DEMS
Department of Economics, Management and Statistics

DEMS WORKING PAPER SERIES

Comparative statics and centrality measures in oligopolies with interdependent preferences

Marco F. Boretto, Fausto Cavalli
and Ahmad K. Naimzada

No. 464 – March 2021

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

Comparative statics and centrality measures in oligopolies with interdependent preferences

Marco F. Boretto, Fausto Cavalli and Ahmad K. Naimzada[1]

Abstract

Considering the Cournot oligopoly with interdependent preferences proposed in [5], we analyze the effects of a change in the network of social interactions. Reconsidering some of the main centrality measures proposed in the literature, we show how intercentrality, Bonacich and Friedkin-Johnsen centrality measures can be related in a network described by a general matrix of interaction. This allows showing under what conditions a firm can benefit, in terms of equilibrium performance, from a change in the weight of interaction with respect to one of its competitors. Extending the approach to the study of a uniform change in the behavior of all the firms, we show that it is collectively beneficial only if the structure of social interaction is characterized by a sufficient degree of homogeneity in terms of weight distributions.

Keywords: Cournot Game, Preference interdependence, Network, Centrality measures, Comparative statics.
JEL: D43, C62, C70

1. Introduction

In recent years game theory has started incorporating increasingly more elements of network theory[2]. Decisions of individuals are recognized as the outcome of strategic interaction among the agents, but it becomes more and more evident that the role played by social interaction can not be neglected. The fact agents are part of a structure of interactions influence their preferences and contribute to evolve them. Such a structure is often characterized by an elevated level of complexity, which requires the investigation of synthetic measures through which provide robust and reliable conclusions on the key economic observables. Among the mass of indexes proposed in the literature and tested on

[1]The authors are indebted to Prof. Paolo Bertoletti for his invaluable comments and suggestions, and to the Professors in the Evaluation Committee for their comments during the dissertation of the PhD Thesis of Dr. Marco Boretto. Both contributions helped to improve the quality of the present contribution, which belongs to a research strand we are pursuing on oligopoly modeling with interdependent preferences and follows the contributions in [5] and [4].

[2]The literature on the topics of networks and of their application to games is too widespread to provide any exhaustive literature review. We limit to refer the interested reader to the survey by Jackson and Zenou [11].

empirical case studies, some centrality measures play a particular role as they can be used to make explicit the effect of a network structure on the characterization of Nash equilibria in games or to describe elements connected to them. As an example, Bonacich centrality measure [3], which basically represents the overall relevance that a player has inside the network of social interaction, can be used to concisely characterize the distribution of equilibrium strategies [7] in network games. Moreover, Bonacich centrality measure can be used to evaluate the contribution to the overall collective outcome that each player provides thanks to the belonging to a structure of interaction characterized by symmetry, giving rise to a new measure, called intercentrality measure, that identifies the key player[3] inside a network [2].

The introduction of a structure of social interaction in a game necessarily dismisses the assumption of agents with independent preferences, namely of agents whose utility function just depends on their own payoff [12, 13]. Individuals, who in a setting with independent preferences would be characterized in terms of selfish behavior, with interdependent preferences have a utility function that positively or negatively depends on the payoff of a competitor, exhibiting a so-called altruistic or spiteful behavior. The classic setting with interdependent preferences is often unsatisfactory in explaining the results of empirical [1, 10, 6] and experimental evidence [9] of game theory, as the players' choices seem to lie above or below the corresponding Nash equilibrium, suggesting the presence of some form of preference interdependence. Focusing in particular on an experiment based on a Cournotian game, Friedman et al. [9] showed that the evolution of players' choices can significantly change from the earlier stages to those later, with strategies that initially overrun the levels corresponding to the classic Nash equilibrium and then evolve toward reduced production choices, consistent with the emergence of some forms of implicit coordination.

In the present contribution we reconsider the Cournotian oligopoly model with interdependent preferences introduced in [5] and we evaluate the effect of changes in the interaction structure on its Nash equilibria, whose characterization in terms of Bonacich centrality measure has been studied in [4]. The question of the emergence and survivability of social behaviors like altruism and/or spitefulness is crucial to understand the possibility to explain phenomena like those in the experiment by Friedman et al., and it is strictly related to the performance of firms in terms of material profits. To this end, we aim at addressing two main questions: what are the effects of a change in the interaction structure on the equilibrium performance of firms? As a consequence of this, what kinds of social interaction structure can be favorable to the equilibrium performance of firms?

To answer the two previous research questions we generalize to a broader family of matrices (in particular, to the asymmetric case), describing the network of interaction, the definition of the intercentrality measure, showing that

[3] More precisely, the player who removed from the network has the largest disruptive effect on the aggregate collective output

it can be expressed in terms of a combination of both Bonacich and Friedkin-Johnsen centrality measures. Intercentrality can then be understood in terms of the joint result of the relevance and of the influence of a firm in the market. As a consequence, comparative statics with respect to a change of any weight of interaction can be expressed through a relation involving the main theoretical centrality indexes, namely Intercentrality, Bonacich and Friedkin-Johnsen centrality measures, and the degree of competitiveness. This allows identifying the situations in which an increase in either altruism or spitefulness results in a beneficial effect for the equilibrium performance of a firm, in particular in terms of its market share. Even if we do not study the way firms form or change their network of social interaction, it is possible to see in the results we find a first, raw suggestion for a "tit-for-tat" scheme in adapting the behavior of a firm with respect to a competitor.

Going further, we show that a collective benefit for every firm is possible from an increase of all the weights of interaction provided that the structure of social interaction is "suitably homogeneous", namely if the weight distribution is sufficiently close to a uniform distribution.

The remainder of the paper is organized as follows. In Section 2 we summarize the model and the main results collected in [5] and [4]. In Section 3 we prove some general results on centrality measures. In Section 4 we study comparative statics. Section 5 bears conclusions. Proofs are collected in Appendix.

2. The model

In this Section we briefly summarize the model under investigation, which was proposed in [5], and the main result for the characterization of the Nash equilibrium, as shown in [4], to which we refer for a complete description of the model and for comments about the propositions reported in this Section.

The model consists of a family of games describing Cournotian oligopolistic competitions in which firms have interdependent preferences. Games are identified by $\Gamma = (\mathcal{N}, S_i, v_i)$, where $\mathcal{N} = \{1, 2, \dots, N\}$ is the set of players corresponding to the firms populating the market, which are assumed to produce homogeneous goods, $S_i \subset [0, +\infty)$ are the sets in which firm can choose their output level q_i and v_i are the payoff functions related to each firm. In particular, the payoff function in Γ does not necessarily correspond to the profits (hereinafter material payoffs) $\pi_i(q_i, Q_{-i}) = q_i(p(Q)-c)$ that firms, facing identical constant marginal costs c, achieve if the market is characterized by a suitably smooth inverse demand function $p : I \to [0, +\infty)$, $Q \mapsto p(Q)$, with Q respectively representing the aggregate output of the industry and $Q_{-i} = Q - q_i$. In this case, game Γ would represent the classic Cournot game with profit maximizing firms. Conversely, we assume that firms' utility function can also depend (either positively or negatively) on the material payoffs of their competitors, being them involved in a network of social interaction arising from their interdependent preferences (see [13] and [5]). In particular, utility v_i is given by

$$\begin{aligned} v_i(q_i, \boldsymbol{q}_{-i}, B) &= \pi_i(q_i, \boldsymbol{q}_{-i}) + \sum_{j=1, i\neq j}^{N} \beta_{ij}\pi_j(q_i, \boldsymbol{q}_{-i}) \\ &= q_i(p(q_i + Q_{-i}) - c) + \sum_{j=1, i\neq j}^{N} \beta_{ij}(q_j(p(q_i + Q_{-i}) - c)), \end{aligned} \tag{1}$$

where $\boldsymbol{q}_{-i} \in [0, +\infty)^{N-1}$ collects the output levels of all firms but the i-th one and B is a hollow $N \times N$ matrix whose elements are weights $\beta_{ij}, i \neq j$, which identify the extent and the kind[4] of social interaction. With the previous utility function, the classic Cournot game corresponds to game $\Gamma_0 = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, O)) = (\mathcal{N}, S_i, \pi_i(q_i, \boldsymbol{q}_{-i}))$, obtained setting B equal to the null matrix O.

In [5] the problem of the existence and uniqueness of a Nash equilibrium to game Γ was investigated, showing that it was possible by assuming that weights fulfill the bound

Assumption 1. $-\frac{1}{N-1} < \beta_{ij} < 1$,

that demand function p is either an isoelastic demand function $p(Q) = 1/Q$ or it satisfies

Assumption 2. *For any* $q_i \in [0, L_i], i \in \mathcal{N}$ *and for* $Q \in \left[0, \sum_{k=1}^{N} L_i\right]$ *we have* $p'(Q) < 0$ *and for any* $z \in \left[0, \sum_{k=1}^{N} L_i\right]$ *we have*

$$\begin{cases} p''(Q)z + p'(Q) < 0, \\ -p''(Q)\frac{z}{N-1} + p'(Q) < 0, \end{cases} \tag{2}$$

and that the structure of interaction is described in terms of a $P-$matrix i.e. when

Assumption 3. *Matrix* $I + B$ *is a* P*-matrix.*

In particular, Assumption 1 guarantees that the aggregate output level of the industry ranges from the output level of a monopoly (monopolistic limit) to that of a competitive market (competitive limit), both indeed characterized by inverse demand function p.

In addition to this, the characterization of a Nash equilibrium of Γ has been investigated in [4]. In the next proposition we summarize the most relevant results in view of the subsequent sections.

[4]In agreement with the literature about interdependent preferences we say that if $\beta_{ij} > 0$, then firm i is altruistic with respect to firm j, if $\beta_{ij} < 0$ firm i is spiteful toward firm j, while firm i is neutral toward firm j if $\beta_{ij} = 0$.

Proposition 1. *Let $\boldsymbol{q}^*$ be an internal Nash equilibrium for game $\Gamma = (\mathcal{N}, S_i, v_i(q_i, \boldsymbol{q}_{-i}, B))$, and let Q^* be the corresponding aggregate equilibrium output of the industry. Then there exists a vector $\boldsymbol{\xi} \in (0, +\infty)^N$, which just depends on coefficients β_{ij}, such that*

$$\boldsymbol{q}^* = Q^* \boldsymbol{\sigma} = Q^* \frac{\boldsymbol{\xi}}{\mu}, \tag{3}$$

with $\mu = \sum_{i=1}^{N} \xi_i$ and where the aggregate equilibrium quantity satisfies

$$Q^* p'(Q^*) = (c - p(Q^*))\mu, \tag{4}$$

while vector $\boldsymbol{\xi}$ is defined by

$$\boldsymbol{\xi} = (I + B)^{-1} \boldsymbol{u}, \tag{5}$$

in which the i-th component represents a measure of the centrality of the i-th firm in the network described by matrix B. Moreover, if the Nash equilibrium is unique, $Q^*(\mu)$ is an increasing function.

3. Centrality measures

Expressions in (3), (4) and (5) show that Nash equilibria can be characterized in terms of the Bonacich centrality measure $\boldsymbol{\xi}$, which quantifies the relevance that a firm has from being in the network of social interaction. It indeed depends on the way the preferences of firm i directly depend on the material payoff of its competitors, but it can be significantly altered by the indirect effects of other firms' preferences structure. The vector of centrality measures has a twofold descriptive power. Firstly, the distribution of centrality measures determines the ordering of firms with respect to their market share, describing how much a firm is dominant inside the market. Moreover, through $\mu = \sum \xi_i$ we are able to quantify the degree of competitiveness characterizing the market, as it determines where the equilibrium production of the industry places between the monopolistic and the competitive limit.

Conversely, how much a firm is taken into account by its competitors (i.e. its influence in the network of social preferences) can be described in terms of Friedkin-Johnsen [8] centrality measure $\boldsymbol{\chi} = (I + B^T)^{-1} \boldsymbol{u}$, which does not directly affect the equilibrium quantities but, as we are going to show, it is relevant to understand the effect of a change in the structure of interaction. We stress that, as evident from Propositions 4 and 5 in [4], the performance of firms and their ordering with respect to their relevance is in general independent of that with respect to their influence, being coincided only in the case of a symmetric network of social interactions.

To summarize, the Bonacich centrality of a firm quantifies the benefits/disadvantages arising from the connections that a firm directly or indirectly has in the network of social interaction and by the (altruistic or spiteful) kind of such interactions. The Friedkin-Johnsen centrality measure quantifies the influence that a firm i

exerts on all the other firms as a consequence of the direct or indirect social connections that they have with firm i. It is clear that each measure describes the role of a firm in the network of social interaction from a different point of view. What is the outcome of the combination of both points of view?

In order to answer this question, we draw our attention to the *intercentrality* measure, which was introduced by Ballester et al. in [2]. The intercentrality measure has been proposed to identify the player providing in a network the largest contribution to the aggregate outcome, namely the player whose removal from the network would lead to the largest disruptive effect to the collectivity performance. For a firm i, it is defined as the sum of firm i relevance (i.e. Bonacich centrality) and of i's contributions to the relevance of all the other firms. For a firm $j \neq i$, such contribution can be quantified by supposing to remove player i from the network (i.e. by setting $\beta_{ij} = \beta_{ji} = 0$ for any $j = 1, \ldots, N$) and by evaluating the difference between the centrality measure achieved by player j when player i is in the network and when player i is not in the network. Let B_{-i} be the network obtained from B by removing any interaction involving firm i and let $\xi_j(B)$ and $\xi_j(B_{-i})$ be the Bonacich centrality of firm j in networks B and B_{-i}. In what follows we will restrict to situations in which $(I+B)^{-1}\boldsymbol{u}$ and $(I+B_{-i})^{-1}\boldsymbol{u}$ both consist of nonnegative elements[5]. The intercentrality of a firm $i = 1, \ldots N$ is then defined by

$$\rho_i = \xi_i(B) + \sum_{j=1, j\neq i}^{N} (\xi_j(B) - \xi_j(B_{-i})), \tag{6}$$

and allows ordering players with respect to the contribution they exert toward the whole set of players. The player with the largest ρ_i is usually addressed as the *key player*.

In [2] the intercentrality measure is introduced for a symmetric matrix[6] whose elements correspond to

$$\rho_i = \frac{\xi_i^2}{\tilde{\beta}_{ii}}. \tag{7}$$

In the next proposition we provide a new characterization of such measure for a general network described by matrix B.

Proposition 2. *Let B be a matrix that satisfies Assumptions 1 and 3 and for which vectors $(I+B)^{-1}\boldsymbol{u}$ and $(I+B_{-i})^{-1}\boldsymbol{u}$ are nonnegative. Then*

$$\rho_i = \frac{\chi_i \xi_i}{\tilde{\beta}_{ii}}, \ i = 1, \ldots N. \tag{8}$$

[5]In the general case, the identities proved in this contribution are still valid provided that we consider also centrality measures with negative elements. However, in this case the connection between the Bonacich index and the Nash equilibrium no more holds.

[6]Rephrased to the present model, the setting in [2] corresponds to a globally symmetric spiteful scenario (i.e. in which the preferences of each firm negatively depend on the material payoffs of all their competitors and $\beta_{ij} = \beta_{ji}$).

The expression of ρ_i indeed coincides with that (7) when B is symmetric, as in such case we have $\chi_i = \xi_i$, namely the relevance of a firm in the network is identical to its influence. The expression of ρ_i provided by (8) is then in line with that in [2], but by removing the symmetry assumption on matrix B (and hence the identity between relevance and influence of a firm), we obtain a more neat social interpretation of the expression of ρ_i. As in (7), from (8) it is immediately evident that, ceteris paribus, if a player has a larger Bonacich centrality, it will contribute to a larger extent to the overall centrality of all the players. However, differently from (7), in (8) it is explicitly specified the role of the influence of player i in the network. The more a player is influential on the collectivity (net of the feedback effect encompassed in $\tilde{\beta}_{ii}$), the more its "achieved centrality" (due to its role in the network) will contribute to the overall centrality of the other players.

We then have that even if a player is central in the network but it is just a few influential, its large centrality will minimally benefit the overall centrality of players. The same occurs when a player has a large influence on the collectivity but it has a small centrality: the resulting contribution to the Bonacich centrality of the collectivity is small. Finally, we stress that since χ_i can be also negative, we have that a player can have a negative intercentrality measure. This is perfectly understandable: player i provides a negative contribution to the Bonacich centrality of the collectivity, i.e., on average, the other players would benefit from a removal of the player from the network.

The economic effects are a straightforward consequence of those social: since the intercentrality represents the contribution to the Bonacich centrality of all the firms, which in turns determines the market share and the degree of competitiveness, it is evident that understanding how ρ_i changes is essential to study the way the equilibrium is affected by the social interaction structure.

We show a possible scenario with respect to the distributions of the different measures in the next example.

Example 1. Distribution of centrality, influence and intercentrality Let consider the 5×5 matrix of negative weights

$$B = \begin{bmatrix} 0 & -0.03 & -0.1 & -0.12 & -0.01 \\ -0.17 & 0 & -0.05 & -0.15 & -0.1 \\ -0.22 & -0.23 & 0 & -0.05 & -0.01 \\ -0.1 & -0.1 & -0.2 & 0 & -0.06 \\ -0.21 & -0.05 & -0.17 & -0.12 & 0 \end{bmatrix} \tag{9}$$

which gives rise to the network shown in Figure 1. We report the matrix $(I +$

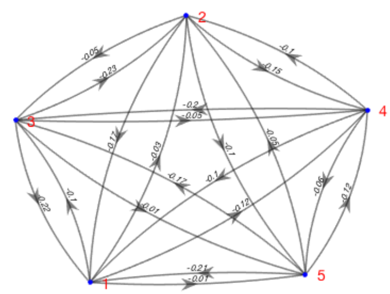


Figure 1: Graphical representation of the network described by weight matrix B_1 in (9)

$B)^{-1}$ of the aggregate effects due to any order dependence of social preferences

$$(I+B)^{-1} = \begin{bmatrix} 1.0674 & 0.0822 & 0.1461 & 0.1513 & 0.0294 \\ 0.2604 & 1.0683 & 0.1432 & 0.2135 & 0.1237 \\ 0.3086 & 0.2741 & 1.0806 & 0.1381 & 0.0496 \\ 0.2134 & 0.1782 & 0.2602 & 1.0758 & 0.0871 \\ 0.3152 & 0.1387 & 0.2528 & 0.1950 & 1.0312 \end{bmatrix} \tag{10}$$

We also report the vector of relative centrality measures ($\boldsymbol{\sigma}$), the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm ($\boldsymbol{\chi}^T$) and the row vector of intercentralities ($\boldsymbol{\rho}^T$), respectively

$$\boldsymbol{\sigma} = \begin{bmatrix} 0.1662 \\ 0.2036 \\ 0.2083 \\ 0.2043 \\ 0.2176 \end{bmatrix}, \boldsymbol{\chi} = \begin{bmatrix} 2.1651 \\ 1.7414 \\ 1.8829 \\ 1.7737 \\ 1.3210 \end{bmatrix} \text{ and } \boldsymbol{\rho} = \begin{bmatrix} 2.9947 \\ 2.9490 \\ 3.2252 \\ 2.9920 \\ 2.4761 \end{bmatrix} \tag{11}$$

We note that, in general, the ordering of the relative centralities, of the influences exerted by each firm and of intercentralities do not necessarily coincide. For instance, given the previous network, firm 5 has the highest centrality measure, but it is firm 3 the one with the highest intercentrality measure.

4. Comparative statics

The goal of the comparative statics is to investigate how a change in the structure of interdependent preferences (i.e. a change in the weight matrix B)

affects the equilibrium. It is clear how internal equilibria are actually characterized in terms of elements related to social interaction (the vector of centrality measures $\boldsymbol{\xi}$ and the related measure μ of the degree of competitiveness) and elements related to market interaction (the inverse demand function). In particular, the aggregate output level at the equilibrium depends on elements related to both market and social interaction, while the way the industry performance is distributed among firms just depends on the structure of interdependent preferences.

The comparative statics of internal equilibria must then be studied in terms of elements related to $\tilde{B} = (I + B)^{-1}$, to which we indeed have to add the characterization due to the inverse demand function. We start focusing on the role of the social interaction structure. Firstly, we investigate how measures ξ_i, χ_i and ρ_i are affected by an increase of a weight characterizing the social preferences of firm i.

Proposition 3. *Let B be a matrix satisfying Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$.*
If we linearly increase[7] coefficient β_{ij}, then ξ_i decreases, χ_i increases provided that $\chi_i \tilde{\beta}_{ji} < 0$ and ρ_i decreases provided that $\rho_i > 0$.

In the previous proposition we are assuming that firm i becomes less spiteful or more altruistic toward firm j. The result regarding ξ_i is unambiguous: the Bonacich centrality of firm i always decreases.
The behavior of χ_i is determined by the kind of influence that firm i has toward the overall industry and toward firm j. If they are of the same kind, then the overall influence of firm i toward the industry decreases as the direct influence that firm j has on firm i increases.
Finally, if firm i exerts a positive effect on the centrality of all firms, if it becomes less spiteful or more altruistic toward another firm, then such effect will decrease. We stress that if firm i becomes more spiteful or less altruistic toward firm j, we have the opposite behaviors, so that the centrality of firm i increases, its influence increases provided that firm i has, toward the overall industry and toward firm j, the same kind of influence and finally the negative effect on the centrality of the overall industry will decrease.

In the next propositions we investigate what happens to the share σ_i of a given firm and the degree of competitiveness μ.

Proposition 4. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$. If we linearly increase coefficient β_{ij}, then*

$$\sigma_i' = \xi_j(\xi_i \chi_i - \tilde{\beta}_{ii} \mu) \tag{12}$$

[7]Hereinafter, with linearly increase we mean that a coefficient increases as a linear function like $\beta(x) = \beta_+ + x$.

and the market share σ_i increases provided that

$$\rho_i = \frac{\xi_i \chi_i}{\tilde{\beta}_{ii}} > \mu. \tag{13}$$

The first part of Proposition 4 focuses on what happens when the structure of interaction of a given firm i changes, due to an increase in one of the weights through which the preferences of such firm depend on the material payoff of another firm. The main result is encompassed in condition (13), which clarifies under what conditions the market share of a given firm increases if its spitefulness decreases or its altruistic behavior becomes stronger We stress that in (13) we find involved all the centrality measures that characterize the outcome of the preference interaction structure at the equilibrium, namely the relevance of firms $\boldsymbol{\xi}$ (and consequently the market share $\boldsymbol{\sigma}$ and the degree of competitiveness μ), the influence $\boldsymbol{\chi}$ and the intercentrality $\boldsymbol{\rho}$. The behavior of the market share of a given firm on increasing β_{ij} basically depends on a comparison of such measures through a simple relation.

From condition[8](13), we can infer that if the overall influence degree is negative ($\chi_i < 0$), then the effect of an increase of altruism (or a decrease in spitefulness) will result in a decrease of the market share of firm i inside the market. Conversely, if $\chi_i > 0$, then the market share of firm i can improve as β_{ij} increases.

Condition (13) is very clear: increasing β_{ij} can result in a strengthening of the position of firm i in the market only provided that its overall contribution to the equilibrium of the industry is large enough. Moreover, the threshold at which this occurs is larger as the degree of competitiveness is higher. This means that if the aggregate equilibrium of game Γ is suitably close to the monopolistic limit, it is more likely that the market share of firm i improves through an increase in the degree of altruism (or a decrease in the degree of spitefulness)

From (13), the joint effect of the overall influence and of the relevance of firm i has to be suitably large. Ceteris paribus, it is more likely for a firm with a large (relative) centrality measure than for a firm with a small (relative) centrality measure to have an improvement in the equilibrium performance with a more altruistic behavior. We stress that condition (13) is independent on j, namely the increase of a weight that defines the social preferences of firm i can be toward any firm.

Conversely, the centrality measure of firm j determines the speed of increase of the market share of firm i, as evident from (12), in which ξ_j is a multiplicative coefficient of the positive term within brackets. In the opposite situation, i.e. when condition (13) is violated, the role of the centrality measure ξ_j conversely has a negative effect on the change of the market share of firm i. In fact, in such case we have that the market share of firm i decreases faster as the firm is more central.

[8]We recall that elements of ξ, the value of μ and the diagonal elements of matrix $\tilde{B}$ are always positive.

To summarize we can say that the more the overall industry gives relevance at the equilibrium to firm i through the network of social interdependent preferences, the more an increase of altruism (or a decrease of spitefulness) can be convenient to firm i, and vice-versa. In fact, in this latter case, if condition (13) is violated, an increase in the market share σ_i realizes if firm i reduces any weight describing its network of social interdependences, i.e. if it becomes less altruistic or more spiteful.

Even if in the model under consideration the distribution of weights is kept exogenous and the case in which firms can decide or change their social preferences is not under investigation, the previous considerations open interesting considerations in view of a possible endogenization and evolution of coefficients β_{ij}. Proposition 4 shows that a firm can improve its performance at the equilibrium if it changes its social preferences in the following way: when the degree χ_i with which the industry, as a whole, takes into account its performance is sufficiently high, the firm can improve its performance by increasing the weight it places on the material payoff of its competitors, while an improvement is obtained by reducing β_{ij} when χ_i is low or even negative. It is easy to read in the previous considerations a first, very prototypical and stylized, justification for a "tit-for-tat" dynamical way to adjust social preferences.

Now we investigate the effect that an increase of the weight another firm places on the material payoff of firm i has on the equilibrium performance of firm i.

Proposition 5. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$. If we linearly increase coefficient β_{ji}, then*

$$\sigma_i' = \xi_i(\xi_i \chi_j - \tilde{\beta}_{ij}\mu) \tag{14}$$

and the market share σ_i increases provided that

$$\chi_j \xi_i > \frac{\tilde{\beta}_{ij}}{\mu}. \tag{15}$$

The condition (15) under which σ_i increases is structurally very similar to that related to the first part of the proposition, and again results in a comparison of χ_j, ξ_i and μ. However, in this case, the discriminant is how much influential is firm j in the network of social interactions. The more firm j is influential, the more the weight that such firm gives to the material payoff of firm i will positively affect the equilibrium performance of firm i.

In line with (13), for the validity of condition (15), also in the present case the greater is the centrality measure of firm i, the smaller is the level of influence that must characterize firm j. Finally, in line with (12), from (14) we have that the greater is the centrality measure of firm i, the faster will increase the market share of firm i when (15) holds.

To summarize we can say that the more the overall industry gives relevance at the equilibrium to firm j through the network of social interdependent preferences, the more an increase of altruism (or a decrease of spitefulness) of firm j

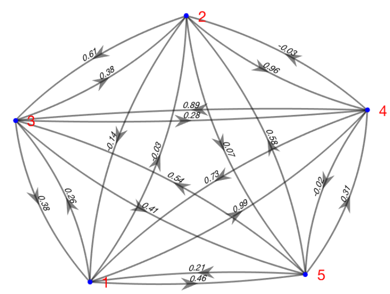


Figure 2: Graphical representation of the network described by weight matrix B in (16)

toward firm i can be convenient to firm i, and vice-versa. In fact, once more, in this latter case, if condition (15) is violated, an increase in the market share σ_i realizes if firm j reduces the weight through which it is linked to firm i, i.e. if firm j becomes less altruistic or more spiteful toward firm i. We deepen the description of the results of Proposition 5 in the next example.

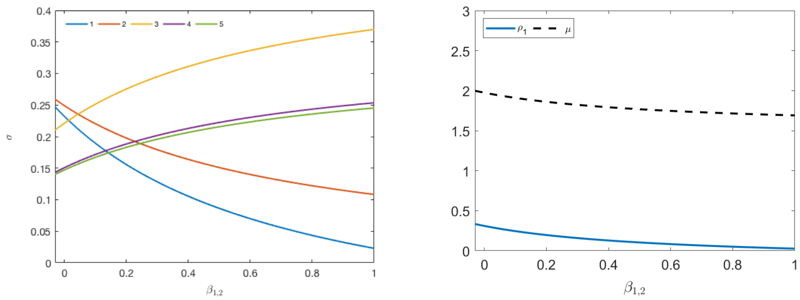
Example 2. Comparative statics: a general case
Let consider the following 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.03 & 0.26 & 0.99 & 0.46 \\ -0.14 & 0 & 0.61 & 0.96 & 0.07 \\ 0.38 & 0.38 & 0 & 0.28 & 0.41 \\ 0.73 & -0.03 & 0.89 & 0 & -0.02 \\ 0.21 & 0.58 & 0.54 & 0.31 & 0 \end{bmatrix} \tag{16}$$

which gives rise to the network shown in Figure 2. We report the matrix that encompasses the aggregate effects due to any order dependence of social preferences, namely

$$(I+B)^{-1} = \begin{bmatrix} 2.4124 & 0.2617 & 2.9176 & -2.7191 & -2.3786 \\ 1.7755 & 1.1569 & 2.5497 & -2.9615 & -2.0023 \\ -1.0866 & -0.3358 & 0.1532 & 1.2049 & 0.4846 \\ -0.7550 & 0.1308 & -2.2194 & 1.8450 & 1.2850 \\ -0.7156 & -0.5852 & -1.4862 & 1.0661 & 2.0008 \end{bmatrix} \tag{17}$$

We also report the column vector of the (absolute) centrality measures ($\boldsymbol{\xi}$) and



(a)

(b)

Figure 3: Figure 3(a) shows firms' market shares given the variation in the coefficient β_{12}. In Figure 3(b) we report the monotonicity relation in terms of ρ_1 and increasing values of the coefficient β_{12} (blue line). Figure 3(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient β_{12} (dark dashed line).

of the relative centrality measures ($\boldsymbol{\sigma}$), respectively

$$\boldsymbol{\xi} = \begin{bmatrix} 0.4940 \\ 0.5182 \\ 0.4204 \\ 0.2864 \\ 0.2799 \end{bmatrix} \quad \text{and } \boldsymbol{\sigma} = \begin{bmatrix} 0.2471 \\ 0.2593 \\ 0.2103 \\ 0.1433 \\ 0.1400 \end{bmatrix} \tag{18}$$

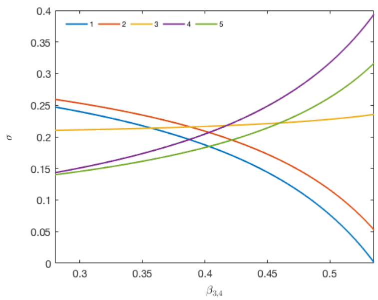
Finally, we report the row vector ($\boldsymbol{\chi}^T$) of aggregate effects due to any order dependence of social preferences of all the firms in the industry on each competitor

$$\boldsymbol{\chi}^T = \begin{bmatrix} 1.6307 & 0.6284 & 1.9148 & -1.5646 & -0.6105 \end{bmatrix} \tag{19}$$

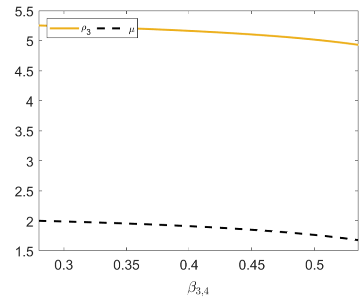
The goal of this example is to show the possible behaviors of market share σ_i of a firm when the firm gives more relevance to the material payoff of one of its competitors, and when one of its competitors increases the relevance given to the i-th firm material payoff.

In general, the relative centrality measure of firm i ($\sigma_i = \frac{\xi_i}{\sum_{i=1}^N \xi_i}$) decreases if any of its social weights β_{ij}, with $j \neq i$, tends towards 1. In other words, acting more altruistically towards any of its opponents tends to disadvantage the firm in terms of centrality. Conversely, if any of its opponent $j \neq i$ tends to increase the weight of the material payoff of firm i into its utility ($\beta_{ji}, j \neq i$), firm i's centrality measure will increase.

As an example of this, we let coefficient β_{12} increase in the interval $[-0.03, 1]$ to see what effect this may have, initially, on the relative centrality measure of firm 1 (blue line) and then on all its opponents, *in primis* on firm 2 (red line). Figure 3(a) shows that an increase in the altruism of firm 1 towards one of its opponents (in this case firm 2) has the direct effect to monotonically decrease the market share of firm 1 but also to decrease the market share of firm 2. We then let coefficient β_{34} increase in the interval $[0.28, 1]$ to see what effect this



(a)



(b)

Figure 4: Figure 4(a) shows firms' market shares given the variation in the coefficient β_{34}. In Figure 4(b) we report the monotonicity relation in terms of ρ_3 and increasing values of the coefficient β_{34} (yellow line). Figure 4(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient β_{34} (dark dashed line).

may have, initially, on the relative centrality measure of firm 3 (yellow line) and then on all its opponents, *in primis* on firm 4 (purple line). As before, we chose β_{34} since it allows for a greater interval of variation compared to the other coefficients of firm 3.

We note that an increase in the altruism of firm 3 towards firm 4 has the direct effect to monotonically increase the market share of firm 3 and the indirect effect to increase the market share of firm 4. Particularly interesting is the fact that for values of β_{34} in the interval $[0.387, 0.417]$ firm 3, which for $\beta_{34} = 0.28$ realized fewer profits than firm 1 and firm 2, is the firm with the highest profits, in the network.

If $\chi_i > 0$, there exists the possibility that an increase in the altruistic level of firm i towards some of its opponent has the effect to increase the relative centrality of firm i in the network. The higher χ_i is the smaller the initial market share firm i needs to own in order to have a positive effect due to an increase in one of its weights β_{ij}. The more central in the network is firm i, the more is probable that, even for a small but positive level of altruism exerted by the industry at the aggregate level towards i, an increase of the level of altruism towards one of its opponents has the effect to increase its centrality in the network. Conversely, the smaller firm i's relative centrality is the more probable, even for high level of aggregate altruism of the industry towards i (χ_i), the effect of a decrease in the centrality of firm i (σ_i) is.

Given the low degree ($\chi_1 = 1.18$), with which the industry as a whole takes into account the performance of firm 3, the increase of its evaluation of the material payoff of firm 2 in its utility has the effect to decrease its centrality in the network. Given the low influential role played by firm 1 in the network, an increase in the coefficient β_{12} also negatively affects the equilibrium performance of firm 2. We stress the fact that the main contribution to the centrality of firm 1 is due to the feedback effect $\tilde{\beta}_{11}$. Instead, given the high degree χ_3, with which the industry as a whole takes into account the performance of firm 3, the increase of its evaluation of the material payoff of firm 4 in its utility has the effect to increase its centrality in the network. Given the influential role played by firm 3 in the network, an increase in the coefficient β_{34} also positively affects the equilibrium performance of firm 4. We stress that in this last scenario the feedback effect on firm 3 is very small ($\tilde{\beta}_{33} \approx 0.15$) .

To show how the role of influence and centrality are both essential to understanding comparative statics of the characterization of the market share at the Nash equilibrium, we study the following structures for which the effect of a change in coefficients β_{ij} is unambiguous. Recalling Propositions 4 and 5 in [4], when the ordering of firms with respect to influence is the reversed one of that with respect to the centrality.

Proposition 6. *Let B be a matrix that satisfies Assumptions 1 and 3. Assume $\beta_{ij} = \beta_i$ for $i = 1, \dots, N$ and $i \neq j$ and that q^* is an internal Nash equilibrium. Then σ_i decreases if β_i increases and increases if $\beta_j, j \neq i$ increases.*

Conversely, assume $\beta_{ij} = \beta_j$ for $i = 1, \dots, N$ and $i \neq j$ and that q^* is an internal Nash equilibrium. Then σ_i increases if β_j increases and decreases if $\beta_i, j \neq i$ increases.

In the former scenario depicted in Proposition 6 we have that if a firm becomes more altruistic or less spiteful, its market share always decreases. This in particular also holds for the most central firm: the reason is that, recalling Proposition 4 in [4], it is also the least influential, and the joint effect of them is too small. In the latter scenario, the situation is instead the opposite one.

The next example focuses on a particular situation of the scenario investigated in Proposition 6.

Example 3. Comparative statics of scenarios in Proposition 6
Let consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.17 & -0.17 & -0.17 & -0.17 \\ -0.13 & 0 & -0.13 & -0.13 & -0.13 \\ -0.04 & -0.04 & 0 & -0.04 & -0.04 \\ 0.15 & 0.15 & 0.15 & 0 & 0.15 \\ 0.16 & 0.16 & 0.16 & 0.16 & 0 \end{bmatrix} \tag{20}$$

which gives rise to the network shown in Figure 5. The main feature of matrix B is that each firm i evaluates the opponents' material payoff the same way in its own utility, either positively or negatively.
We report the matrix $(I + B)^{-1}$ of the aggregate effects due to any order de-

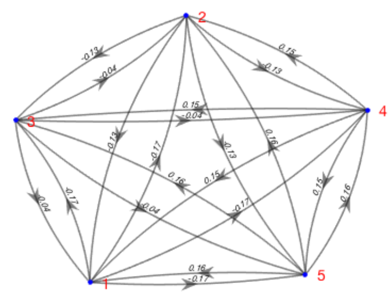


Figure 5: Graphical representation of the network described by weight matrix B in (20)

pendence of social preferences

$$(I+B)^{-1} = \begin{bmatrix} 0.9710 & 0.1204 & 0.1308 & 0.1600 & 0.1619 \\ 0.0921 & 0.9803 & 0.1036 & 0.1267 & 0.1282 \\ 0.0308 & 0.0319 & 0.9962 & 0.0424 & 0.0429 \\ -0.1412 & -0.1462 & -0.1589 & 0.9821 & -0.1967 \\ -0.1524 & -0.1578 & -0.1715 & -0.2098 & 0.9782 \end{bmatrix} \tag{21}$$

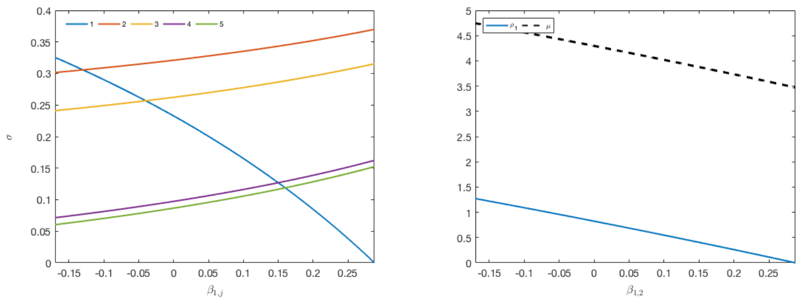
We compute the column vector of the centrality measures $\boldsymbol{\xi}$ and the column vector of relative centrality measures $\boldsymbol{\sigma}$, respectively

$$\boldsymbol{\xi} = \begin{bmatrix} 1.5441 \\ 1.4308 \\ 1.1440 \\ 0.3391 \\ 0.2867 \end{bmatrix} \text{ and } \boldsymbol{\sigma} = \begin{bmatrix} 0.3254 \\ 0.3016 \\ 0.2411 \\ 0.0715 \\ 0.0604 \end{bmatrix} \tag{22}$$

and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix $\tilde{B}$

$$\boldsymbol{\chi}^T = \begin{bmatrix} 0.8002 & 0.8285 & 0.9002 & 1.1014 & 1.1145 \end{bmatrix} \tag{23}$$

We let coefficients β_{1j}, with $j \neq i$ to increase in the interval $[-0.0725, 0.3]$ to see what effect this may have, initially, on the relative centrality measure of firm 1 (blue line) and then on all its opponents. Firm 1 represents an interesting case since it is, *a priori*, the most central firm in the network ($\xi_1 = \max(\boldsymbol{\xi}) = 1.5441$), given the highest overall outgoing degree of spitefulness in the network



(a) (b)

Figure 6: Figure 6(a) shows firms' market shares given the variation in coefficient β_{1j}, with $j \neq i$. In Figure 6(b) we report the monotonicity relation in terms of ρ_1 and increasing values of the coefficient $\beta_{1,j}$ (blue line). Figure 6(b) also reports the graph of the degree of competitiveness μ for increasing values of the coefficient $\beta_{1,j}$ (dark dashed line).

$((B\boldsymbol{u})_1 = \max((B\boldsymbol{u})) = -0.29)$, but is the less influential firm in the network $(\chi_1 = \min(\boldsymbol{\chi}) = 0.8002)$. The blue line in Figure 6(a) shows that a linear increase in the level of altruism exerted by firm 1 towards all firms (β_{1j}) has the direct effect to decrease its centrality in the network and therefore the market share. Note that firm 1 is initially spiteful, and as β_{1j} increases, it becomes less and less spiteful, turning into the most altruistic firm on $(0.17, 0.3)$. The market share lost by firm 1 is then redistributed among all its opponents whose profits firm 1 evaluates in its utility the same way.

We then consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.04 & 0.07 & 0.22 & 0.26 \\ -0.23 & 0 & 0.07 & 0.22 & 0.26 \\ -0.23 & -0.04 & 0 & 0.22 & 0.26 \\ -0.23 & -0.04 & 0.07 & 0 & 0.26 \\ -0.23 & -0.04 & 0.07 & 0.22 & 0 \end{bmatrix} \tag{24}$$

which gives rise to the network shown in Figure 7. The main feature of matrix B is that firm i's material payoff is considered with the same weight in each of its opponents' utility, either positively or negatively.

We report the matrix $(I + B)^{-1}$ of the aggregate effects due to any order dependence of social preferences

$$(I + B)^{-1} = \begin{bmatrix} 0.9155 & 0.0211 & -0.0413 & -0.1546 & -0.1926 \\ 0.1212 & 0.9865 & -0.0488 & -0.1828 & -0.2278 \\ 0.1356 & 0.0279 & 1.0207 & -0.2045 & -0.2547 \\ 0.1616 & 0.0332 & -0.0651 & 1.0383 & -0.3037 \\ 0.1704 & 0.0350 & -0.0686 & -0.2570 & 1.0312 \end{bmatrix} \tag{25}$$

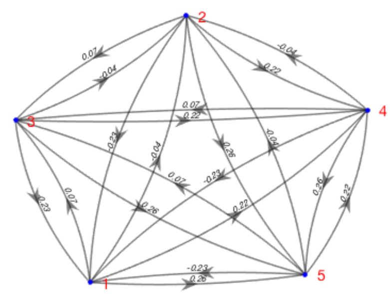


Figure 7: Graphical representation of the network described by weight matrix B in (24)

We compute the column vector of the centrality measures $\boldsymbol{\xi}$ and of market shares $\boldsymbol{\sigma}$

$$\boldsymbol{\xi} = \begin{bmatrix} 0.5481 \\ 0.6483 \\ 0.7250 \\ 0.8644 \\ 0.9111 \end{bmatrix} \quad \text{and } \boldsymbol{\sigma} = \begin{bmatrix} 0.1483 \\ 0.1754 \\ 0.1961 \\ 0.2338 \\ 0.2465 \end{bmatrix} \tag{26}$$

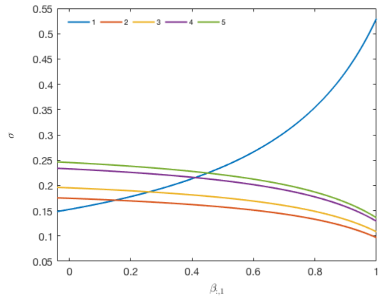
and the row vector of the aggregate effects due to any order dependence of social preferences of the industry on each firm $i = 1, 2, \cdots, N$ made by the column summations of matrix $\tilde{B}$

$$\boldsymbol{\chi}^T = \begin{bmatrix} 1.5043 & 1.1037 & 0.7970 & 0.2394 & 0.0525 \end{bmatrix}. \tag{27}$$

Vector $\boldsymbol{\sigma}$ shows that firm 1 is the least central while is firm 5 the one to own the largest market share. Looking at vector $\boldsymbol{\chi}$, we note that firm 1 is the most influential while firm 5 is the least one in the network.
As shown in Figure 8(a) by increasing homogeneously the influence of firm 1 over its opponents the market share also increases, to the point that firm 1, from being the least powerful oligopolist in the market, becomes the most central. The previous pattern is confirmed by looking at firm 5 situation shown in Figure 9(a). Increasing the influence of firm 5 into its opponents' utility preserves its leadership in the network.

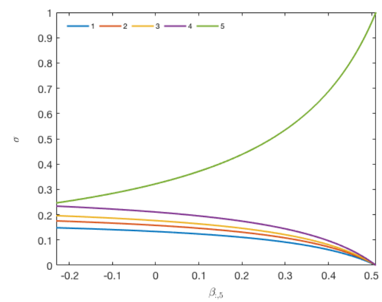
In general, it is not possible to have monotonicity results in a completely heterogeneous structure. The unique unambiguous situation is that in which all firms are spiteful, as shown in the next proposition.

Proposition 7. *Assume that all firms are spiteful or selfish with respect to all the other firms, then if* $|B_1| \geq |B_2|$ *we have* $\xi_1 \geq \xi_2$ *and* $\chi_1 \geq \chi_2$.



(a)

Figure 8: Figure 8(a) shows firms' market shares given the variation in coefficients β_{j1}, with $j \neq i$.



(a)

Figure 9: Figure 9(a) shows firms' market shares given the variation in coefficients β_{j5}, with $j \neq i$.

The following example reports a situation described by Proposition 7.

We move from the study of the effects on a single individual of the change of a single weight to the investigation of the effects on the collectivity of the change in the collective behavior. To this end, we first need to focus on what happens to the degree of competitiveness as the social preference structure changes.

Proposition 8. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and be $1 \leq i, j \leq N$ with $i \neq j$. If a given β_{ij} linearly increases, the degree of competitiveness μ increases provided that $\chi_j < 0$, or, equivalently, if $\rho_j < 0$. If all coefficients β_{ij} linearly increase, the degree of competitiveness μ decreases.*

From Proposition 8 we have that a firm takes into account more and more the material payoff of a firm j in its utility, the degree of competitiveness embedded in the game increases if firm j has a negative influence.

As predictable, when all the players become more altruistic or less spiteful, the degree of competitiveness decreases.

Now, the main question is: when is a *collective* change of the social preference structure beneficial to *every* firm? This is clarified in the next proposition for two relevant inverse demand functions.

Proposition 9. *Let B be a matrix that satisfies Assumptions 1 and 3 and to which corresponds an internal Nash equilibrium and let $p(Q) = \max\{a - QB, 0\}$ or $p(Q) = 1/Q$. Then as all β_{ij} linearly increase, the profits of all firms simultaneously increase provided that the distribution of firms with respect to centrality is suitably close to the uniform distribution.*

The previous proposition shows that a beneficial effect in terms of the achieved profits is possible provided that firms are suitably "homogeneous" in terms of their relevance at the equilibrium.
We deepen the investigation through the next example.

Example 4. Collective effects of a collective increase of altruism
Let consider the 5×5 matrix

$$B = \begin{bmatrix} 0 & -0.1467 & -0.0061 & -0.0858 & 0.0086 \\ -0.1323 & 0 & -0.0272 & 0.1520 & -0.2224 \\ 0.1391 & -0.1852 & 0 & -0.1602 & -0.0236 \\ -0.0472 & -0.1801 & -0.2298 & 0 & 0.2271 \\ 0.2231 & -0.2648 & -0.1599 & -0.0284 & 0 \end{bmatrix} \tag{28}$$

that describes a scenario in which there is homogeneity in the firms' centralities, i.e. the row summations are all equals.
Let us consider a perturbation matrix B_0, given by

$$B_0 = \begin{bmatrix} 0 & 0.0879 & -0.0043 & 0.0854 & 0.0870 \\ 0.0580 & 0 & 0.0909 & 0.0530 & 0.0285 \\ 0.0394 & -0.0076 & 0 & 0.0689 & 0.0956 \\ 0.0931 & 0.0759 & 0.0689 & 0 & 0.0579 \\ 0.0483 & 0.0247 & 0.0657 & 0.0087 & 0 \end{bmatrix} \tag{29}$$

whose network is represented in Figure 10

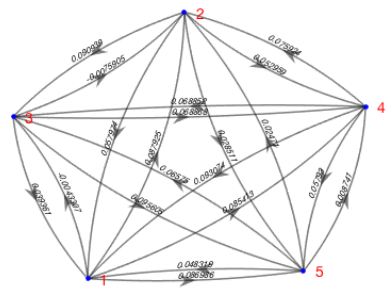


Figure 10: Graphical representation of the network described by weight matrix B_0 in (29)

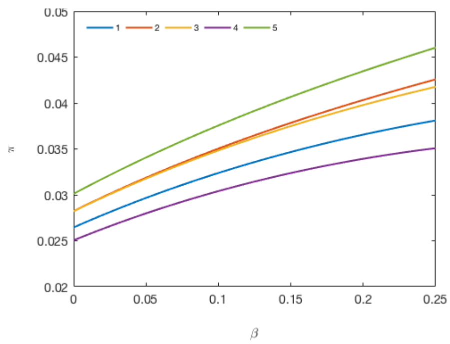
The goal is to study the behavior of profits of the firms in the network described by $B + \alpha B_0 + \beta(U - I)$ where $\alpha \geq 0$ and β ranges from 0 to a suitable maximum value. Matrix $B + \alpha B_0$ is an initial network consisting of a perturbation of the homogeneous scenario given by B. The larger is α, the greater is the degree of heterogeneity encompassed in $B + \alpha B_0$. Moreover, we stress that heterogeneity also increases as β grows up.
We report the matrix $(I + B + \alpha B_0)^{-1}$ of the aggregate effects due to any order dependence of social preferences both for values of $\alpha \approx 0.1$ and $\alpha \approx 0.5$

$$(I + B + 0.1B_0)^{-1} = \begin{bmatrix} 1.0274 & 0.0380 & -0.0015 & -0.0093 & -0.0881 \\ 0.0283 & 1.0282 & -0.0747 & -0.2123 & 0.2625 \\ -0.1561 & 0.1787 & 0.9921 & 0.0532 & -0.0370 \\ 0.0123 & 0.0616 & 0.1296 & 0.9937 & -0.2818 \\ -0.2865 & 0.2546 & 0.0784 & -0.0239 & 1.0779 \end{bmatrix} \quad (30)$$

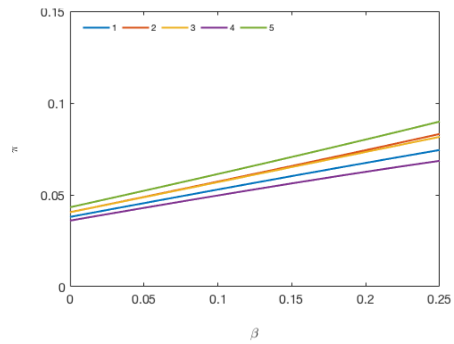
and

$$(I+B+0.5B_0)^{-1} = \begin{bmatrix} 1.3765 & -0.3380 & 0.3257 & -0.3799 & -0.5892 \\ -0.1079 & 1.0563 & -0.5071 & -0.3181 & 0.5270 \\ -0.1637 & 0.2730 & 0.9412 & -0.2279 & -0.2156 \\ -0.2129 & -0.2370 & 0.0550 & 1.1698 & -0.5539 \\ -0.6241 & 0.2638 & -0.3827 & 0.1523 & 1.3931 \end{bmatrix} \quad (31)$$

We report the vector of centrality measures ($\boldsymbol{\xi}$), for values of $\alpha \approx 0.1$ and



(a)



(b)

Figure 11: Profits for each firm in the network described in Figure 10 for increasing values of β (i.e. increasing the degree of altruism) given a linear demand function (in Figure 11(a)) and given an isoelastic demand function (in Figure 11(b)) for value of the perturbation parameter $\alpha \approx 0.1$

$\alpha \approx 0.5$, respectively

$$\boldsymbol{\xi}_{0.1} = \begin{bmatrix} 0.9666 \\ 1.0320 \\ 1.0309 \\ 0.9154 \\ 1.1005 \end{bmatrix}, \boldsymbol{\xi}_{0.5} = \begin{bmatrix} 0.3952 \\ 0.6502 \\ 0.6071 \\ 0.2210 \\ 0.8023 \end{bmatrix} \tag{32}$$

We note that the values in $\boldsymbol{\xi}_{0.1}$ are suitably close, representing an initial situation in which the firms own almost the same market share, while the values in $\boldsymbol{\xi}_{0.5}$ are much more sparse. We run the experiment for both the linear demand (for parameters' values of $a = 2$, $b = 1$) and the isoelastic demand function. In both cases we set marginal costs $c = 1$.

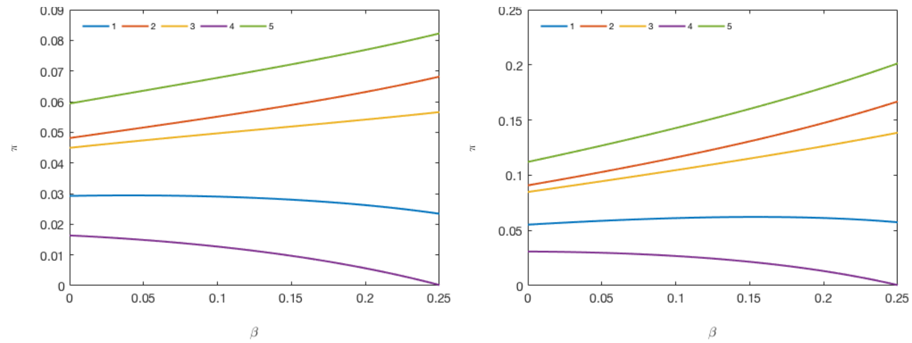
Proposition 9 shows that linearly increasing each coefficient β_{ij} of the matrix B, the profits of all firms simultaneously increase provided that the distribution of firms with respect to centrality is suitably close to the uniform distribution and matrix B satisfies 1-3.

In Figure 11(a) and Figure 11(b) we report achieved profits for the scenario with the perturbation parameter $\alpha = 0.1$. We note how all firms' profits are increasing as β increases.

Conversely, if the initial distribution of centralities is too heterogeneous, there is no chance for a firm with a low centrality to increase its profits by increasing its altruism toward the industry, as evident from Figure 12(a) and Figure 12(b), related to the case with $\alpha \approx 0.5$

5. Conclusions

The comparative statics analysis provided has highlighted the crucial role that the elements, characterizing the social interaction structure, play on the



(a) (b)

Figure 12: Profits for each firm in the network described in Figure 10 for increasing values of β (i.e. increasing the degree of altruism) given a linear demand function (in Figure 12(a)) and given an isoelastic demand function (in Figure 12(b)) for value of the perturbation parameter $\alpha \approx 0.5$

equilibrium. Among these elements we focused on the vector of centrality measures $\boldsymbol{\xi}$, known in the literature as the Bonacich centrality index, on the degree of competitiveness of the market μ, the vector of the degree of influence $\boldsymbol{\chi}$ that a single firm exerts in the network of social interactions, known in the literature as the Friedkin-Johnsen centrality measure, and the vector of intercentrality measure $\boldsymbol{\rho}$ which identifies in a network the player providing the largest contribution to the aggregate outcome, which in this case is represented by the aggregated utility. The contribution of each element to the equilibrium, in terms of market share and/or profits, has been studied in order to understand the reasonableness of an individualistic behavior or a collusive one. For instance, a given firm i can improve its individual performance, in terms of market share, at the equilibrium by increasing the weight it places on the material payoff of its competitors if the degree χ_i is sufficiently high or by reducing β_{ij} when χ_i is low or even negative. Even if in the proposed model the distribution of weights is kept exogenous and the case in which firms can decide or change their social preferences is not under investigation, it is easy to read in the previous considerations a first, very prototypical and stylized, justification for a "tit-for-tat" dynamical way to adjust social preferences in order to achieve higher market share. Moreover, we investigated the effects on the industry of the change in the collective behavior. Comparative statics shows that a beneficial effect in terms of the achieved profits is possible provided that firms are suitably "homogeneous" in terms of their relevance at the equilibrium.
To conclude, the provided local analysis allows us to suggest a unitary answer to the question of what drives, in the first stages of the game, the individual participant in Friedman laboratory experiment. That is, given the interaction structure of the game, how it is possible for a player to individually increase its performance. Simultaneously, the global analysis can explain why, from a certain stage of Friedman's laboratory experiment, emerges a decision behavior

that is fundamentally collective, passing from a heuristic that tries to improve the performance of the single player to a heuristic that tries to improve the performances of all the players, simultaneously. Friedman himself concludes that, in a triopoly context, the change from an individualistic heuristic to a collective one is facilitated when players interact, in groups or as a whole, in a more homogeneous social structure. In line with Friedman's conclusions, we showed that an increase in the collective performance, in terms of profits, is possible when the industry acts more altruistically, given that the interaction structure is characterized by less heterogeneity as possible between the players.

The results of the proposed model then offers the possibility to observe a sequence of static "snapshots" that can potentially describe the dynamical evolution observed in experiments. The next step of the present research strand is to introduce dynamics into the model, with a particular focus on the way the agents can adapt their preferences, i.e. on the way the structure of social interaction evolves. Just to fix ideas, a possible approach could consist of a gradient-like adjusment mechanism for β coefficients, in which agents, on the basis of a profitability signal, modify their social interaction structure to improve their preformance. However, such mechanism should take also in account some kind of reciprocity, namely should depend on the behavior of the competitors, in line with the approach, carried on in a static framework, in [13].

Appendix

Proof of Proposition 2. The proof is essentially the same as that of Theorem 3 in [2]. From the definition of ρ_i in (6) we can write

$$\rho_i = \xi_i + \sum_{j=1,j\neq i}^{N} \sum_{k=1}^{N} (\tilde{\beta}_{jk} - \tilde{\beta}_{-i,jk}) = \xi_i + \sum_{j=1,j\neq i}^{N} \sum_{k=1}^{N} \frac{\tilde{\beta}_{ji}\tilde{\beta}_{ik}}{\tilde{\beta}_{ii}}$$

We have

$$\rho_i = \xi_i + \sum_{j=1,j\neq i}^{N} \frac{\tilde{\beta}_{ji} \sum_{k=1}^{N} \tilde{\beta}_{ik}}{\tilde{\beta}_{ii}} = \xi_i + \sum_{j=1,j\neq i}^{N} \frac{\tilde{\beta}_{ji}\xi_i}{\tilde{\beta}_{ii}} = \xi_i \left(1 + \sum_{j=1,j\neq i}^{N} \frac{\tilde{\beta}_{ji}}{\tilde{\beta}_{ii}} \right)$$

which allows concluding. □

Proof of Proposition 3. Let $A_z = I + B + zE$ and $A = I + B$, where E is a matrix in which the unique non-null element is $(E)_{ij} = 1$. Let $\boldsymbol{e}_i$ the i-th vector of the euclidean basis of $\mathbb{R}^N$. We recall that

$$\frac{dA_z^{-1}}{dz}(0) = -A^{-1}\frac{dA}{dz}(0)A^{-1} = -A^{-1}EA^{-1}$$

In what follows we drop the evaluation at $z = 0$: we implicitly mean that all the involved functions depend on z and are evaluated at $z = 0$.

We have $\xi_i = \boldsymbol{e}_i^T A^{-1}\boldsymbol{u}$, so $\partial\xi_i/\partial z = -\boldsymbol{e}_i^T A^{-1}EA^{-1}\boldsymbol{u} = -\tilde{\beta}_{ii}\boldsymbol{e}_j^T\boldsymbol{\xi} = -\tilde{\beta}_{ii}\xi_j$. Since both $\tilde{\beta}_{ii}$ and ξ_j are positive, we conclude $\partial\xi_i/\partial z < 0$.

We have $\chi_i = \boldsymbol{u}^T A^{-1} \boldsymbol{e}_i$, so $\partial \chi_i / \partial z = -\boldsymbol{u}^T A^{-1} E A^{-1} \boldsymbol{e}_i = -\chi_i \tilde{\beta}_{ji}$, which allows concluding.

We have $\rho_i = (\boldsymbol{e}_i^T A^{-1} \boldsymbol{u} \boldsymbol{e}_i^T A^{-1} \boldsymbol{u}) / (\boldsymbol{e}_i^T A^{-1} \boldsymbol{e}_i)$, so

$$\begin{aligned}\frac{\partial \rho_i}{\partial z} &= \frac{(-\boldsymbol{e}_i^T A^{-1} E A^{-1} \boldsymbol{u} \boldsymbol{e}_i^T A^{-1} \boldsymbol{u} - \boldsymbol{e}_i^T A^{-1} \boldsymbol{u} \boldsymbol{u}^T A^{-1} E A^{-1} \boldsymbol{e}_i) \boldsymbol{e}_i^T A^{-1} \boldsymbol{e}_i + \boldsymbol{e}_i^T A^{-1} \boldsymbol{u} \boldsymbol{e}_i^T A^{-1} \boldsymbol{u} \boldsymbol{e}_i^T A^{-1} E A^{-1} \boldsymbol{e}_i}{(\boldsymbol{e}_i^T A^{-1} \boldsymbol{e}_i)^2} \\ &= \frac{(-\tilde{\beta}_{ii} \xi_j \chi_i - \xi_i \chi_i \tilde{\beta}_{ji}) \tilde{\beta}_{ii} + \xi_i \chi_i \tilde{\beta}_{ji} \tilde{\beta}_{ii}}{(\tilde{\beta}_{ii})^2} = -\frac{\xi_j \chi_i}{\tilde{\beta}_{ii}}\end{aligned}$$

□

Proof of Propositions 4,4 and 8. Let $A_z = I + B + zE$ and $A = I + B$.

Without loss of generality, we can focus on what happens to component 1. Let $\boldsymbol{e}_1$ the first vector of the euclidean basis of $\mathbb{R}^N$. We recall that

$$\frac{dA_z^{-1}}{dz} = -A_z^{-1} \frac{dA_z}{dz} A_z^{-1} = -A_z^{-1} E A_z^{-1} \Leftrightarrow \frac{dA^{-1}}{dz} = -A^{-1} E A^{-1}$$

From

$$\bar{\xi}_1(z) = \frac{\xi_i(z)}{\sum_{k=1}^N \xi_k(z)} = \frac{\boldsymbol{e}_1^T A_z^{-1} \boldsymbol{u}}{\boldsymbol{u}^T A_z \boldsymbol{u}}$$

we have

$$\frac{d\bar{\xi}_1}{dz}(0) = \frac{-\boldsymbol{e}_1^T A^{-1} E A^{-1} \boldsymbol{u} \boldsymbol{u}^T A^{-1} \boldsymbol{u} + \boldsymbol{e}_1^T A^{-1} \boldsymbol{u} \boldsymbol{u}^T A^{-1} E A^{-1} \boldsymbol{u}}{(\boldsymbol{u}^T A^{-1} \boldsymbol{u})^2} \tag{33}$$

We consider the case in which the unique non-null element is $(E)_{ij} = 1$, with $i \neq j$. It is easy to see that $A^{-1} \boldsymbol{u} = \boldsymbol{\xi}(0)$, $\boldsymbol{e}_1^T A^{-1} \boldsymbol{u} \boldsymbol{u}^T = \xi_1(0) \boldsymbol{u}^T$, $\boldsymbol{e}_1^T A^{-1} E = \tilde{\beta}_{1i} \boldsymbol{e}_j^T$ and $\boldsymbol{u}^T A^{-1} \boldsymbol{u} = \mu$, so we have that the numerator of the previous expression can be rewritten as (we drop evaluation at $z = 0$)

$$-\tilde{\beta}_{1i} \boldsymbol{e}_j^T \boldsymbol{\xi} \mu + \xi_1 \boldsymbol{u}^T A^{-1} E \boldsymbol{\xi} = -\tilde{\beta}_{1i} \xi_j \mu + \xi_1 \boldsymbol{u}^T A^{-1} E \xi_j$$

Since $\boldsymbol{u}^T A^{-1} \boldsymbol{e}_i = \chi_i$, we have that the right hand side in the last expression can be rewritten as

$$-\tilde{\beta}_{1i} \xi_j \mu + \xi_1 \chi_i \xi_j \tag{34}$$

from which we can obtain the corresponding conditions.

The case of $E = U - I$ can be obtained by summing all terms in (34). We have

$$\begin{aligned}\sum_{i=1}^N \sum_{j=1, j \neq i}^N (-\tilde{\beta}_{1i} \mu + \xi_1 \chi_i) \xi_j &= \sum_{i=1}^N (-\tilde{\beta}_{1i} \mu + \xi_1 \chi_i)(\mu - \xi_i) \\ &= \sum_{i=1}^N -\tilde{\beta}_{1i} \mu^2 + \xi_1 \chi_i \mu + \tilde{\beta}_{1i} \mu \xi_i - \xi_1 \chi_i \xi_i \\ &= -\xi_1 \mu^2 + \xi_1 \mu^2 + \tilde{\boldsymbol{\beta}}_1^T \boldsymbol{\xi} \mu - \xi_1 \boldsymbol{\chi}^T \boldsymbol{\xi}\end{aligned}$$

which allows concluding.

For the comparative statics on μ, we have

$$\frac{d\mu}{dz} = -\boldsymbol{u}^T A^{-1} E A^{-1} \boldsymbol{u}$$

If the unique non-null element of E is $(E)_{ij} = 1$, we have

$$\frac{d\mu}{dz} = -\boldsymbol{\chi}_j \boldsymbol{\xi}_i$$

If $E = U - I$, we have

$$\frac{d\mu}{dz} = -\boldsymbol{\chi}^T(\mu\boldsymbol{u} - \boldsymbol{\xi}) = \boldsymbol{\chi}^T\boldsymbol{\xi} - \mu^2$$

Noting that both the sum of the elements of $\boldsymbol{\chi}$ and $\boldsymbol{\xi}$ provides μ we can conclude that the previous difference is always negative. $\square$

Proof of Proposition 6. For both cases considered in the current proposition, we have already computed the values of ξ_i and μ in the proof of Propositions 4 and 5 in [4], so we refer to it for the related expressions.

We study the first scenario, in which B is such that $\beta_{ij} = \beta_i$ for $i \neq j, i, j \in \mathcal{N}$. The goal is to study the sign of

$$\frac{\partial\left(\frac{\xi_i}{\sum_{k=1}^N \xi_k}\right)}{\partial\beta_i} = \frac{\frac{\partial\xi_i}{\partial\beta_i}\left(\sum_{k=1}^N \xi_k\right) - \xi_i \sum_{k=1}^N \frac{\partial\xi_k}{\partial\beta_i}}{\left(\sum_{k=1}^N \xi_k\right)^2}$$

We have

$$\begin{aligned}
\frac{\partial\xi_i}{\partial\beta_j} &= \frac{-\displaystyle\sum_{k=1}^N \frac{\beta_i}{(1-\beta_i)(1-\beta_k)}}{1 + \displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}} \\
&= -\frac{\dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}\left(1 + \displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right) - \dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}\displaystyle\sum_{k=1}^N \frac{1}{1-\beta_k}}{\left(1 + \displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\dfrac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}(N-1)}{\left(1 + \displaystyle\sum_{k=1}^N \frac{\beta_k}{1-\beta_k}\right)^2} > 0
\end{aligned}$$

As a consequence we can compute

$$
\begin{aligned}
\sum_{k=1}^{N} \frac{\partial \xi_k}{\partial \beta_i} &= \frac{\frac{1}{(1-\beta_i)^2}\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-\frac{1}{(1-\beta_i)^2}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\frac{1}{(1-\beta_i)^2}\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}
\end{aligned}
$$

The derivative of the relative centrality index of firm i with respect to β_i is then

$$
\frac{\partial\left(\frac{\xi_i}{\sum_{k=1}^{N}\xi_k}\right)}{\partial \beta_i} = \frac{X}{Y}
$$

where

$$\begin{aligned}
X &= \frac{(1-N)\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)}{(1-\beta_i)^2\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad -\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\displaystyle\sum_{j=1}^{N}\frac{1}{1-\beta_j}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\frac{(1-N)}{(1-\beta_i)^2\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{(1-N)}{(1-\beta_i)^2\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\left[\frac{\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right. \\
&\quad \left. -\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\displaystyle\sum_{j=1}^{N}\frac{1}{1-\beta_j}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\right]
\end{aligned}$$

in which the first factor is negative and

$$Y=\left(\frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)^2>0 \tag{35}$$

The sign of the fraction is then provided by

$$
\begin{aligned}
&-\frac{\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+\left(\tfrac{1}{1-\beta_i}-\frac{\tfrac{\beta_i}{1-\beta_i}\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\\
&=-\frac{\tfrac{\beta_i}{1-\beta_i}\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+\tfrac{1}{1-\beta_i}-\frac{\left(1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}-\frac{\beta_i}{1-\beta_i}\right)\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\\
&=\frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\left(-\tfrac{\beta_i}{1-\beta_i}-1-\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}+\frac{\beta_i}{1-\beta_i}\right)+\tfrac{1}{1-\beta_i}\\
&=\frac{\displaystyle\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\displaystyle\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\left(-1-\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)+\tfrac{1}{1-\beta_i}\\
&=\tfrac{1}{1-\beta_i}-\sum_{k=1}^{N}\frac{1}{1-\beta_k}<0
\end{aligned}
$$

Similarly, the derivative of the relative centrality index of firm i with respect to β_j is

$$
\frac{\partial\frac{\xi_i}{\sum_{k=1}^{N}\xi_k}}{\partial\beta_j}=\frac{X}{Y}
$$

where Y is defined by (35) while

$$
\begin{aligned}
X &= \frac{\frac{\beta_i}{(1-\beta_i)(1-\beta_j)^2}(N-1)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad -\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\frac{(1-N)}{(1-\beta_j)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\frac{1}{(1-\beta_j)^2}(N-1)}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\left[\frac{\beta_i}{1-\beta_i}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right. \\
&\quad \left.+\left(\frac{1}{1-\beta_i}-\frac{\frac{\beta_i}{1-\beta_i}\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right)\right]
\end{aligned}
$$

Simplifying X/Y we have

$$
\frac{\partial\frac{\xi_i}{\sum_{k=1}^{N}\xi_k}}{\partial\beta_j}=\frac{N-1}{(1-\beta_j)^2(1-\beta_i)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)^2}>0
$$

Now we consider the second part of the proposition, in which B is such that $\beta_{ij}=\beta_j$ for $i\neq j, i,j\in\mathcal{N}$. Note that we can write $C=B^T=D+\boldsymbol{u}\boldsymbol{b}^T$, where D and $\boldsymbol{b}$ are respectively a diagonal matrix in which $d_{ii}=1-\beta_i$ for $i\in\mathcal{N}$ and a vector with $b_i=\beta_i$ for $i\in\mathcal{N}$. We indeed have $(I+C)^{-1}=((I+B)^{-1})^T$ and applying again Sherman-Morrison formula we can write $(I+C)^{-1}=D^{-1}-\frac{D^{-1}\boldsymbol{u}\boldsymbol{b}^TD^{-1}}{1+\boldsymbol{u}^TD^{-1}\boldsymbol{b}}$. It is easy to see that the elements of $D^{-1}\boldsymbol{b}\boldsymbol{u}^TD^{-1}$ are given by $a_{ij}=\frac{\beta_j}{(1-\beta_i)(1-\beta_j)}$ while $1+\boldsymbol{b}^TD^{-1}\boldsymbol{u}=1+b^TD^{-1}\boldsymbol{u}=1+\sum_{j=1}^{N}\frac{\beta_j}{1-\beta_j}$

We have $\boldsymbol{\xi}=D^{-1}\boldsymbol{u}-\frac{D^{-1}\boldsymbol{b}\boldsymbol{u}^TD^{-1}\boldsymbol{u}}{1+\boldsymbol{u}^TD^{-1}\boldsymbol{b}}$ and $u^T\boldsymbol{\xi}=\boldsymbol{u}^T(I+C)^{-1}\boldsymbol{u}=\boldsymbol{u}^T((I+$

$B)^{-1})^T\boldsymbol{u} = \boldsymbol{u}^T((I+B)^{-1})\boldsymbol{u}$

$$
\begin{aligned}
\xi_i &= \frac{\frac{1}{1-\beta_i} - \sum_{k=1}^{N} \frac{\beta_k}{(1-\beta_i)(1-\beta_k)}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)} \\
&= \frac{1}{1-\beta_i} - \frac{\frac{1}{1-\beta_i}\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&= \frac{1}{1-\beta_i}\left(1 - \frac{\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right) \\
&= \frac{1}{1-\beta_i}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}
\end{aligned}
$$

We note that $\boldsymbol{u}^T\boldsymbol{\xi}$ provides the same result for B and B^T. We have

$$
\begin{aligned}
\frac{\partial\xi_i}{\partial\beta_i} &= \frac{1}{(1-\beta_i)^2}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} - \frac{1}{1-\beta_i}\cdot\frac{\frac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\cdot\left(1-\frac{1}{1-\beta_i}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\right) \\
&= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}
\end{aligned}
$$

and

$$
\frac{\partial\xi_i}{\partial\beta_j} = -\frac{1}{1-\beta_i}\cdot\frac{\frac{1}{(1-\beta_j)^2}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}
$$

which allows writing

$$\frac{\partial \xi_j}{\partial \beta_i} = -\frac{1}{1-\beta_j} \cdot \frac{\frac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2}$$

We need to evaluate the sign of

$$\frac{\partial \left(\frac{\xi_i}{\sum_{k=1}^{N} \xi_k} \right)}{\partial \beta_i} = \frac{\frac{\partial \xi_i}{\partial \beta_i} \left(\sum_{k=1}^{N} \xi_k \right) - \xi_i \sum_{k=1}^{N} \frac{\partial \xi_k}{\partial \beta_i}}{\left(\sum_{k=1}^{N} \xi_k \right)^2} = \frac{X}{Y}$$

where Y is provided by (35). We have

$$\begin{aligned}
\sum_{k=1}^{N} \frac{\partial \xi_k}{\partial \beta_i} &= -\frac{1}{(1-\beta_i)^2 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \sum_{k=1, k\neq i}^{N} \frac{1}{1-\beta_k} \\
&\quad + \frac{1}{(1-\beta_i)^2 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \cdot \left(1 - \frac{1}{1-\beta_i} \cdot \frac{1}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} \right) \\
&\quad + \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right) - 1}{(1-\beta_i)^3 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \\
&= -\frac{1}{(1-\beta_i)^2 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)^2} \sum_{k=1}^{N} \frac{1}{1-\beta_k} + \frac{1}{(1-\beta_i)^2 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \\
&= \frac{1}{(1-\beta_i)^2 \left(1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}\right)} \left[-\frac{\sum_{k=1}^{N} \frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N} \frac{\beta_k}{1-\beta_k}} + 1 \right]
\end{aligned}$$

which allows obtaining

$$
\begin{aligned}
X &= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}} \\
&\quad -\frac{1}{1-\beta_i}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\cdot\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right] \\
&= \frac{\left((1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)-1\right)\cdot\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right]\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3} \\
&= \frac{(1-\beta_i)\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)+\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3} \\
&= \frac{(1-\beta_i)\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)-1}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} \\
&= \frac{\sum_{k=1,\neq i}^{N}\frac{1}{1-\beta_k}}{(1-\beta_i)^3\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2} > 0
\end{aligned}
$$

Similarly, we have

$$
\frac{\partial \frac{\xi_i}{\sum_{k=1}^{N}\xi_k}}{\partial\beta_j} = \frac{\frac{\partial\xi_i}{\partial\beta_j}\left(\sum_{k=1}^{N}\xi_k\right)-\xi_j\sum_{k=1}^{N}\frac{\partial\xi_k}{\partial\beta_i}}{\left(\sum_{k=1}^{N}\xi_k\right)^2} = \frac{X}{Y}
$$

where Y is provided by (35) and

$$\begin{aligned}
X &= -\tfrac{1}{1-\beta_j}\cdot\frac{\tfrac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\cdot\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\\
&\quad -\tfrac{1}{1-\beta_j}\cdot\frac{1}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}\frac{1}{(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right]\\
&= -\tfrac{1}{1-\beta_j}\cdot\frac{\tfrac{1}{(1-\beta_i)^2}}{\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3}\cdot\left(\sum_{k=1}^{N}\frac{1}{1-\beta_k}\right)\\
&\quad -\frac{1}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}\left[-\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}}+1\right]\\
&= -\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3}+\frac{\sum_{k=1}^{N}\frac{1}{1-\beta_k}}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^3}\\
&\quad -\frac{1}{(1-\beta_j)(1-\beta_i)^2\left(1+\sum_{k=1}^{N}\frac{\beta_k}{1-\beta_k}\right)^2}<0
\end{aligned}$$

□

Proof of Proposition 9. Let $p(Q) = a - bQ$. Using (4), the aggregate equilibrium is

$$Q^* = \frac{\mu(a-c)}{b(\mu+1)}$$

so profits can be written as

$$\pi_i = \frac{\xi_i(a-c)^2}{b(\mu+1)^2}$$

Assume that $B = \bar{B} + Z + \beta E$, where $E = U - I$, $\bar{B}$ is matrix with off-diagonal constant elements and Z is an hollow matrix whose off-diagonal elements describe the departure of elements of matrix B from the homogeneous weights distribution in $\bar{B}$. We assume that $\bar{B} = \bar{\beta}(U - I)$ is chosen in such a way the elements of Z have zero mean.

Computing the derivative of profits with respect to β we have

$$\pi_i' = \frac{(a-c)^2(\xi_i' + \mu\xi_i' - 2\xi_i\mu')}{b(\mu+1)^3}$$

The monotonicity of π_i' is determined by the sign of $\xi_i' + \mu\xi_i' - 2\xi_i\mu'$. Let $\bar{\boldsymbol{\xi}} = (I + \bar{B} + \beta E)^{-1}\boldsymbol{u}, \bar{\mu} = \boldsymbol{u}^T\bar{\boldsymbol{\xi}}, \boldsymbol{\varepsilon} = \boldsymbol{\xi} - \bar{\boldsymbol{\xi}}$ and $\mu_\varepsilon = \mu - \bar{\mu}$, we have

$$\begin{aligned}
\xi_i' + \mu\xi_i' - 2\xi_i\mu' &= (\bar{\xi} + \varepsilon_i)' + (\bar{\mu} + \mu_\varepsilon)(\bar{\xi} + \varepsilon_i)' - 2(\bar{\xi} + \varepsilon_i)(\bar{\mu} + \mu_\varepsilon)' \\
&= \bar{\xi}' + \bar{\mu}\bar{\xi}' - 2\bar{\xi}\bar{\mu}' + \varepsilon_i' + \mu_\varepsilon\bar{\xi}' + \bar{\mu}\varepsilon_i' + \mu_\varepsilon\varepsilon_i' \\
&\quad -2(\bar{\xi}\mu_\varepsilon' + \varepsilon_i\bar{\mu}' + \varepsilon_i\mu_\varepsilon')
\end{aligned}$$

We know that

$$\bar{\xi} = \frac{1}{(N-1)\bar{\beta}+1}, \; \bar{\mu} = \frac{N}{(N-1)\bar{\beta}+1}$$

from which

$$\bar{\xi}' = -\frac{N-1}{((N-1)\bar{\beta}+1)^2}, \; \bar{\mu} = -\frac{N(N-1)}{((N-1)\bar{\beta}+1)}$$

and hence

$$\bar{\xi}' + \bar{\mu}\bar{\xi}' - 2\bar{\xi}\bar{\mu}' = \frac{(N-1)^2(1-\bar{\beta})}{((N-1)\bar{\beta}+1)^3}$$

To have $\xi_i' + \mu\xi_i' - 2\xi_i\mu' > 0$ we then need

$$\left|\varepsilon_i' + \mu_\varepsilon\bar{\xi}' + \bar{\mu}\varepsilon_i' + \mu_\varepsilon\varepsilon_i' - 2(\bar{\xi}\mu_\varepsilon' + \varepsilon_i\bar{\mu}' + \varepsilon_i\mu_\varepsilon')\right| < \frac{(N-1)^2(1-\bar{\beta})}{((N-1)\bar{\beta}+1)^3}$$

We note that all the elements related to ε and their derivatives depend with continuity on the elements of Z, so, provided that Z is suitably small in some norm, the previous inequality holds.

Now let $p(Q) = 1/Q$. Using (4), the aggregate equilibrium is

$$Q^* = \frac{\mu - 1}{c\mu}$$

so profits can be written as

$$\pi_i = \frac{\xi_i}{\mu^2}$$

The derivative of π_i with respect to β is then

$$\pi_i' = \frac{\mu(\xi_i'\mu - 2\xi_i\mu')}{\mu^4}.$$

Repeating the last part of the proof for the linear case we obtain a similar conclusion. □

References

[1] R. Antle and A. Smith. An Empirical Investigation of the Relative Performance Evaluation of Corporate Executives. *Journal of Accounting Research*, 24:1–39, 1986.

[2] C. Ballester, A. Calvó-Armengol, and Y. Zenou. Who's Who in Networks. Wanted: The Key Player. *Econometrica*, 74(5):1403–1417, 2006.

[3] P. Bonacich. Simultaneous group and individual centralities. *Social Networks*, 13:155–168, 1991.

[4] M. Boretto, F. Cavalli, and A. K. Naimzada. Characterization of Nash equilibria in Cournotian oligopolies with interdependent preferences. DEMS Working paper series 463, University of Milano-Bicocca, 2021.

[5] M. Boretto, F. Cavalli, and A. K. Naimzada. Oligopoly model with interdependent preferences: existence and uniqueness of Nash equilibrium. DEMS Working paper series 462, University of Milano-Bicocca, 2021.

[6] M.J. Clayton and B.N. Jorgensen. Optimal Cross Holding with Externalities and Strategic Interactions. *The Journal of Business*, 78(4):1505–1522, 2005.

[7] J. Corbo, A. Calvó-Armengol, , and D.C. Parkes. The importance of network topology in local contribution games. In Springer, editor, *Internet and newtork economics*, 2007. Proceeding to the International Workshop on Web and Internet Economics.

[8] N.E. Friedkin and E.C. Johnsen. Social positions in influence networks. *Social Networks*, 19:209–222, 1997.

[9] D. Friedman, S. Huck, R. Oprea, and S. Weidenholzer. From imitation to collusion: Long-run learning in a low-information environment. *Journal of Economic Theory*, 155:185–205, 2015.

[10] R. Gibbons and K.J. Murphy. Relative performance evaluation for chief executive officers. *Industrial and Labor Relations Review*, 43, 1990.

[11] M.O. Jackson and Y. Zenou. Games on Networks. *Handbook of Game Theory with Economic Applications*, 4(3):95–163, 2015.

[12] D.K. Levine. Modeling altruism and spitefulness in experiments. *Rev. Econ. Dynam.*, 1:593–622, 1998.

[13] R. Sethi and E. Somanathan. Preference Evolution and Reciprocity. *Journal of Economic Theory*, 97:273–297, 2001.