begin{aligned}
\frac{\partial \xi_i}{\partial \beta_i} &= \frac{1}{(1-\beta_i)^2} - \frac{\left(\sum_{k=1}^{N} \frac{1}{(1-\beta_i)(1-\beta_k)} + \frac{\beta_i}{(1-\beta_i)^2(1-\beta_k)} + \frac{\beta_i}{(1-\beta_i)^3}\right)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2 - \left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}+\frac{\beta_i}{1-\beta_i}\right)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right) + \frac{\beta_i}{1-\beta_i}\sum_{j=1}^{N}\frac{1}{1-\beta_k}}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\left[1-N-\frac{\beta_i}{1-\beta_i}\right]\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)+\frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{(1-\beta_k)}}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} < 0
\end{aligned}
$$

we can conclude the ordering of elements in vector $\boldsymbol{\xi}$.

The influence vector $\boldsymbol{\chi}$ is given by $\boldsymbol{\chi}^T = \boldsymbol{u}^T D^{-1} - \frac{\boldsymbol{u}^T D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$

Moreover,

$$
\begin{aligned}
\mu = u^T\xi &= \sum_{k=1}^{N}\frac{1}{1-\beta_i} - \frac{\displaystyle\sum_{k=1}^{N}\sum_{j=1}^{N}\frac{\beta_k}{(1-\beta_k)(1-\beta_j)}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \sum_{k=1}^{N}\left(\frac{1}{1-\beta_k}\right) - \frac{\left(\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)\left(\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \sum_{k=1}^{N}\left(\frac{1}{1-\beta_k}\right) - \left(1-\frac{1}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right) \\
&= \frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} = \frac{\displaystyle\sum_{k=1}^{N}\frac{1-\beta_k+\beta_k}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{N+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}
\end{aligned}
$$

Similarly we have $\chi = \boldsymbol{u}^T D^{-1} - \frac{\boldsymbol{u}^T D^{-1}\boldsymbol{b}\boldsymbol{u}^T D^{-1}}{1+\boldsymbol{u}^T D^{-1}\boldsymbol{b}}$ so

$$
\begin{aligned}
\chi_i &= \frac{1}{1-\beta_i} - \frac{\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{(1-\beta_k)(1-\beta_j)}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i} - \frac{1}{1-\beta_j}\left(1-\frac{1}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)
\end{aligned}
$$

Noting that

$$\frac{\partial \chi_i}{\partial \beta_i} \quad = \quad \frac{1}{(1-\beta_i)^2} + \frac{1}{1-\beta_j} \cdot \frac{1}{\left(1 + \sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \cdot \frac{1}{(1-\beta_i)^2} > 0$$

we can conclude the ordering of elements in vector $\boldsymbol{\chi}$.

We use the previous results to prove Proposition 3. The expressions for $\tilde{\beta}_{ii} = \tilde{\beta}_1$ and $\tilde{\beta}_{ij} = \tilde{\beta}_2$ can be easily found setting $\beta_i = \beta$ in (14). The elements of $\boldsymbol{\xi}$ are obtained by computing $\xi_i = \tilde{\beta}_1 + (N-1)\tilde{\beta}_2$, from which it is straightforward to compute $\sigma_i = \xi_i / N\xi_i = 1/N$ and $\mu = N\xi_i$. When the Nash equilibrium is internal and the demand is linear, from (10) we have that Q^* must fulfill

$$-bQ^* = (c - a + bQ^*)\mu = (c - a + bQ^*)\frac{N}{(N-1)\beta + 1}$$

while if $p(Q) = 1/Q$, from (10) we have that Q^* must fulfill

$$-\frac{1}{Q^*} = \left(1 - \frac{1}{Q^*}\right)\frac{N}{(N-1)\beta + 1}$$

Solving by Q^* and then setting $q^* = Q^*/N$ provides the equilibrium values.

The proof of Proposition 5 is obtained by noting that it is simply the transposed case of that in Proposition 4, so we simply have to swap $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$. $\square$

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