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Dynamical analysis of evolutionary transition toward sustainable technologies

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Abstract

We propose a model for exploring the feasibility of the green transition between dirty and clean technologies. It relies on an evolutionary framework for the technology selection interacting with the environmental domain, which describes the evolution of pollution. A regulator charges an ambient tax to the producers, and the agents can choose between the less profitable clean technology and the more profitable dirty one, which however is taxed to a greater extent with respect to the clean one. The environmental tax depends endogenously on the level of pollution, which rises because of the producers' emissions. The pollution stock also naturally decays, and can be abated by involving the resources collected from the taxation. We analytically study the resulting two-dimensional model from both the static and the dynamical points of view, to understand under what conditions the green transition can take place and results in an improvement for the environmental quality. We show that excessive over-taxation of the dirty technology may be not always beneficial, as steady state pollution level can increase above a certain taxation threshold and multiple steady states can emerge. Moreover, dynamics can result in persistent endogenous oscillations that systematically lead to a significant increase in pollution levels. Finally, we discuss the economic rationale for the results also in the light of possible policy suggestions.

1 Introduction

Worldwide pollution and anthropogenic climate changes are two of the most dramatic and challenging issues of the beginning of the 21st century [18]. Despite a number of already undertaken actions, see for instance the 2015 Paris Agreement in which 194 nations committed themselves to keep the increase in temperature below 1.5°C, this problem still seems very far to be consistently managed. A measure of the importance of this effort is evident in the yearly hosted United Nations (UN) Climate Change Conference of the Parties (COP); the aim of these meetings is "... to accelerate action towards the goals of the Paris Agreement and the UN Framework Convention on Climate Change"[1]. The 'EU fit 55 package' [17] is a further agreement aiming at reducing emissions in Europe by at least 55% before 2030 and achieving climate neutrality by 2050.

Global warming is becoming worse and worse on a yearly basis. Some authors (see, as an example, [2]) claim that the world is reaching a threshold temperature above which changes aiming at reverting the actual trend will be highly likely irreversible. As of 2022, it has been measured that the quantity of carbon dioxide (one of the main component of greenhouse gas and the one with the largest radiative forcing [12]) yearly emitted into the atmosphere has reached the enormous quantity of 50 gigatonnes[2].

If, on one hand, substantial improvements have been successfully achieved in terms of lead pollution (a remarkable example is the one regarding lead in gasoline that has been eliminated worldwide, being Algeria in 2021 the last country to comply with [8]), on the other plastic waste released into the oceans is still an open issue.

However, if we observe the scenarios realized in terms of pollution evolution through the years, we can ascertain substantially differences from country to country, with either increasing, decreasing or persistently oscillating pollutant levels (also comparing neighboring or with similar development levels countries, as reported e.g. in [16]), making the issue of planning effective strategies both compelling and complex. The relevant point to raise in order to tackle this worrisome outlook is how to enforce a "green transition" toward less polluting best practices. Fiscal policies are relevant, here (see chapter 5 in [1]). Firms do not usually promptly adapt their behavior right after some economic changes occur. Switching between technologies carries relevant abandonment and replacement costs that might suggest producers to maintain obsolete or waste emitting machinery and devices.

The theoretical economic literature regarding issues related to environmental quality and climate changes is vast, encompasses many research strands and is virtually impossible to summarize. We limit ourselves to mentioning the portion of contributions that studies the interaction between the environmental quality and economic growth (for surveys on earlier contributions we refer to [5,11,21]). In this regard, we make reference to seminal contribution by John and Pecchinino [10], in which the conflict between economic growth and environmental sustainability is studied by taking into account an OLG economy. Here, the agent preferences depend on both consumption choices and a dynamically evolving index of environmental quality. The same model has been then studied from the dynamical point of view by Zhang [24], who showed that cyclical and chaotic trajectories are possible. Dynamical aspects of the interaction between environmental and economic spheres have been recently

[1] https://ukcop26.org/

[2] https://www.bbc.com/news/science-environment-63200589

studied also by Matsumoto and Szidarovszky [14], and Matsumoto et al. [13, 15].

Through years, the modelling approach in [10] has been reconsidered and enriched; to provide just some examples, we recall the contributions by Seegmuller and Verchére [20], in which the occurrence of endogenous fluctuations is again investigated, and Fodha and Seegmuller [7], where the authors examine the effectiveness of policies to improve the quality of the environment through government debt.

The previous discussion should have made clear that the most appropriate setting where technological transition can be analyzed is the dynamical one. It seems natural to introduce an evolutionary framework for studying the evolutionary feasibility of the green transition toward sustainable technologies. Still, this modelling step has been taken in just a handful of contributions.

To the best of our knowledge, we can mention [26] and [25], in which emissions abatement and public pollution governance are analyzed by applying evolutionary game theory, and the contribution by Zeppini [23]. In his article, this author studies the transition from dirty to clean technologies by considering a discrete choice model, in which the adoption of a technology is the consequence of an evolutionary selection. The selection mechanism is driven by the respective profitability of each production process, by positive externalities due to social interactions, by technological progress and by a pollution tax charged on dirty producers. Such levy is gauged by a regulator with the goal of promoting the transition toward the clean technology. In [23], the aim of taxation policies is simply to reduce their profitability and force them to opt for less polluting methods. The main results of this contribution are related to the existence of multiple coexisting steady states, due to the "lock-in" effects generated by the imitative process and unstable dynamics characterized by period-2 cycles triggered by the tax level. However, in [23], the environmental domain is not part of the model and, consequently, it is not possible to assess how the green transition affects (and is affected by) the environmental quality.

In the present contribution, starting from an evolutionary setting close to that proposed in [23], we want to investigate the role of the environmental domain on the effective possibility of achieving a transition toward sustainable technologies by adopting environmental taxation. Differently from [23], we neglect the influence of network externalities on the adoption of a particular technology[3], shifting our focus on the interaction between the discrete choice model and the dynamics characterizing the environmental domain. As in [23], agents can choose to adopt a clean or a dirty technology, in which the latter one is inherently more profitable. Differently from [23], we assume that the regulator can levy taxes on both kinds

[3] In [23], a first, basic model is progressively enriched by considering taxation and then technological progress. Since the focus of the present contribution is on the role of the environmental domain on the effectiveness of environmental policies, in this first research step we do not discuss the effects of technological progress, which would better fit in a framework in which the economic sphere is explicitly modelled. To this end, we can say that the model we propose departs from that in Section 3 in [23].

of producers, charging more the dirty ones. Moreover, in [23] tax is levied regardless of the actual environmental quality, being taxation proportional to the share of dirty producers. Conversely, in the present contribution the regulator can implement an ambient (or environmental) tax[4] that charges the adopted technologies, so that agents are taxed consistently to the ambient level of the pollutant.

The dynamics related to the environmental domain are modelled by introducing a variable that corresponds to environment pollutant level. The stock of pollutant increases due to the emissions of both clean and dirty producers, while it decreases thanks to natural decay and absorption achieved by employing the resources collected through taxation[5]. The main research questions we seek to tackle relate to the feasibility through an environmental taxation policy of a green transition that leads to sustainable levels of environmental quality. In particular:

Under what conditions an environmental taxation is able to trigger the evolutionary green transition toward clean technology?

To what extent is the green transition always able to improve the environmental quality?

We analyze the previous issues from both the static and dynamical point of view, and we show how, from both perspectives, managing the previous issues can be a complicated task. From the static point of view, the initially beneficial effect of charging more heavily the dirty producers allows starting a migration toward clean technologies that reduces the pollutant levels, but this can turn into a reversed situation if the taxation levels becomes too unbalanced. Moreover, scenarios characterized by high efficiency in abatement can result in multiple coexisting steady states. In this case, scenarios characterized by a larger share of producers adopting cleaner technologies not necessarily correspond to those characterized by an improved environmental quality. In addition to this, both the taxation level and the effectiveness of abatement can have a destabilizing effect on dynamics, being the source of the endogenous quasi-periodic oscillating trajectories observed in real contexts (see e.g. [16]). This also suggests that the static analysis can be misleading. We show that even when the ambient taxation triggers an evolutionary selection of technologies promoting the adoption of clean technologies and the green transition leads to an improvement of the environmental quality, the dynamical analysis can depict a substantially different scenario. In particular, when pollution trajectories exhibit strongly oscillating behaviors, they possibly reach from time-to-time peaks that are much more significant than the average level consistent with the steady state investigation. In this regard, including the environmental sphere into the model and allowing for their interaction with the evolutionary framework is essential to provide

[4]For references on ambient taxation, we refer the interested reader to [19, 22]

[5]In [23], clean producers are assumed not to pollute, and, consequently, they are not taxed. Besides representing limit, simplifying assumptions, they can also be too restrictive, and conceal relevant scenarios that can be shown by removing them. In addition to this, since the aim of [23] is mainly on studying the effects of social externalities and hence no environmental dynamics are not modelled, and collected resources can not used to improve the environmental quality.

reliable policy proposals. To stress this, we show how the proposed modelling approach paves the way for some initial, early-stage discussion in view of policy implications.

The structure of this contribution is as follows: in Section 2 we introduce the model, in Section 3 we analyze its steady states and their properties. In Section 4 we illustrate the possible dynamical behaviors while in Section 5 we discuss the results also in view of some green policy management insights. Section 6 concludes and provides some possible future developments of the research. In Appendix we report proofs of Propositions.

2 The model

We consider a productive environment in which a number of manufacturers, assumed to amount to a unit mass, can choose between two technologies, a '*c*lean' one or a '*d*irty' one. The share of firms that, at each discrete time t, exploits the first (latter) technology is denoted with $x_{c,t}$ ($x_{d,t} = 1 - x_{c,t}$).

Polluting technologies are more profitable than environmental friendly ones. Regulators that want to establish an effective environmental policy must then act so that dirty manufacturing becomes less rewarding, hinting agents that exploit it to shift toward cleaner, more sustainable technologies.

In line with Zeppini [23], this can be modelled through an evolutionary selection mechanism, in which the profitability, or utility, of adhering to one of the two technologies is described by means of fitness measures $u_{i,t}, i = c, d$ that positively depend on the intrinsic profitability λ_i of the chosen technology and negatively depend on the taxation level τ_i charging the adoption of a particular technology. As the dirty technology is assumed to be more profitable of the clean one, $\lambda_d - \lambda_c > 0$. In the present framework, the goal of the policy taxation intervention is to foster a "grid parity" by charging more the dirty technology. This is realized through an *environmental tax*[6], i.e. by introducing a "price for a permit to pollute" (see e.g. [16]), which allows adapting taxation to pollution levels. Greater amount of taxes should be paid by all producers when pollution levels are large while this burden can be reduced if pollution is low. This translates into the model the effect of an increasingly strong public intervention when the quality of environment worsen.

Let p_t be a measure of the pollution stock observed in the environment at time t; the resulting environmental tax (see [16]) can be modelled by $\tau_i p_t$, with $\tau_d > \tau_c \geq 0$, where $\tau_i p_t$, $i = c, d$, represents the taxation level charged to a representative agent adopting the

[6]More precisely, we assume that the regulator is not able to have precise information about the emissions of each agent or just the aggregate effects of pollution are observable (as in the case of non-point source pollution) or it is by far to costly to gather precise information about polluters. Since we are not interested in modelling a particular kind of pollutant dynamics, this is the most natural assumption. We refer to [22] for a discussion on the economics of non-point pollution, and to [6, 9, 19] for ambient taxation. We stress that, conversely, it is reasonable to assume that the regulator can distinguish between dirty and clean producers, as this just requires the knowledge of the adopted production technology.

$i-$th technology and proportional to p_t. Further, each τ_i represents the levy charged for a unitary amount of pollution stock. This modelling choice conveys that taxation depends on the environmental situation.

The resulting fitness measures[7] are then

$$u_{c,t} = \lambda_c - \tau_c p_t, \qquad u_{d,t} = \lambda_d - \tau_d p_t \tag{1}$$

Following Brock and Hommes [4], setting $\lambda_0 = \lambda_d - \lambda_c$ and $\Delta u_t = u_{d,t} - u_{c,t} = \lambda_0 - p_t(\tau_d - \tau_c)$, the time evolution of the share of clean producers $x_t = x_{c,t}$ can be described by the discrete time choice modelling

$$x_{t+1} = x_{c,t+1} = \frac{e^{\beta u_{c,t}}}{e^{\beta u_{c,t}} + e^{\beta u_{d,t}}} = \frac{1}{1 + e^{\beta \Delta u_t}} = \frac{1}{1 + e^{\beta(\lambda_0 - p_t(\tau_d - \tau_c))}}, \tag{2}$$

from which indeed $x_{d,t+1} = 1 - x_{c,t+1}$. In (2), parameter $\beta \geq 0$ measures the intensity of choice or evolutionary pressure of the selection mechanism. It implicitly encompasses the degree of rationality[8] of the agents in adopting a particular technology, so that the larger β is, the more the agents base their choices on the profitability gap Δu_t. The sign of term Δu_t denotes which technology performs better and depends on the distance between the profitability of the two technologies and on the gap between taxes levied on agents exploiting either the dirty or the clean productive facilities.

When $\beta = 0$, $x_{t+1} = 1/2$ regardless of the sign of Δu_t (in this case, the agents do not take into account Δu_t to make their decisions, and so they randomly adopt a technology, which results in a uniform share distribution). Conversely, if $\beta \to +\infty$, $x_{t+1} \to 1$ when $\Delta u_t < 0$ (the clean technology is preferred) and $x_{t+1} \to 0$ when $\Delta u_t > 0$ (the dirty technology carries larger profits). Otherwise stated, only if $\beta = +\infty$ all members of the entire population of agents is perfectly rational and adopt the most performing technology, while, conversely, the reduced rationality excludes the possibility that all the agents adopt the most profitable technology.

Even if we are interested in studying what happens when $\tau_d > \tau_c$, in what follows we allow for $\tau_c = \tau_d$, which will be used as a benchmark, limit situation.

As said in the Introduction, a contribution of this article abides in the use of resources collected through taxation to tame pollution.

In line with the existing literature [7, 10, 20], a simple way of modelling dynamics for the

[7] Recalling what we said in the Introduction, we stress that Zeppini [23] considered in $u_{i,t}$ a social interaction term $\rho_i x_{i,t}$, and taxes only charged the adoption of the dirty technology, modelled by $-\tau(1 - x_t)$ and interpreted as a average pollution emission tax.

[8] It is also worth mentioning that this article abides in the vein of rational decision making analysis. A comprehensive review of scientific contributions to rationality and sustainable development can be found in [3].

environmental sphere is

$$p_{t+1} = \max\{p_t - \alpha p_t + \varepsilon_c x_t + \varepsilon_d(1 - x_t) - \theta(\tau_c p_t x_t + \tau_d p_t(1 - x_t)), 0\} \tag{3}$$

where p_t is the pollution stock at time t. The pollution level at the next time period is influenced by three factors. The first one is the natural pollution decay, encompassed in term $-\alpha p_t$, where $\alpha \in (0; 1)$ represents the rate at which pollution naturally decreases.

Firms are assumed to have a constant pollution intensity of emissions, different just with respect to the adopted technology. During the production process, that takes place throughout the time interval $[t, t+1)$, the clean and the dirty technology respectively pollute at a constant rate $\varepsilon_c \geq 0$ and $\varepsilon_d > \varepsilon_c$, emit pollution stocks respectively corresponding to $\varepsilon_c x_t$ and $\varepsilon_d(1 - x_t)$, and raise the pollution level p_t. We stress that ε_i is actually the constant stock of pollution emitted by a single producer adopting technology i.

Finally, the last term in the right hand side of (3) displays how the aggregate amount of taxes collected in t levying the clean (dirty) technology $\tau_c p_t x_t$ $(\tau_p p_t(1 - x_t))$ is used and affects pollution abatement, with parameter $\theta \geq 0$ that gauges the effectiveness of resources adopted in pollution reduction policies[9]. We remark this last term in (3) both endogenously depends on the actual pollution level and the share of manufacturers resorting to each of the technologies. The case of $\theta = 0$ represents the extreme situation in which either it is impossible to reduce the pollution or no measures to abate the pollution are taken.

If no natural decay and abatement took place, the pollution level at time $t + 1$ would correspond to $E_{t+1} = p_t + \varepsilon_c x_t + \varepsilon_d(1 - x_t)$, consisting of the aggregated pollution level already present at time t and the stock of that emitted during the production activity. Moreover, the aggregated level of pollution that is removed from the environment during time interval $[t, t+1)$ thanks to natural decay and abatement amounts to $D_{t+1} = \alpha p_t + \theta(\tau_c p_t x_t + \tau_d p_t(1 - x_t))$. We assume that the "virgin", unpolluted state in which there is no contamination in the environment is set at $p = 0$ and hence p_t can not become negative. As a consequence of this, if $D_{t+1} > E_{t+1}$, the resulting pollution level would correspond to that of the virgin scenario, with $p_{t+1} = 0$, which explains the $\max\{\}$ function on the right hand side of (3).

Introducing function $M : (0, 1) \times [0, +\infty) \to (0, 1) \times [0, +\infty), (x, p) \mapsto M(x, p)$ defined by

$$M : \begin{cases} x_{t+1} = \frac{1}{1+e^{\beta(\lambda_0 - p_t(\tau_d - \tau_c))}} \\ p_{t+1} = \max\{(1 - \alpha)p_t + \varepsilon_c x_t + \varepsilon_d(1 - x_t) - \theta(\tau_c p_t x_t + \tau_d p_t(1 - x_t)), 0\}. \end{cases} \tag{4}$$

we obtain the two-dimensional dynamical system that describes the coevolution of shares of

[9] From the analytical view point, the choice of a simple, linear dependence of natural decay, emissions and abatement on, respectively, the pollution stock, the share of producers and the collected resources is to avoid that the emergence of (both static and dynamical) outcomes could be ascribed as an effect of the non linear terms. In any case, it is in line with the possible modelling of the environmental domain proposed in the literature (e.g. see [20]).

technology adoption and environmental sphere.

3 Static analysis

In this Section, we focus on the study of steady states of model (4) and on their properties. To better explain this and to understand the economic rationale of their occurrence, we subdivide the analysis into different steps. We start focusing on two simplified benchmark problems, consisting of the two uncoupled models related to the environmental domain and to the evolutionary mechanism.

Static analysis of uncoupled models

If we assume constant values for $x^* \in (0,1)$, model (4) reduces to the one dimensional, linear recurrence equation defined through function $\rho : [0,+\infty) \to [0,+\infty), p \mapsto \rho(p)$ by

$$p_{t+1} = \rho(p_t) = \max\{(1-\alpha)p_t + \varepsilon_c x^* + \varepsilon_d(1-x^*) - \theta\tau_d p_t(1-x^*) - \theta\tau_c p_t x^*, 0\} \tag{5}$$

for which we have the following result.

Proposition 1. *Model (5) has a unique, strictly positive steady state*

$$p^* = \frac{\varepsilon_d - (\varepsilon_d - \varepsilon_c)x^*}{\alpha + \tau_d\theta - \theta(\tau_d - \tau_c)x^*}, \tag{6}$$

which is decreasing with respect to both τ_d and θ, and for which we have

$$\frac{\partial p^*}{\partial x^*} \lesseqgtr 0 \Leftrightarrow \frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c} \lesseqgtr \frac{\alpha}{\theta}\left(\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_d}\right) \tag{7}$$

The environmental dynamics with a constant exogenous technology distribution always has a unique steady state. Its comparative statics with respect to taxation and technology efficiency are predictable as, ceteris paribus, an increase of either τ_d or θ raises the amount of abatable pollution (through an increase of collected resources or an improved effectiveness), and this contributes to a reduced steady state pollution stock. What is more counterintuitive is that, as the share of clean producers increases, the value of p^* does not necessarily decrease. Making the former condition in (7) more explicit, though, we can identify three possible scenarios on the behaviour of the share of clean firms (see Figure 1):

(a) a decrease of steady state pollution if $\frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c} < \frac{\alpha}{\theta}\left(\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_d}\right)$ (blue line)

(b) steady state pollution does not change if $\frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c} = \frac{\alpha}{\theta}\left(\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_d}\right)$ (red line)

(c) an increase of steady state pollution if $\frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c} > \frac{\alpha}{\theta}\left(\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_d}\right)$ (black line)

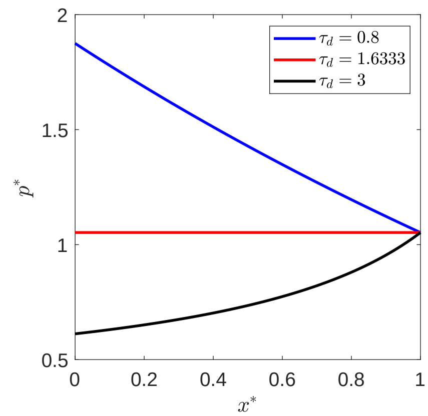


Figure 1: Steady state values for the pollution p^* as the share of clean producers x^* is exogenously increased, for different values of τ_d and setting $\theta = 0.75$.

It is worth noting that each ratio τ_i/ε_i in the left hand side of (7) represents the tax for each unit of pollutant emitted by each technology $i = c, d$, and correspond to the ratio between the positive (by supporting abatement through taxation) and negative (by polluting) effects on the environment of the presence of each technology. The left hand side in (7) then represents the (positive or negative) gap between the relative intensity of taxation of dirty and clean producers. To understand the occurrence of scenarios (b) and (c), let us assume that we are at a steady state, so that the amount of pollutant emitted and that naturally decays and is abated balance out. In the following comments we make reference to Figure 1. When a fraction of dirty producers is replaced by clean producers, two concurrent phenomena take place. The amount of emitted pollutant decreases, but collected resources from the taxation of a unit of pollutant reduce as well. If the marginal reduction in emissions is greater than the marginal reduction in removal, the steady state pollution will decrease, since disadvantage coming from the reduced potential capability to abate pollution is more than compensated for the reduced stock of pollution released. However, the opposite effect on p^* is obtained if the marginal reduction in emissions is not significant with respect to the marginal loss in abatement capability due to the decrease in collected resources from the reduced share of dirty producers. This occurs if the difference in emissions between the clean and the dirty producers is small relative to the taxes charged upon them. The effect of this (exogenous) transition is to replace dirty and heavily charged producers with not-so-clean and, proportionally, under-taxed producers and the final effect is a deterioration of the environmental scenario. This counterintuitive result resides in the fact that the clean producers still generate some pollution ($\varepsilon_c = 1$). In the numerical example reported in Figure 1, the dirty emission rate is 50% more then the clean one. The realized shift to a less polluting, but still not sufficiently clean, technology does not carry the expected result.

We can summarize the previous result as follows.

Outcome 1. *An increase of the share of clean producers can lead to an increase of the steady state pollution level if the taxation of dirty producers is larger than taxation of the clean one, proportionally to their respective emissions.*

Conversely, if we assume a constant value for p^*, equation (2) is actually a static process, with a constant distribution of shares $x^* = \frac{1}{1+e^{\beta(\lambda_0 - p^*(\tau_d - \tau_c))}}$. It is immediate to see that x^* increases both with respect to p^* and τ_d, as they both foster a green transition by penalizing the profitability of the dirty technology and driving agents to adopt the clean one.

Possible steady states of coupled model

We start characterizing conditions under which conditions $\mathbf{s}^* = (x^*, p^*)$ are steady states of (4), and their possible number. In what follows, we assume that solutions to equations and steady states are counted with their multiplicities. To this end, let us introduce functions $f_1 : (0, 1) \to [0, +\infty)$, $x \mapsto f_1(x)$ and $f_2 : [0, +\infty) \to (0, 1)$, $p \mapsto f_2(p)$.

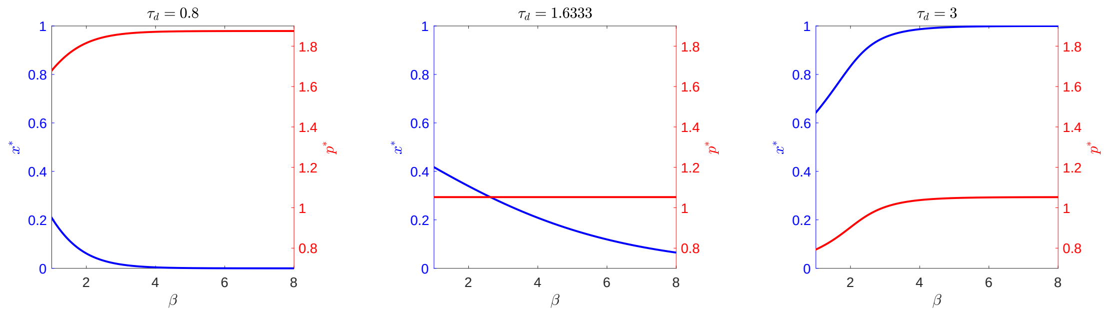
Proposition 2. *Model (4) always has either a unique or three steady states. At any steady state $\mathbf{s}^* = (x^*, p^*)$ we have*

$$\begin{aligned} p^* = f_1(x^*) &= \frac{\varepsilon_d - (\varepsilon_d - \varepsilon_c)x^*}{\alpha + \tau_d\theta - \theta(\tau_d - \tau_c)x^*}, \\ x^* = f_2(p^*) = \frac{1}{e^{\beta(\lambda_0 - (\tau_d - \tau_c)p^*)} + 1} &= \frac{1}{e^{\beta\left(\lambda_0 - \frac{(\tau_d - \tau_c)(\varepsilon_d - (\varepsilon_d - \varepsilon_c)x^*)}{\alpha + \tau_d\theta - \theta(\tau_d - \tau_c)x^*}\right)} + 1}. \end{aligned} \tag{8}$$

A necessary condition for the occurrence of multiple steady states is

$$\frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c} > \frac{\alpha}{\theta}\left(\frac{1}{\varepsilon_c} - \frac{1}{\varepsilon_d}\right). \tag{9}$$

The explicit analytical expression of steady states $\mathbf{s}^* = (x^*, p^*)$ of (4) is not available. The share component x^* is implicitly defined by the latter identity in (8) and determines the steady state value of pollution level through the former expression in (8). Proposition 2 shows that, even if for the two uncoupled dynamical mechanisms a unique steady state is possible, when they are coupled a multiplicity of steady states can occur. We stress that a similar scenario occurred in [23] as well, but in that case it was necessary to take into account an additional element in the model, represented by a (suitably large intensity of) positive externality of agent's decisions due to social interaction. Proposition 2 shows that, in an evolutionary framework, the simple mechanisms characterizing the environmental domain can be the source of steady state multiplicity. This is possible only if condition (9) holds true; otherwise, a unique steady state exists. It is relevant to underline that condition under which p^* negatively depends on the share of clean producers (scenario (c) after Proposition 1) is the same necessary condition that can induce multiplicity of steady states. What drives



(a) (b) (c)

Figure 2: Steady state values for the share of clean producers (left scale, blue color) and pollution (right scale, red color) for different values of τ_d and setting $\theta = 0.75$.

both behaviors are not the absolute values of pollution and taxation; rather, this depends on their relative values enter in the thresholds defining concurrent scenarios. We stress that such a phenomenon is completely ascribed to interaction between the evolutionary mechanisms and the environmental domain, and it is driven by elements that are related to the regulator choices (encompassed in the taxation levels τ_d and τ_c), to the agent choices (the adopted technology, depending on the intensity of choice β and the profitability gap λ_0) and to the environmental domain ($\varepsilon_c, \varepsilon_d, \alpha$ and θ).

To deepen the economic rationale of this phenomenon and to understand the role of each steady state in the case of coexisting ones, we need to investigate some additional features of model (4). At the present point, we can just remark that multiple steady states are possible only if the increase of clean producers brings about an increase of the steady state pollution levels. Moreover, the greater the effectiveness of abatement technology is, the more likely scenario (c) occurs because, *c*eteris paribus, this magnifies the marginal loss in the pollution abatement as the number of clean producers increases.

We remark that the right hand side in (9) is always positive. If both taxation levels were equally proportional to the rates of emissions, i.e. the relative intensity of taxation gap were null, just a unique steady state would be possible. The same occur if such a gap is negative, while a relative over-taxation of the polluting technology paves the way for the occurrence of multiple steady states.

We stress that even if Proposition 1 is not a comparative statics result for model (4), it implicitly provides the behavior of p^* on increasing β in the evolutionary model (4). In fact, p^* does not directly depends on β, while it is indirectly affected by the intensity of choice through the dependence of x^* on β. In Figure 2, we report a simulation obtained for the same parameter setting used in Figure 1 but obtained for model (4) on increasing β.

Graphs 2a, 2b, and 2c represent the effect of increasing τ_d levels (left to right) on steady states x^* (measured on the left vertical axis and plotted in blue) and p^* (measured on the right vertical axis and plotted in red) with variable β as the independent one. As previously said, the larger β is, the more rational-oriented is the choice in terms of tax-corrected profitability. In Figure 2a, the portion of clean producers vanishes and the level of pollution increases. This occurs when taxes for dirty technology are sufficiently low that applying this technology becomes more profitable. This also leads to an increase in pollution. The case with no change in pollution (Figure 2b) displays a reduction in producers adopting the clean technology. This unexpected result is even more pronounced in the right plot, where disappearance of dirty producers still lead to larger pollution.

In the next propositions we study how the uniqueness/multiplicity scenarios of steady states evolve on varying the parameter settings on their possible range of values. If we always have a unique steady state, we identify it by $\mathbf{s}^* = (x^*, p^*)$. Conversely, when scenarios characterized by a unique or three steady states alternate, we identify steady states by $\mathbf{s}_i^* = (x_i^*, p_i^*)$, with $i \in \{1, 2, 3\}$. In this case, the index related to steady states is such that, if $i < j$, we have $x_i^* < x_j^*$, namely steady states are ordered with respect to the share of clean agents. Moreover, as we will see from Section 4, when multiple steady states $\mathbf{s}_1^*, \mathbf{s}_2^*$ and $\mathbf{s}_3^*$ coexist, steady state $\mathbf{s}_2^*$, characterized by an intermediate share of clean producers, is always locally asymptotically unstable. For this reason, we just focus on $\mathbf{s}_1^*$ and $\mathbf{s}_3^*$. As otherwise specified, in all the simulations reported in this Section we set $\beta = 8, \lambda_0 = 1, \alpha = 0.2, \varepsilon_c = 1, \varepsilon_d = 1.5$ and $\tau_c = 1$.

The first set of results concerns the behavior of steady states with respect to τ_d. We start considering two limit cases, corresponding to $\theta = 0$ (i.e. the is no pollution abatement) and to $\varepsilon_c = \tau_c = 0$ (i.e. clean producers do not pollute at all, and hence they are not levied any tax).

Proposition 3. *Let $\theta = 0$. A unique steady state $\mathbf{s}^*$ exists for $\tau_d \in (\tau_c, +\infty)$. As τ_d increases, we have that x^* increases and approaches 1 as $\tau_d \to +\infty$, while p^* decreases.*

Increasing taxation of dirty producers promotes a gradual green transition. Since $\theta = 0$, the shares of dirty and clean producers just affect the stock of emissions, and hence the steady state pollution goes on decreasing as τ_d becomes larger and larger thanks to the transition from dirty to clean producers. We stress that even if in this case the complete green transition can take place with a persistent decrease of pollution levels, p^* can be large if the clean producers are significantly polluting, since no abatement policy is implemented.

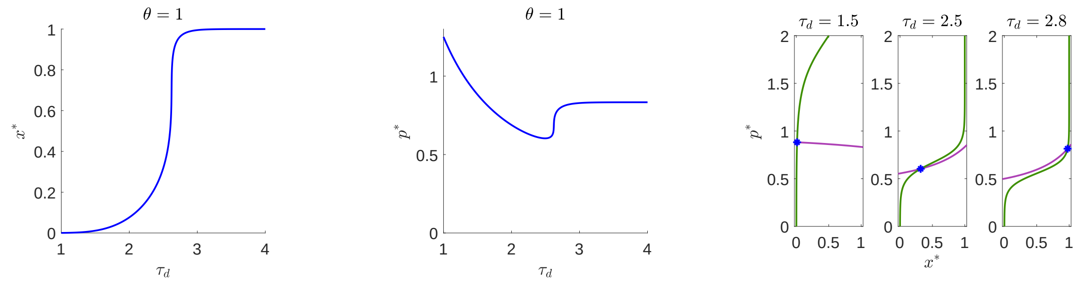
Proposition 4. *Let $\varepsilon_c = \tau_c = 0$ and $\theta > 0$. A unique steady state $\mathbf{s}^*$ exists for $\tau_d \in (\tau_c, +\infty)$. As τ_d increases, we have that x^* increases and approaches some $\bar{x}^* < 1$ as $\tau_d \to +\infty$, while p^* decreases.*

If clean producers do not pollute at all and they are not taxed, a unique steady state is possible. However, differently from Proposition 3, in this setting a complete green transition does not occur, and an environmental taxation is able to drive only a fraction of producers to adopt the clean technology. The reason is that as the green transition starts, since an increasingly share of the producers does not pollute, the steady state pollution level decreases, and this occurs faster than the increase of taxation. At some point, the penalization through τ_d of the dirty technology is balanced out by the decrease in p^*, and hence, also recalling the bounded rationality of the agents encompassed in the evolutionary selection mechanism, the green transition stops.

Now we consider the case of $\varepsilon_c > 0$, and we study it by distinguishing three different cases depending on θ.

Proposition 5. *Let $\varepsilon_c > 0$. There is $\theta_a > 0$ such that if $\theta \in (0, \theta_a]$, a unique steady state $\mathbf{s}^*$ exists for $\tau_d \in (\tau_c, +\infty)$. As τ_d increases, we have that x^* increases and there exists $\tilde{\tau}_d > \tau_c$ such that p^* decreases on $(\tau_c, \tilde{\tau}_d)$ and increases on $(\tilde{\tau}_d, +\infty)$.*

The behavior described in Proposition 5 is graphically depicted in Figure 3. Figure 3a shows that the share of clean producers is strictly increasing and converges to 1. Here, the presence of the dirty technology becomes negligible for a sufficiently large value of τ_d, and we can say that the green transition completely takes place. However, Figure 3b illustrates that the steady state pollution level, for increasing values of τ_d, correspondingly decreases only up to threshold $\tilde{\tau}_d$. This phenomenon is a direct consequence of the mechanisms that leads to Outcome 1. If τ_d is small, we know from (9) that increasing the share of clean producers leads to a decrease of the pollution level. In this case, as τ_d increases, the pollution emitted decreases thanks to the green transition, and the pollution abatement still increases, as the transition is initially slow (leftmost part of the curve in Figure 3a), and hence the small decrease of resources collected by the taxation of dirty producers is more than offset by the increase of the taxation level. However, if we further increase τ_d, the green transition accelerates, as the profitability of the dirty technology starts to be penalized by the environmental tax. Moreover, we come to a point at which increasing the share of clean producers leads to an increase of the pollution level, due to the reduction in the abatement. This initially slows down the decrease in the pollution level, and when the reduction in abatement is stronger than the decrease in the emission, and the steady state pollution starts increasing (middle parts of the curves in Figures 3a and 3b). From this level on, p^* increases reaching a steady pattern. Depending on the intensity of choice, the reversal in profitability order of technologies can lead to a very quick acceleration of the green transition as dirty producers are more significantly charged. We eventually come to a situation in which most producers adopt the clean technology. In this case, the marginal effect of an increase of τ_d is small, and we come to an approximately steady situation (right parts of the curves in Figure 3a and 3b). A qualitative comparison between these two graphs pinpoints that, for sufficiently large values of τ_d, switching from a more polluting to a less (but not substantially so) polluting technologies might lead to an unwanted outcome when dealing with taxation policies. We summarize the previous discussions as follows.



(a) (b) (c)

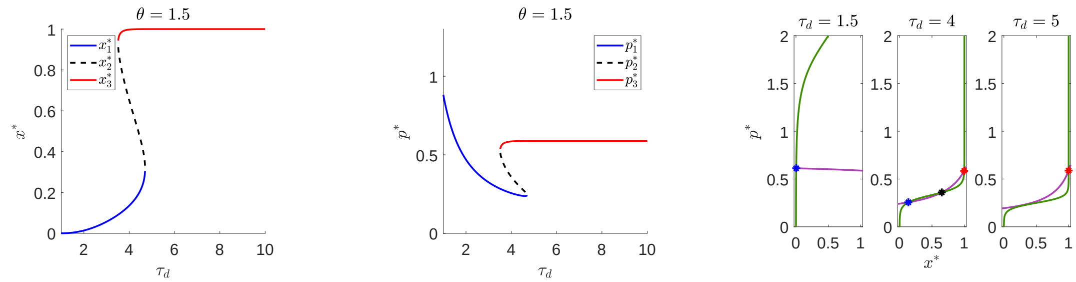
Figure 3: Behavior of steady state $\mathbf{s}^*$ on increasing τ_d when $\theta = 1$ lies in the range considered in Proposition 5. Component x^* and p^* are reported in panels (a) and (b), respectively, while panel (c) accounts for functions $p^* = f_1(x^*)$ (purple line) and $p^* = f_2^{-1}(x^*)$ (green line) defined through (8), with asterisks representing steady states.

Outcome 2. *An increase of the taxation of dirty producers with a consequent increase of the share of green producers not necessarily leads to a decrease for the steady state pollution levels.*

We stress that Proposition 5 also shows that if the effectiveness of pollution abatement measures is small, we always have a unique steady state. To give an insight on this, we make reference to Figure 3c. The red curve describes the effect of a change of the share of clean producers on the steady state pollution level. The blue curve represents the steady state stock of pollution for which a given share of clean producers is selected by the evolutionary mechanism[10]. Indeed, $\mathbf{s} = (x, p)$ is a steady state if the two curves intersect, namely when given the share x of clean producers, the corresponding steady state pollution level is such that the evolutionary selection mechanism induces a share distribution in which the corresponding share of clean producers is exactly x. As τ_d increases, the profitability of the dirty technology decreases, as the environmental tax $\tau_d p_t$ that each dirty producer is charged increases. Even when the environmental situation improves (i.e. p_t decreases) thanks to the larger amount of resources collected for the pollution abatement and to the increased presence of clean producers, the small effectiveness is not sufficient to allow for a decrease in the pollution level at least proportionally to the increase of τ_d, and hence more producers adopt the clean technology. For small θ, this is the unique possible scenario, which is not the case when effectiveness increases as shown in the next results.

Proposition 6. *Let $\varepsilon_c > 0$. There exist $\theta_b > \theta_a > 0$ such that if $\theta \in (\theta_a, \theta_b)$, there exists $0 < \tau_{d,1} < \tau_{d,2}$ such that a unique steady state $\mathbf{s}_1^*$ exists for $\tau_d < \tau_{d,1}$, three steady states $\mathbf{s}_1^*, \mathbf{s}_2^*$ and $\mathbf{s}_3^*$ with $x_1^* < x_2^* < x_3^*$ and $p_1^* < p_2^* < p_3^*$ for $\tau_d \in (\tau_{d,1}, \tau_{d,2})$, and a unique steady state $\mathbf{s}_3^*$ exists for $\tau_d > \tau_{d,2}$. As τ_d increases, x_1^* and x_3^* increase; there exists $\tilde{\tau}_d \in (\tau_c, \tau_{d,1}]$ such that p_1^* decreases on $(\tau_c, \tilde{\tau}_d)$ and increases on $(\tilde{\tau}_d, \tau_{d,1})$, while p_3^* increases on $(\tau_{d,1}, +\infty)$.*

[10] From the analytical point of view, it is the inverse of function connecting p^* to x^* reported in the former expression in (8).



(a) (b) (c)

Figure 4: Behavior of steady states $\mathbf{s}_i^*, i = 1, 2, 3$ on increasing τ_d when $\theta = 1.5$ is in the range considered in Proposition 6. Components x_i^* and p_i^* are reported in panels a and b, respectively, while panel (c) accounts for functions $p^* = f_1(x^*)$ (purple line) and $p^* = f_2^{-1}(x^*)$ (green line) defined through (8), with asterisks representing steady states.

The behavior described in Proposition 6 is graphically depicted in Figure 4. Results abiding in this and the following Propositions should be taken into account carefully as now more than a steady state coexist. In Figures 4a and 4b colors identify different steady states. Looking at the blue lines both figures, the share of clean manufacturers increases and the pollution level decreases, which is what a green transition policy strives to obtain, while an increase in p^* characterizes the red lines. The fulfillment of the reduction in pollution will then depend on the initial status of the system, involving dynamical effects of path dependency. Note that, differently from the case reported in Figure 4, it is possible that the steady state characterized by the small share of clean producers has a non-monotonic behavior with respect to p^*, but we observed that the extent of this is in general much less significant than in the case of Proposition 5.

For intermediate values of θ, we have that three steady states can exist, but only for intermediate values of τ_d (see also Figure 4c). To explain this, let us assume that the environmental situation is initially characterized by a suitably small pollution level. As τ_d increases, we have that even if the taxation of dirty producers increases, the effectiveness of abatement is enough to considerably lower the pollution level. The result is that the taxation level $\tau_d p_t$ remains low enough to allow dirty technology to be profitable. This results in steady state $\mathbf{s}_1^*$, which is characterized by a population dominated by dirty producers and a low level of pollution. Note that if environmental situation is such that the pollution level is initially more consistent, this scenario can not occur, as the environmental tax charging dirty producers is heavy, and in this case we have a green transition toward the clean technology. However, since this is just a “less dirty” technology that pollutes as well and is under-levied, this leads to few resources collected for pollution abatement, and this results in a worse environmental situation, and the green transition has not the desirable effects. This results in a steady state $\mathbf{s}_3^*$ that is characterized by a population dominated by clean producers and a large level of pollution. If τ_d is small, the taxation level is too small to trigger the green transition, and hence this latter scenario can not take place. However, the abatement technology is not so efficient to keep on decreasing the pollution level as the number of dirty producers increases and the collected resources decrease. Also, if τ_d is very large, the former scenario is not self-sustaining and steady state $\mathbf{s}_1^*$ disappears, leading to the existence of the unique steady state $\mathbf{s}_3^*$. Note that, from the previous discussion, we can also perceive why $\mathbf{s}_2^*$ does not play an active role as a steady state, as, in some sense, it just discriminates between the occurrence of the two scenarios.

Finally, we consider the case of large values of θ.

Proposition 7 (t). *Let $\varepsilon_c > 0$. There is $\theta_b > 0$ such that, if $\theta \in [\theta_b, +\infty)$, there is $0 < \tau_{d,1}$ such that a unique steady state $\mathbf{s}_1^*$ exists for $\tau_d < \tau_{d,1}$ and three steady states $\mathbf{s}_1^*, \mathbf{s}_2^*$ and $\mathbf{s}_3^*$ exist for $\tau_d > \tau_{d,1}$, with $x_1^* < x_2^* < x_3^*$ and $p_1^* < p_2^* < p_3^*$. As τ_d increases, we have that x_1^* and x_3^* increase; there exist $\tilde{\tau}_d \in (\tau_c, +\infty]$ such that p_1^* decreases on $(\tau_c, \tilde{\tau}_d)$ and increases on $(\tilde{\tau}_d, +\infty)$, while p_3^* increases on $(\tau_{d,1}, +\infty)$.*

We note that Proposition 7 allows for a decreasing behavior of p_1^* for any $\tau_d > \tau_c$ (this occurs when $\tilde{\tau}_d = +\infty$). In all the numerical simulations we performed, we always observed this latter scenario, but in any case the change for p_1^* for suitably large values of τ_d is negligible.

The behavior described in Proposition 7 is graphically depicted in Figure 5.

In Figures 5a and Figures 5b, the steady state denoted in blue leads to a reduction in the pollution; this is achieved with very small changes in the share of clean producers. This fact can be explained with the capability of the resources collected by means of taxation to positively impact on pollution reduction. The resulting scenario is very similar to that of Proposition 6, with the unique difference that, in the present case, the existence of multiple steady states is persistent. The reason is that the strong capability to abate the pollution is able to constantly reduce the pollution level and this consequently keeps moderate the taxation for the dirty technology, so that each dirty producer is moderately levied, and the green transition may not take place.

We summarize the previous discussions as follows.

Outcome 3. *For suitably large effectiveness in pollution abatement, multiple coexisting steady states are possible, characterized by either a large share of dirty producers and a low pollution level or by a small share of clean producers and a high pollution level.*

We stress that for both $\theta \in (\theta_a, \theta_b)$ and $\theta \in (\theta_b, +\infty)$, the realization of one of the coexisting scenarios is allowed by the kind of initial environmental situation, but studying this requires dynamical investigations, so we will return on it in Section 4.

Finally, we study comparative statics with respect to the remaining parameters, assuming that each parameter varies in the range of values for which the steady state exists. We avoid to show comparative statics with respect to β, as we want to just focus on the elements related to the environmental sphere.

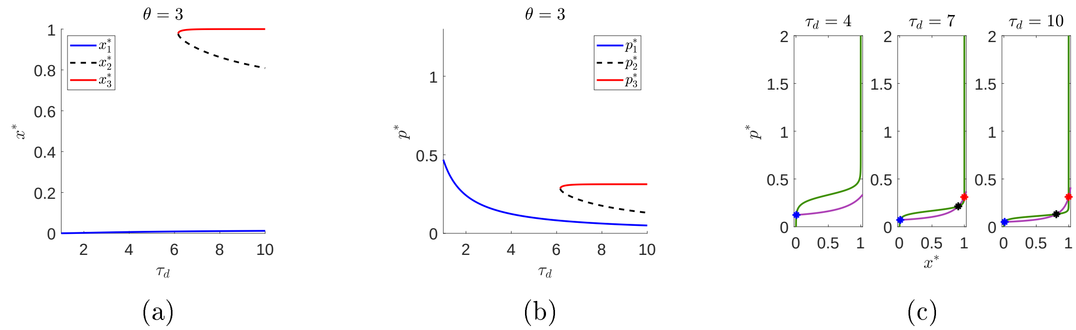


Figure 5: Behavior of steady states $\mathbf{s}_i^*, i = 1, 2, 3$ on increasing τ_d when $\theta = 3$ is in the range considered in Proposition 7. Components x_i^* and p_i^* are reported in panels a and b, respectively, while panel (c) accounts for functions $p^* = f_1(x^*)$ (purple line) and $p^* = f_2^{-1}(x^*)$ (green line) defined through (8), with asterisks representing steady states.

Proposition 8. *On increasing θ and α, both shares of clean producers x^*, x_1^* and x_3^* the corresponding pollution levels p^*, p_1^* and p_3^* decrease.*

On increasing ε_c or ε_d, both shares of clean producers x^*, x_1^* and x_3^* the corresponding pollution levels p^*, p_1^* and p_3^* increase.

On decreasing τ_c from τ_d to 0, the shares of clean producers x^*, x_1^* and x_3^* increase while the corresponding steady state pollution levels are either increasing or initially decreasing and then increasing.

In Proposition 8 we found the expected behavior of steady states with respect to $\alpha, \theta, \varepsilon_c$ and ε_d. Conversely, decreasing the taxation level of the clean technology τ_c has a "symmetric" effect with respect to that obtained by increasing τ_d. Its rationale can be then understood along the lines of the comments related to Proposition 1.

4 Dynamical analysis

In this section we study the dynamical properties of model (4), providing both analytical characterization of the stability regions and numerical investigation of non convergent dynamics. The goal is not to provide a systematic characterization of any possible dynamical behaviors occurring in model (4), but to pay specific attention to those dynamics from which we can infer additional economic insights with respect to the static perspective. In the proposed simulations[11], we set $\lambda_0 = 1, \beta = 25$ and $\varepsilon_d = 1.5$ and we focus on two scenarios, a

[11] The two dimensional bifurcation diagrams are obtained by setting the initial values of x and p suitably close to $\mathbf{s}^*$ (when a unique steady state exists) or $\mathbf{s}_3^*$ (when multiple steady states coexist). We note that the parameter setting considered in this section is different from that used for the static analysis. Numerically investigating the model, we observed that parameter configurations that are significant to discuss outcomes from the static point of view do not exhibit interesting behaviors from the dynamical point of view, and vice-versa. As we will discuss in Section 5, both the static and dynamical results alone provide relevant insights on the economic phenomena occurring, so it's worth considering two distinct parameter settings for static and dynamical simulations.

former one in which clean producers do not pollute at all (in this case we set $\varepsilon_c = \tau_c = 0$) and a latter one in which the clean technology pollutes (obtained setting $\varepsilon_c = 0.5$ and $\tau_c = 1.2$).

To better understand the role of share and pollution dynamics on the stability of the steady states, we discuss the possible dynamical evolution of p_t and x_t considering two benchmark situations.

The role of pollution equation

To study the dynamical effects related to the environmental side, we focus on equation (5), in which the unique steady state is locally asymptotically stable provided that $|\rho'(p^*)| < 1$. A simple direct check shows that, since $x^* \in (0, 1)$, we have that $\rho'(p^*) < 1$ is always fulfilled, while inequality $\rho'(p^*) > -1$ requires condition

$$2 - \alpha - \theta\tau_d(1 - x^*) - \theta\tau_c x^* > 0 \;\Leftrightarrow\; x^* > \bar{x} = \frac{\alpha + \tau_d\theta - 2}{\theta(\tau_d - \tau_c)} \tag{10}$$

It is straightforward to see that $\bar{x} \leq 0$ if and only if $\alpha + \tau_d\theta - 2 \leq 0$; here, p^* is stable independently of x^*. Conversely, p^* is stable only for suitably large values of the share of clean producers. We note that the fulfillment of stability condition is negatively affected by term $\alpha + \theta\tau_d(1 - x^*) + \theta\tau_c x^*$, which represents the stock of pollution that is removed from the environment if $p_t = 1$. This means that, with no evolutionary mechanism on shares, the natural decay, the effectiveness of technology for the abatement, and the taxation levels of each kind of producers have a destabilizing effect, in addition to the share of dirty producers.

To explain this, to fix ideas, let us assume that the initial pollution level is large, in particular above the steady state p^*. We note that p^* is the pollution level for which the amount of new pollutant emitted from one epoch to the next one is exactly compensated by the natural decay and abatement. From the dynamical viewpoint, we can have three possible scenarios. In the first scenario the aggregated effects of natural decay and abatement is small, so from time to time the stock of pollution that is removed from the environment is small, in particular less than $p_t - p^*$; as a consequence, the pollution level monotonically decreases until it reaches the steady state level p^*. If the aggregated effect of natural decay and abatement is more consistent, at the next time period the pollution level can fall below p^*. This reduces the amount of collected taxes at time $t + 1$ for pollution abatement, so pollution in $t + 2$ will increase above p^*. If natural decay and abatement effects are still moderate (such that $p^* - p_{t+1} < p_t - p^*$), this sequence of rebounds above and below p^* gives rise to dampening oscillations and p_t again converges toward the steady state level p^*. Conversely, if the aggregated effects of natural decay and abatement are further increased, (such that $p^* - p_{t+1} > p_t - p^*$), these oscillations self-sustain and become persistent. In this case, due to the piecewise linear nature of the equation, it is easy to see that the pollution level alternates between the two values 0 and $\varepsilon_c x^* + \varepsilon_d(1 - x^*)$. We can summarize this as follows.

Outcome 4. *If the joint effect of the taxation level of clean producers, share of dirty producers and their taxation level, the effectiveness of abatement technology and the natural decay is so significant to allow removing a large amount of pollutant, it can give rise to self-sustained and persistent oscillating behavior leading pollution level to not converge toward a constant level.*

The role of evolutionary selection of shares

If we consider an exogenous, constant in time pollution level p^*, the share distribution would result constant in time, and, consequently, the uncoupled share equation can not be the source of endogenous non convergent dynamics. However, the coupling between shares and pollution dynamics can give rise to "second order", indirect effects as a consequence of which persistent oscillations can arise even if the two uncoupled dynamics were stable, in particular if the evolutionary pressure is suitably strong. To show this, we consider the simplified model obtained by setting $\alpha = 1$ and $\theta = 0$. This allows focusing on the role of the evolutionary selection mechanism alone, excluding the emergence of dynamical behaviors arising from Outcome 4 related to the environmental sphere. In fact, in this case, the pollution equation reduces to $p_{t+1} = \varepsilon_c x_t + \varepsilon_d(1 - x_t)$, and hence the pollution stock at time $t + 1$ does not (directly) depend on the pollution level at time t. The resulting model can be rewritten as the one-dimensional second order difference equation defined through function $\sigma : (0, 1) \to (0, 1), x \mapsto \sigma(x)$ by

$$x_{t+1} = \sigma(x_{t-1}) = \frac{1}{1 + e^{\beta(\lambda-(\tau_d-\tau_c)(\varepsilon_c x_{t-1}+\varepsilon_d(1-x_{t-1})))}}. \tag{11}$$

For equation (11), we have the next Proposition[12].

Proposition 9. *The unique steady state $x^*$of* (11) *is locally asymptotically stable provided that*

$$1 - \beta x^*(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c)(1 - x^*) > 0 \tag{12}$$

When (12) *is violated, a Neimark-Sacker bifurcation occurs.*

Equation (11) describes how a change in the shares has a delayed effect on the share evolution itself. In fact, the distribution of adopted technologies at time $t - 1$ uniquely determines the pollution level at time $t + 1$, which in turns determines the share of clean producers at time $t + 2$. To fix ideas, we assume that the initial share of clean producers is very small, e.g. close to 0. This means that the pollution stock $(\varepsilon_c - \varepsilon_d)x_0 + \varepsilon_d$ at $t = 1$ is approximately equal to ε_d, i.e. to that produced by a group of firms adopting the

[12] We remark that, in Proposition 9 and those subsequent, stability conditions are expressed in terms of x^* (related to a steady state $\mathbf{s}^* = (x^*, p^*)$ of (4))), as it is just implicitly defined. Since x^* depends on all the parameters of the model, in reading stability conditions, we must be careful to have in mind that when a parameter changes, x^* changes as well.

dirty technology. Moreover, we assume that the value of β is not too small, so that the agents, in choosing which technology to adopt, suitably take into account the difference in the profitability measures of the two strategies.

If ε_d is suitably small, the pollution level is hence small and, consequently, if the two technologies are charged to a similar extent, we have that the fitness measure of the clean producers is smaller than that of dirty manufacturers, so that the majority of producers will still adopt the dirty technology. Consequently, the pollution level does not significantly change and the evolutionary mechanism selects a stable population of most dirty producers.

Now let τ_d increase, so that the difference in taxation between the two technologies is significant. We come to a point at which fitness measure of the dirty technology is smaller than that of the clean one, and most dirty producers would adopt the clean technology, leading p_t to decrease. As p_t decreases, the disadvantage of dirty technology more and more reduces as well, since the taxation of dirty producers progressively dampens with p_t. Hence the share of dirty producers increases, slowing down the decrease of pollution, which however eventually falls below the steady state value. Due to the evolutionary selection mechanism, this would drive part of the clean producers to reverse to the dirty technology, giving rise to a rebound of the pollution levels. If $\tau_d - \tau_c$ is still moderate these fluctuations are small as well, and we observe oscillating dynamics that dampen toward the steady state. If τ_d is further increased but still suitably moderate, persistent overreaction phenomena in switching between the two technologies lead to self sustained large oscillations in shares, which drive smooth alternating transitions between scenarios in which a majority of clean/dirty producers occur. In this case, the competition among technologies is strong, since the evolution of the difference in the performance of the two technologies is strongly affected by the large changes in the pollution levels, especially when $\varepsilon_d - \varepsilon_c$ is large. However, if τ_d is further increased, the reduction in the rebound of the fitness measure of the dirty producers obtained when the pollution decreases is smaller, since the adoption of dirty technology is heavily charged. This leads to oscillations in shares around larger values of x_t, and with smaller amplitudes. Above a certain level of τ_d, these oscillations diminish with respect to time and we again have convergence, which is now toward a population of almost any clean producers.

A second possible source of instabilities essentially due to the evolutionary selection of shares is remarked as follows.

Outcome 5. *The evolutionary selection mechanism can give rise to persistent overreaction phenomena that are due to a second order effect of the change in the shares on the share evolution themselves, even when mediated by potentially stable pollution dynamics alone. In particular, these occurs for intermediate taxation levels of the dirty technology.*

4.1 Local Stability

We now turn our attention on the analytical study of stability for model (4). We start from the case of a non-polluting clean technology in which, as shown in Proposition 4, we have a unique steady state $\mathbf{s}^*$ for any taxation level of the dirty technology.

Proposition 10. *Let $\varepsilon_c = \tau_c = 0$. The unique steady state $\mathbf{s}^* = (x^*, p^*)$ is locally asymptotically stable provided that*

$$\begin{cases} 2 - \alpha - \tau_d\theta(1-x^*) + \frac{\alpha\beta\varepsilon_d\tau_d x^*(1-x^*)}{\alpha+\tau_d\theta(1-x^*)} > 0, \\ 1 - \frac{\alpha\beta\varepsilon_d\tau_d x^*(1-x^*)}{\alpha+\tau_d\theta(1-x^*)} > 0. \end{cases} \tag{13}$$

When the former condition in (13) *becomes an equality, instability can just occur through a flip bifurcation, while when the later condition in* (15) *becomes an equality, instability can only occur through a Neimark-Sacker bifurcation.*

In particular, $\mathbf{s}^*$ is locally asymptotically stable for $\tau_d = \tau_c = 0$ and it is unstable for $\tau_d > \bar{\tau}_d$, for some suitable $\bar{\tau}_d > 0$.

According to (13), $\mathbf{s}^*$ is stable for suitably small values of τ_d. In fact, if $\tau_d = 0$, since no taxes are collected, there is no pollution abatement but only the natural decay, and the pollution level monotonically increases/decreases toward the steady state level. Moreover, we note that the former addend in the former condition in (13) corresponds to the left-hand side in (10), while the second addend is positive. This means that, in the case of $\varepsilon_c = \tau_c = 0$, the introduction of the evolutionary selection of shares has the potential effect of hindering the instabilities arising from the dynamics of the pollution, as high levels of pollution drive agents to adopt the clean technology, and this progressively reduces the pollution level and soften the endogenous large oscillations in p_t explained by Outcome 4. However, along the lines of Outcome 5, the evolutionary selection of shares can be itself the source of instabilities, and this explains the latter, new stability requirement in (13). The main difference with the case of $\varepsilon_c = 0$ is that, when $\varepsilon_c > 0$ and if $\alpha + \tau_c\theta < 2$, we have dynamics persistently not converging to the steady state as the taxation of the dirty technology increases. It may seem that there is a contradiction in the dynamical behaviors described by Propositions 10 and both Outcome 5 and subsequent Proposition 11 as $\tau_d \to +\infty$, in the case of a whatever small $\varepsilon_c > 0$ and $\alpha+\tau_c\theta < 2$. However, it is possible to show that as ε_c decreases approaching 0, the range of intermediate taxation levels for which the steady state is stable becomes increasingly larger, and its upper boundary diverges, which depicts a continuous changeover from the case of polluting and non-polluting clean technologies.

For the next explanations, we make reference to simulations reported in Figure 6. In the first row of Figure 6 we report two dimensional bifurcation diagrams[13] with respect to

[13]Different colors are used to distinguish between the number of points characterizing different attractors

variables τ_d and θ. Note that Proposition 10 shows that $\mathbf{s}^*$ is stable on a right neighborhood of $\tau_d = 0$ and unstable on a neighborhood of $\tau_d = +\infty$, but for intermediate values of τ_d we may have in principle several transitions from stability to instability and vice-versa. Since similar scenarios are more evident in the case of $\varepsilon_c > 0$, we discuss that situation after Proposition 13.

We start focusing on the case of small and intermediate effectiveness θ, corresponding to the lower and middle parts of Figures 6a-6c. In this case, as τ_d increases, the steady state becomes unstable by means of a Neimark-Sacker bifurcation (see also the bifurcation diagrams reported in Figures 6d-6e). The quasi-periodic nature of dynamics is related to the delay due to the second order effect of share changes on themselves described in Outcome 5. For example, let us focus on the time series reported in Figure 6f. If the pollution level at period t is larger than that at period $t-1$, the share of clean producers at time $t+1$ increases, but this has a beneficial effect on pollution just at time $t+2$, so it is just able to initially slow down the pollution increase, and only subsequently reverse it. The resulting trajectories quite smoothly oscillate above and below p^*. The bifurcation diagram for small values of θ resembles that reported in the Figure 6d. We note that in this case, the larger is the natural decay, the greater is the taxation level of dirty producers that triggers instabilities, since in the presence of reduced pollution levels, the oscillations on p^* has less effect on taxation and on the fitness measures of the two producers.

Conversely, if θ is suitably large (upper part of in Figures 6a-6c), for small values of τ_d the oscillations in dynamics can arise directly from the pollution dynamics, which would be unstable even in the case of constant shares, and are transmitted to the share evolution. In this case, oscillations inherit the cyclical nature induced by those characterizing the pollution levels, and a flip bifurcation occurs.

Looking at Figures 6a-6c, we note that, as θ initially increases, the taxation level τ_d that triggers instability increases as well. This can be explained recalling that, as θ increases, p^* decreases, so dynamics take place around a reduced level of pollution, and the overall taxation $\tau_d p_t$ of dirty producers is consequently reduced. For this reason, the fitness measure of dirty producers prevails only in the case of larger taxation levels with respect to that when θ is small, and we observe strong changes in their shares only for larger values of τ_d. A similar phenomenon occurs as α increases: the explanation is the same, with the unique difference that dynamics take place around a reduced level of pollution thanks to a stronger natural decay. Conversely, above a certain threshold of θ, we observe that instability occurs for increasingly small taxation levels. This is due to the change in the source of instability, that now arises from the environmental sphere (Outcome 4). In particular, it is easy to show

toward which convergence occurred. White color is used for convergence toward the steady state, red color toward a period 2 cycle and so on, with cyan color representing an attractor consisting of more than 32 points. In cyan regions we then can then find evidence of either quasi-periodic, chaotic attractors or high period cycles.

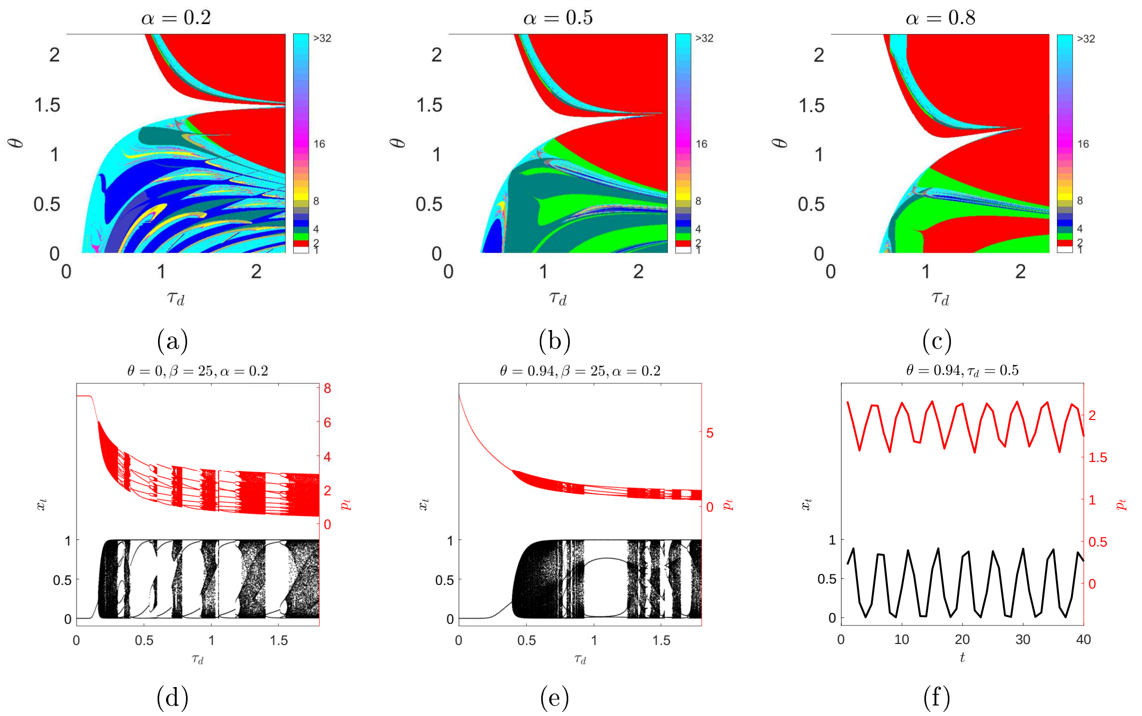


Figure 6: Simulations related to the case of non polluting clean technology, obtained setting $\varepsilon_c = \tau_c = 0$ (Proposition 10). Top row: two-dimensional bifurcation diagrams for different absorption rates on varying τ_d and θ. Bottom row: panels (d,e) bifurcation diagrams on varying τ_d; panel (f) times series. The share of clean producers and pollution are respectively represented using black (left scale) and red (right scale) color.

that even if p^* decreases with respect to θ, θp^* increases with respect to θ. This is reasonable, since, as the pollution stock decreases, the environmental situation improves, and it is simpler to abate pollution level than in a polluted environment. As a consequence of this, the amount of pollution $\theta \tau_i p_t$ that can be abated thanks to the taxation of a single producer increases with θ, and, in line to the discussion related to Outcome 4, the steady state becomes unstable for reduced values of τ_d. We note that, in this case, the larger is the natural decay, the smaller is the stability region, since a greater natural decay enforces the decrease in the pollution levels, and this fosters the emergence of instabilities in the pollution dynamics.

We now turn our attention on the case of polluting clean technology. Firstly, we focus on the role of the limit values for the taxation of the dirty technology, namely when both dirty and clean technologies are charged the same extent ($\tau_c = \tau_d$) and when $\tau_d \to +\infty$.

Proposition 11. *Let $\mathbf{s}^* = (x^*, p^*)$ be a steady state of (4) for $\varepsilon_c > 0$. If $\tau_c = \tau_d$, $\mathbf{s}^*$ is locally asymptotically stable provided that $\tau_c \theta + \alpha < 2$. If $x^* \to 1$ as $\tau_d \to +\infty$, there is $\bar{\tau}_d > 0$ such that for $\tau_d > \bar{\tau}_d$, we have that $\mathbf{s}^*$ is locally asymptotically stable provided that $\tau_c \theta + \alpha < 2$. If x^* does not approach 1 as $\tau_d \to +\infty$, we have that $\mathbf{s}^*$ is unstable.*

In discussing the previous proposition, we make reference to the two-dimensional bifurcation diagrams reported in Figures 7a-7c, in particular to the left boundary of each figure, which corresponds to the case of $\tau_c = \tau_d$. We stress that the black boundary denotes the region inside which we have multiple steady states. We start from the situation in which the effectiveness of pollution abatement is small ($\theta < \theta_a$ in Proposition 5, bottom parts of Figures 7a-7c), and hence there exists a unique steady state for any $\tau_d \in [\tau_c, +\infty)$ and a complete transition from dirty to clean technology could in principle take place, since $x^* \to 1$ as $\tau_d \to +\infty$.

We note that condition $\tau_c\theta + \alpha < 2$ is exactly the stability condition for p^* in the pollution equation with exogenous shares since, if $\tau_c = \tau_d$ or if $\tau_d = +\infty$ (in which case $x^* = 1$), the left-hand side in (10) simplifies as $2 - \alpha - \theta\tau_c > 0$. This means that instabilities arise from the environmental side, and can be explained accordingly to Outcome 4.

In this scenario, $\mathbf{s}^*$ is either both stable at $\tau_c = \tau_d$ and $\tau_d = +\infty$ or both unstable at $\tau_c = \tau_d$ and $\tau_d = +\infty$.

We stress that $\tau_c\theta + \alpha < 2$ can be rewritten as $\tau_c < (2 - \alpha)/\theta$, so stability requires a sufficiently small taxation of the clean technology. In the limit case of $\tau_d = \tau_c$, the shares of clean and dirty producers do not depend on the pollution level, and their distribution depends only on the profitability difference λ_0 and on the intensity of choice β. In this case, since the taxation level is the same for both technologies, the dirty producers do not benefit from adopting a clean technology, so most producers immediately decide to adopt the dirty technology (in the limit $\beta \to +\infty$ we would have dirty producers only). If τ_d is slightly larger than τ_c, the transient or persistent oscillations in the pollution levels induce oscillations in the distribution of clean producers, as the fitness measure of the clean technology is slightly more favored (resp. hindered) when the pollution level is large (resp. small). These oscillations can dampen or be persistent, depending on the underlying dynamics of the pollution equation and according to the discussion related to Outcome 4.

Let us now consider the case of $\tau_d \to +\infty$. In the presence of a whatever small level of pollution, all producers would adopt a clean technology (i.e. $x^* = 1$). Also, in this case no oscillation arises from the share dynamics, and dynamics are either stable or not depending on those arising from the environmental sphere.

If the efficiency of pollution abatement is intermediate ($\theta_a < \theta < \theta_b$ in Proposition 6), we have that, for intermediate values of τ_d, three steady states coexist but, at the extreme values of the range of variation of τ_d, we still have a unique steady state so that dynamics can be explained similarly to case $\theta < \theta_a$. We simply note that, in this case, the maximum taxation level for the clean technology that guarantees convergent dynamics is smaller than in the previous case. The reason is that $\alpha + \tau_c\theta$, for the same value of τ_c, becomes larger and larger as θ increases and, thanks to the larger efficiency in pollution abatement, significant variations in the pollution levels occur for reduced taxation level.

If the efficiency of pollution abatement is large ($\theta > \theta_b$ in Proposition 6, middle and top parts Figures 7a-7c), we have that, for suitably large values of τ_d, three steady states always coexist. The above discussion indeed still applies for $\tau_d \to +\infty$ at $\mathbf{s}_3^*$, as this case a complete green transition takes place and $x_3^* = 1$. Conversely, at $\mathbf{s}_1^*$, since a complete green transition does not occurs, we have that a share of dirty producers is still present in the market which, according to the dynamics discussed with Outcome 4, has a destabilizing effect.

As τ_d increases above a certain threshold, the amount of collected resources is so large that it is possible to remove all the pollution from the environment. However, this brings down the taxes collected from the dirty producers, and the environmental situation immediately worsen. Simultaneously, due to the improved pollution level, most producers can decide to revert to the dirty technology, which is not taxed since the pollution level is null. This starts a cyclical behavior between scenarios characterized by no pollution and a population of most dirty producers and high pollution level and a population of most clean producers. This means that, for suitably large values of τ_d, $\mathbf{s}_1^*$ is unstable and can coexist with a stable steady state $\mathbf{s}_3^*$.

We now consider the case of positive emissions for the clean technology. We start considering the limit case of $\theta = 0$.

Proposition 12. *Let $\theta = 0, \varepsilon_c > 0$. The unique steady state $\mathbf{s}^* = (x^*, p^*)$ of* (4) *is locally asymptotically stable provided that*

$$\beta x^*(1 - x^*)(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c) < 1 \tag{14}$$

In particular, condition (14) *is fulfilled for $\tau_d = \tau_c$ and for $\tau_d > \bar{\tau}_d$, for some suitable $\bar{\tau}_d > 0$.*

When condition (14) *is violated, instability can occur only through a Neimark-Sacker bifurcation.*

A bifurcation diagram related to the previous Proposition is reported in Figure 7d. As already noted, when $\theta = 0$ the pollution dynamics with exogenous shares converges toward the steady state. Recalling the discussion after Proposition 11, we understand why, for the limit values of τ_d, we have that $\mathbf{s}^*$ is stable. Conversely, when $\tau_d \in (\tau_c, +\infty)$, the unique source of instability is related to the second order effect arising from the interdependence between pollution and share dynamics (Outcome 5). When τ_d is small, we have convergent dynamics, while for intermediate values of τ_d the interaction between pollution and share dynamics can give rise to a Neimark-Sacker bifurcation with consequent quasi-periodic dynamics. The explanation of these two scenarios is basically the same of that provided for the case of $\varepsilon_c = 0$, with the unique difference that, since in the setting of Proposition 12 the clean producers are taxed, the difference between fitness measures of clean and dirty technology is favorable to the latter one only for large values of τ_d, and hence instability occurs for larger values of $\tau_d - \tau_c$.

The main difference between the cases of $\varepsilon_c = 0$ and $\varepsilon_c > 0$ is that, in this latter

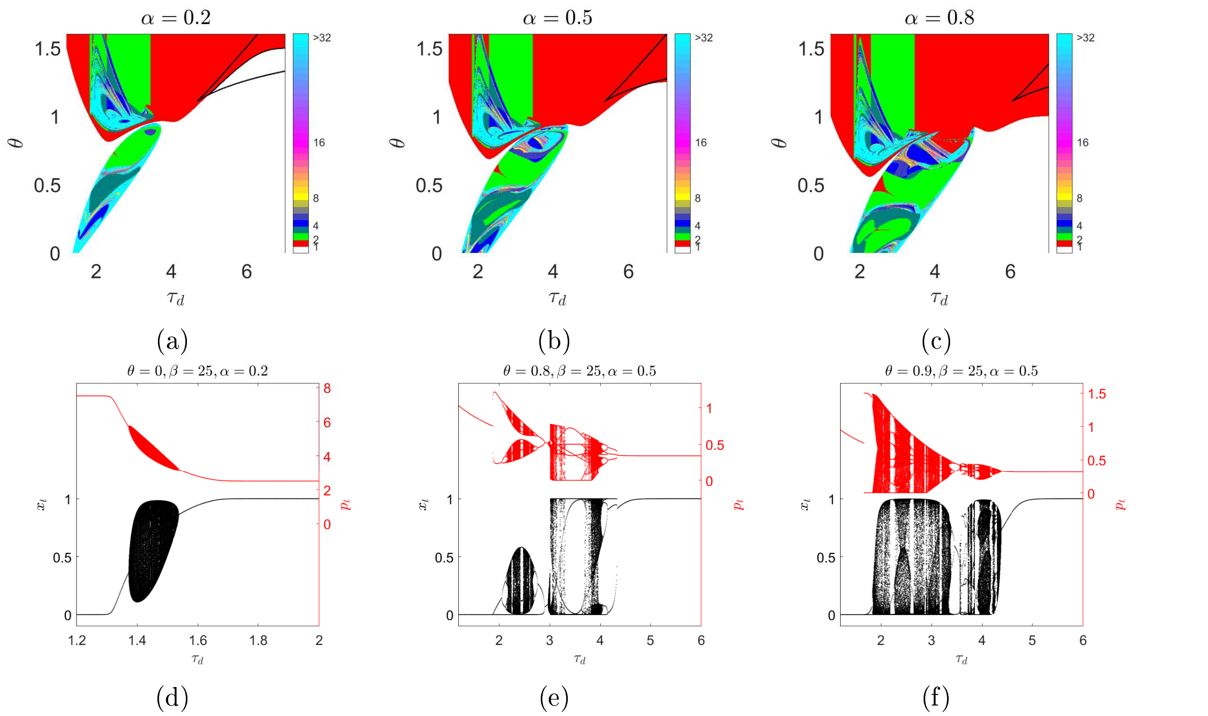


Figure 7: Simulations related to the case of polluting clean technology, obtained setting $\varepsilon_c = 0.5$ and $\tau_c = 1.2$ (Proposition 12 and 13). Top row: two-dimensional bifurcation diagrams for different absorption rates on varying τ_d and θ. Bottom row: bifurcation diagrams on varying τ_d. The share of clean producers and pollution are respectively represented using black (left scale) and red (right scale) color.

case, a complete green transition occurs when $\tau_d = +\infty$, with $x^* = 1$. This means that, as τ_d increases, the share of the clean producers can become as close as we want to 1. This progressively reduces the profitability advantage of the dirty technology, which in turns firstly reduces the extent of oscillatory phenomena and hence leads to their disappearance, stabilizing dynamics. A scenario in which, on increasing a parameter, a stable steady state becomes unstable and finally regains stability is often called *bubble* in the bifurcation diagram. Stability conditions in the general case are reported in the following proposition.

Proposition 13. *A steady state* $\mathbf{s}^* = (x^*, p^*)$*is locally asymptotically stable provided that*

$$\begin{cases} \alpha + \beta x^*(1-x^*)(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c) + \theta(\tau_d(1-x^*) + \tau_c x^*) - \frac{\beta\theta x^*(1-x^*)(\tau_d-\tau_c)^2(\varepsilon_d(1-x^*)+\varepsilon_c x^*)}{\alpha+\tau_d\theta(1-x^*)+\tau_c\theta x^*} > 0 \\ 2 - \alpha - \theta\tau_d(1-x^*) - \theta\tau_c x^* - \frac{\beta(\tau_d-\tau_c)x^*(1-x^*)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)}{\alpha+\tau_d\theta(1-x^*)+\tau_c\theta x^*} > 0 \\ 1 + \frac{\beta(\tau_d-\tau_c)x^*(1-x^*)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)}{\alpha+\tau_d\theta(1-x^*)+\tau_c\theta x^*} > 0 \end{cases} \tag{15}$$

When the second condition in (15) *is violated, instability can just occur through a flip bifurcation, while when the third condition in* (15) *is violated, instability can occur only through a Neimark-Sacker bifurcation.*

In particular, when three steady states exist, $\mathbf{s}_2^*$ *is always unstable.*

We start noting that, when the first condition in (15) is the only one that becomes an equality, we have the emergence/disappearance, through a saddle-node bifurcation, of a couple of new steady states, in line with the static analysis carried on in Section 3. When three steady states coexist, we have that $\mathbf{s}_2^*$ is always unstable, in particular because the first condition in (15) is violated. For comments in Proposition 13 we make reference to the two dimensional bifurcation diagrams reported Figures 7a-7c, in which we stress that the left and lower boundaries have been already studied in the previous Propositions.

If we look at horizontal sections of the lower parts of Figures 7a-7c, also recalling the bifurcation diagram reported in Figure 7d, we can see that as τ_d increases, we have a Neimark-Sacker bubble. We stress that as α increases the loss of stability occurs at larger values of τ_d, in line with what we observed and commented in the case of $\varepsilon_c = 0$. However, the set of values of τ_d for which we have unstable dynamics become larger. Thanks to the larger natural decay, in the presence of a reduced pollution level, endogenous oscillations arising due to the differences in the fitness measures of the two technologies (Outcome 5) start occurring for greater values of τ_d and, similarly, a marginal increase of τ_d induces a reduced change on the difference between the fitness measures, so oscillations last for a longer values of τ_d.

As already explained in the case where $\theta = 0$, a Neimark-Sacker bubble can occur, due to an overreaction phenomenon in the evolutionary selection mechanism, which lessens and disappears as the profitability of the dirty technology decreases. However, we also remarked how, for $\theta > 0$ and τ_d suitably large, the dynamics of pollution can become unstable. Depending on the parameter setting, we can have that, in addition to the Neimark-Sacker bubble, a flip bubble emerge due to the oscillations induced by the dynamics of the pollution, as we can infer from the central horizontal part of the two dimensional bifurcation diagrams reported in Figures 7a-7c, an example of which is reported in Figure 7e. Note that as τ_d increases, the share of clean producers increases as well, and this, recalling (10) and the subsequent comments, has a stabilizing effect on the dynamics of the environmental side, and so differently from the pollution equation with fixed exogenous shares, we observe a return to stability. However, an increase of τ_d can trigger the second order effect on the share dynamics (Outcome 5) that gives rise to a Neimark-Sacker bubble. If θ is further increased, the two bubbles can merge, and we can have the effect that, as τ_d increases, stability is lost through a period doubling and recovered through a Neimark-Sacker bifurcation (see Figure 7f). We summarize the previous discussions as follows.

Outcome 6. *Intermediate taxation levels can be the source of endogenous oscillating or complex dynamical behavior in the trajectories of pollution levels.*

Outcome 7. *Stable steady states can become unstable as efficiency in pollution abatement increases.*

The last set of simulations we report are related to what happens when the rate of

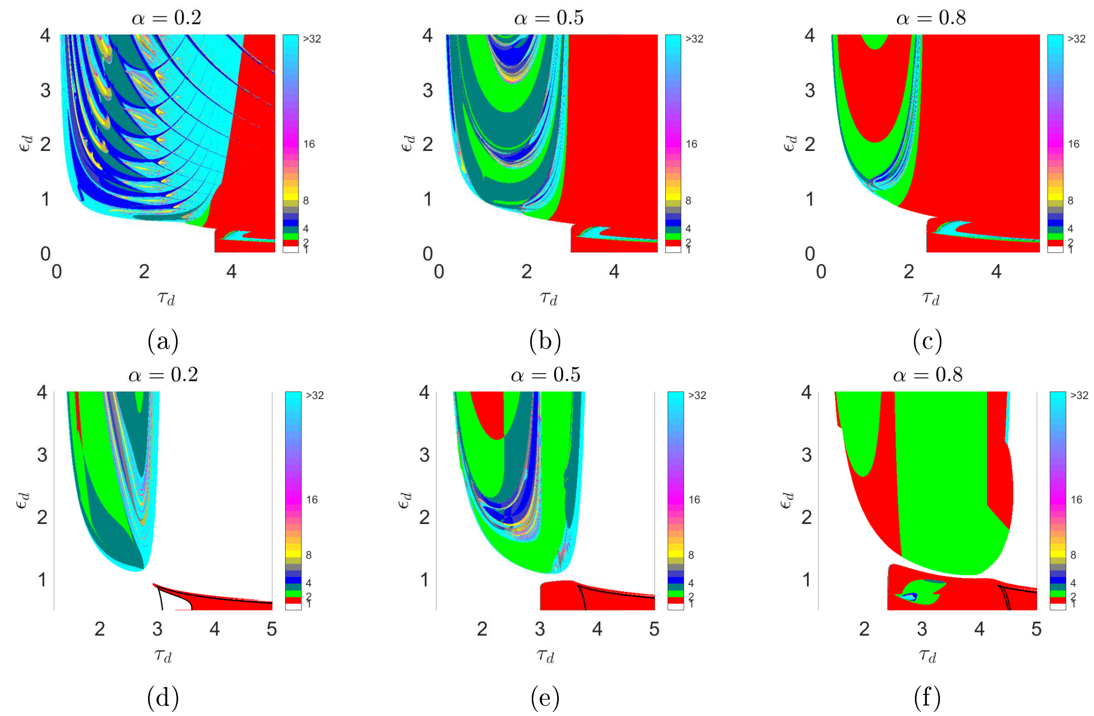
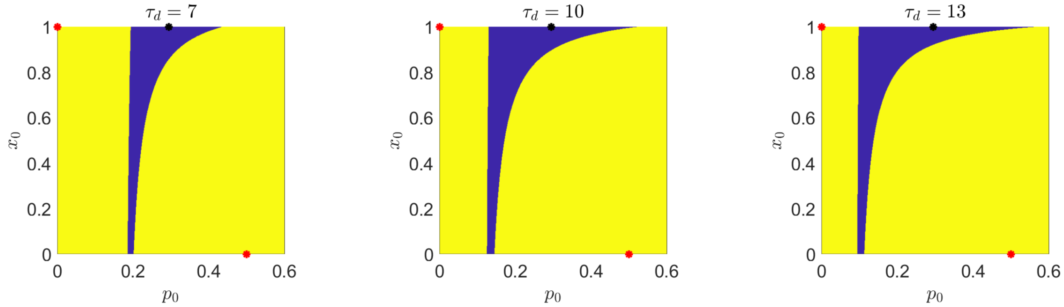


Figure 8: Two-dimensional bifurcation diagrams for different absorption rates on varying ε_d and τ_d. Top row: non polluting clean technology ($\varepsilon_c = \tau_c = 0$). Bottom row: polluting clean technology ($\varepsilon_c = 0.5, \tau_c = 1.2$). In both rows we set $\theta = 0.5$.

emissions of dirty producers change, increasing from the minimum level ε_c. Looking at Figure 8, when dirty producers have low emission rates a reduced taxation τ_d can effectively stabilize dynamics (in this case, increasing τ_d would introduce instability phenomena related to over-taxation, as previously discussed). When ε_d grows, increasing values for τ_d allows for a complete stabilization only if $\varepsilon_c > 0$, while conversely it just lead to a qualitative stabilization. The reported simulations suggest that, from the stability point of view, scenarios with reduced pollution emissions are more advisable than those with large abatement efficiency, since a suitable taxation policy can recover steady state stability.

Before concluding this section, we cast a quick glance on the dynamical effects of multiple steady states coexistence. In the next discussion, we refer to Figure 9. We numerically checked through intensive simulations that when $\mathbf{s}_1$ and $\mathbf{s}_3$ coexist, $\mathbf{s}_1$ is unstable, so we actually observe coexistence between $\mathbf{s}_3$ and a period-2 cycle attractor. As we can see, the basin related to $\mathbf{s}_3$ lies around it, and $\mathbf{s}_3$ is more likely reached if the initial share of clean producers is suitably large, as otherwise convergence toward it realizes only starting from particular initial pollution levels.



(a) (b) (c)

Figure 9: Basins of attractions related to steady state $\mathbf{s}_3$ (black asterisk, blue region) and period-2 cycle (red asterisk, yellow region) arisen from the instability of $\mathbf{s}_1$. We set $\theta = 1$, while the remaining parameters are those used for the simulation reported in Figure 7b.

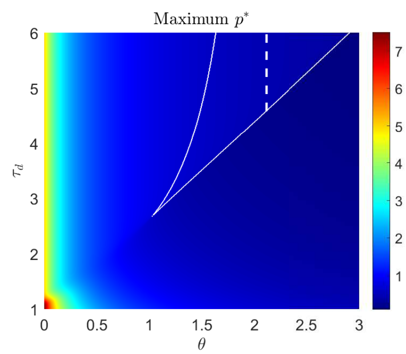


Figure 10: Value of p^* for different taxation levels and abatement efficiency. Inside the white region, multiple steady states occurs and $\max p_i^* = p_3^*$ is displayed.

5 Discussion and insights on policy issues

In the last step of this contribution, we want to carry an explanation of the relevance of the static and dynamical properties of the model, in order to examine the findings in light of environmental policy insights.

The static analysis suggests that policymakers should increase the taxation of the dirty technology wisely. Diversifying the taxation of technologies by placing more burden on the dirty producers has an initial positive effect, both on the promotion of technological change and on improving in the quality of the environment. However, there exist scenarios for which such an action will backfire, with either a despicable increase in pollution levels (Outcome 2) or failures in obtaining transition toward the clean one (Outcome 3). Moreover, green transitioning does not necessarily imply an ameliorated environment if the clean technology has some polluting capability (Outcome 3). Another caveat is that even in the limit case of a perfectly clean technology, the transition might be incomplete. At least, differently from the case above, in this scenario pollution decreases. Likewise, improved abatement technology

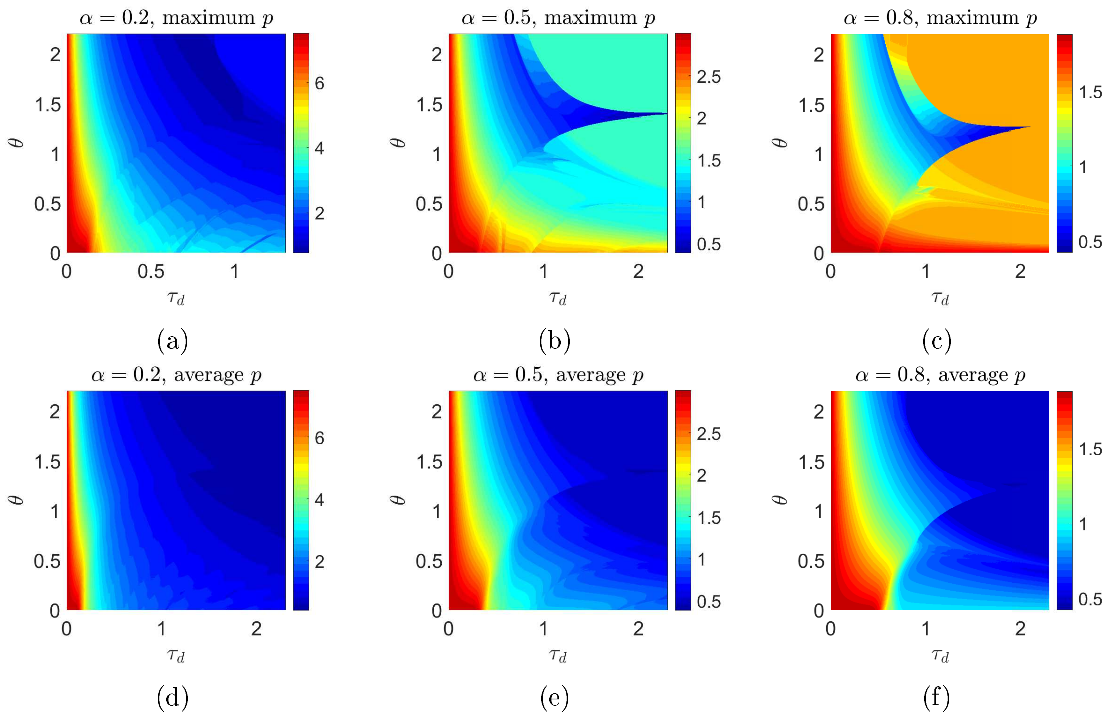


Figure 11: Comparison on plane (τ_d, θ) between maximum (top row) and average (bottom row) pollution levels related to the case of non polluting clean technology. Parameter setting is the same as that of simulations reported in Figure 6.

may give way to a number of unintended consequences. Similarly to what discussed before, the green transition might dampen and a 'lock-in' situation ensues, where multiple steady states give rise to a less than expected reduction in pollution. If we look at Figure 10, we can see that the maximum pollution level increases (we pass from dark blue to light blue) when moving horizontally and entering the region with white boundary, corresponding to the parameter settings providing steady states multiplicity. This means that if the efficiency of abatement is large, increasing the taxation level can actually increase the maximum possible level of pollution. Moreover, recalling Figure 9, it is not simple for the regulator to adjust τ_d in order to promote convergence toward a desirable steady state, as basins of attractions evolve as τ_d changes.

On top of this, the interaction between the two variables in the proposed model might drive pollution to values that are hard to forecast by means of the static analysis only, which turns out to be misleading in elaborating effective policies. In some parameter settings, endogenous oscillations in pollution dynamics could lead p_t significantly above its steady state values. Green transition could end up in similar behaviors, with oscillation in the shares of manufacturers that comply with the clean/dirty technologies.

This occurs when taxes imposed on the dirty ones are moderate (Outcome 6), but dynamical instabilities are also prone to occur when, regardless the level of emission for the

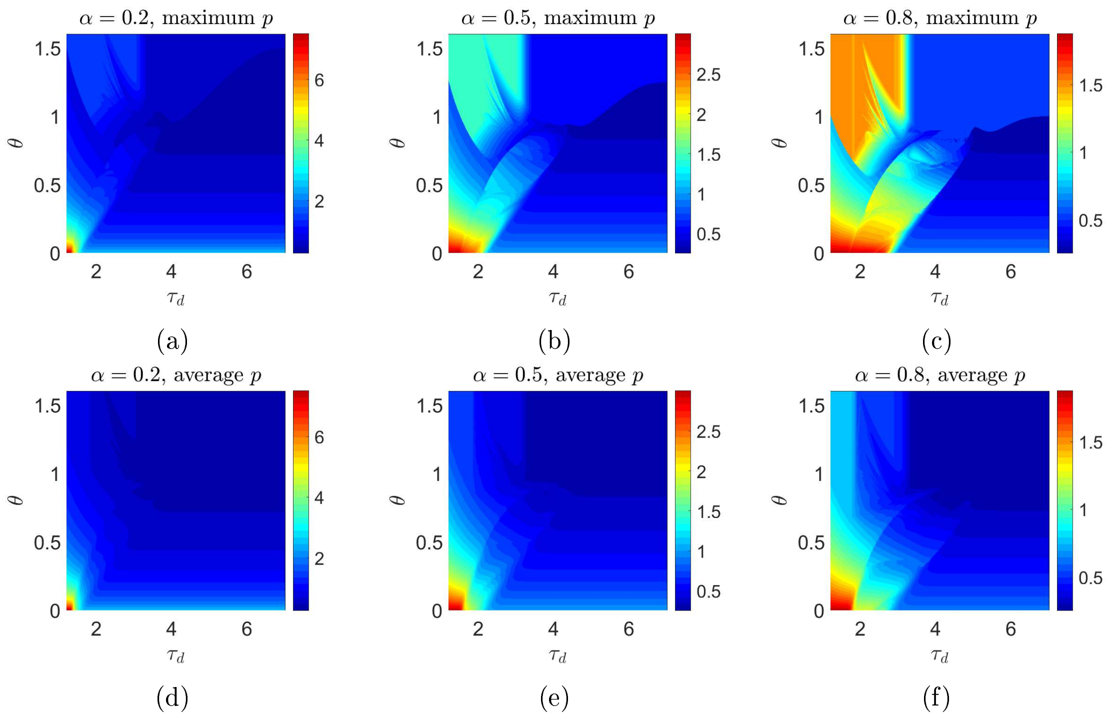


Figure 12: Comparison on plane (τ_d, θ) between maximum (top row) and average (bottom row) pollution levels related to the case of polluting clean technology. Parameter setting is the same as that of simulations reported in Figure 7.

clean technology, effectiveness of abatement is large (Outcome 7). The extent of this can be detected from Figures 11-14, in which we compare the maximum (top rows) and average (bottom rows) level of p^* evaluated considering time series of 1 000 values, discarding an initial transient of 4 000 values. The parameter settings are those used in Figures 6-8. Even when the average pollution is comparable to its static counterpart, its extreme realizations are comparable with those corresponding to much lower taxation levels. In a real context, the consequences of this could have an overall negative impact in the social and economic senses. It is worth pointing out how the effectiveness of various healthcare or economic systems suffers sudden degradation when environmental conditions deteriorate below a certain threshold. High levels of pollution lead to more people becoming ill, with the result that healthcare facilities experience congestion effects when the number of patients exceeds certain levels. There is also a sharp decline in the efficiency of production processes when the pollution exceeds a certain level (as well as if the workforce falls too much due to the consequences on health of a compromised environmental quality). As a consequence of this, it is evident that constantly keeping pollution levels below a certain threshold or keeping it below that threshold only on average are not equivalent scenarios. This requires careful consideration of the dynamical aspects of the problem.

The discussion of the results of the model is certainly not meant to argue that the green

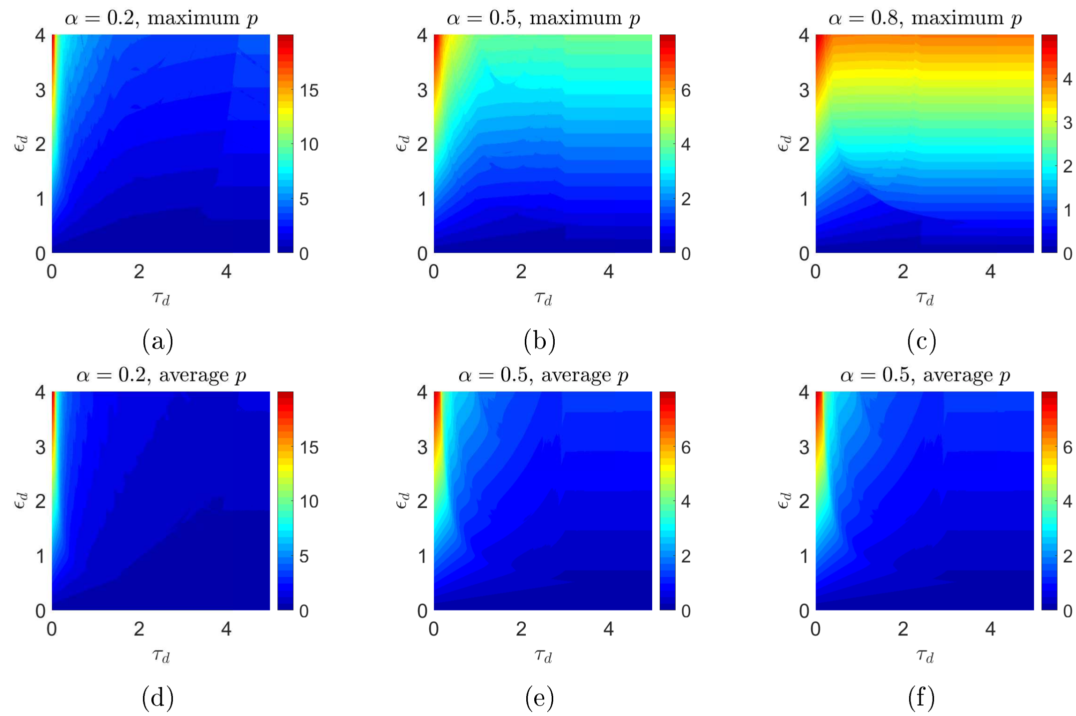


Figure 13: Comparison on plane (τ_d, ε_d) between maximum (top row) and average (bottom row) pollution levels related to the case of non polluting clean technology. Parameter setting is the same as that of simulations reported in the top row of Figure 8.

transition is bad or harmful to the quality of the environment. The results show that environmental taxation can be an effective tool for converting production technologies towards greener ones. Nevertheless, its effectiveness is closely linked to the extent to which these production systems are actually cleaner and to how the resources collected through taxation are invested. In particular, the findings demonstrate how a more efficient green transition can be achieved when the gap between the polluting power of production technologies is smaller. This is what emerges from the dynamical analysis of Section 4 (see Figure 8), but that discussion is supported by the simulations reported in Figures 13 and 14. The aforementioned phenomena occur on the assumption that the collected resources are only used to reduce pollution, namely to reduce its current level. This suggests testing the possibility of reducing these undesirable phenomena by developing policies aimed at improving production systems through R&D and incentives.

6 Conclusions and future perspectives

The model we have proposed is a first attempt to highlight some key points that should be taken into consideration in modelling a green transition process. The pursued approach departs from that presented by Zeppini [23] in several ways. Resources collected through taxes

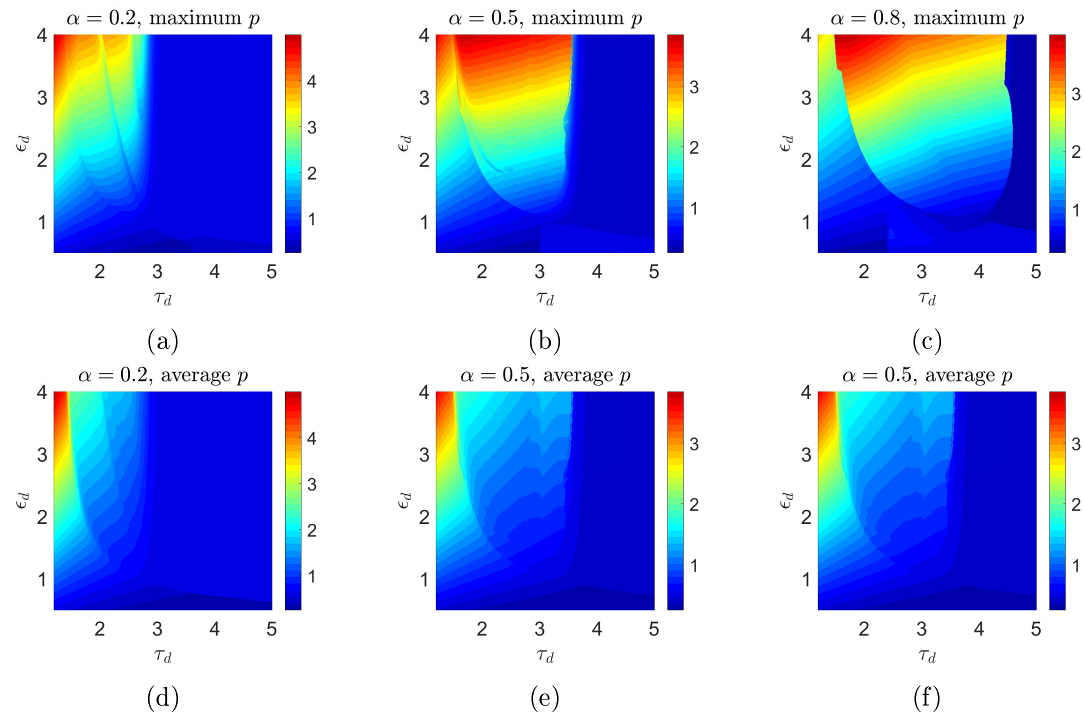


Figure 14: Comparison on plane (τ_d, ε_d) between maximum (top row) and average (bottom row) pollution levels related to the case of polluting clean technology. Parameter setting is the same as that of simulations reported in the bottom row of Figure 8.

play a twin role. Along with a sheer penalty for pollution, the amount collected contributes to mitigate the environmental deterioration. Moreover, we explicitly consider environmental dynamics, which play a central role in the possible migration to clean technology adoption. Furthermore, simple steady state analysis gives interesting results, but it is not sufficient and could be misleading in many cases. From both static and dynamical points of view, the model revealed thorny issues related to a green technological transition. A first theoretical contribution is that pollution reduction based solely on ambient taxes may not be an effective policy. Environmental policy should be careful in increasing taxation for dirty technology when the clean one is less dirty but only to a minor extent, and relying exclusively on excessive taxation for dirty technology may lead to a deterioration in the quality of the environment. Otherwise stated, the focus should be not only to which technology pollutes less, but what is the overall impact on the environment of the two production methodologies. This can occur both in terms of an increase of the steady state pollution stock, through the emergence of a multiplicity of coexisting steady states and by means of endogenous persistent oscillations in the pollutant levels. Furthermore, an improvement in the efficiency of abatement technology may not solve all these problems. Indeed, the simple model we have proposed is only a first step towards a better understanding of how to make an efficient ecological transition. A conclusion this article leads to is that this intertwined and sometimes contradictory effect may be solved by tackling the technology issue. Regardless of their label, clean and dirty producers should be prompted to achieve a transition toward minimum levels of pollution. To investigate the effect of this, it is essential that the economic sphere be incorporated into the model. As in [23], in the present contribution the economic side is represented in terms of exogenous, larger profitability of the most polluting technology. Taking into account the economic dynamics of production would make it possible to include in the model subsidies and R&D investments for a technological progress that reduces emitted pollution. Thanks to this, it would be possible to encompass into the model different ways for the regulator to use the resources collected through taxation. In addition to pollution abatement, resources can be allocated to technological research and subsidies aiming to reduce emissions. The first, sketched out results obtained in the present settings suggest that operating in this way could be effective when combined with abatement. This makes interesting the study of the optimization, possibly through endogenous self-adjustment, of the allocation of resources between the different policy interventions, in order to improve at best the quality of the environment.

Appendix

To prove propositions in Section 3 we introduce the next two two Lemmas, which require two ancillary functions. The first one is function $g : (0, 1) \to \mathbb{R}, x \mapsto g(x)$ defined as

$$g(x) = \beta \left(\lambda_0 - \frac{(\tau_d - \tau_c)(\varepsilon_c x + \varepsilon_d (1 - x))}{\alpha + \tau_d \theta (1 - x) + \tau_c \theta x} \right) - \ln \left(\frac{1}{x} - 1 \right). \tag{16}$$

The second one is function $g : (0, 1) \times [\tau_c, +\infty), (x, \tau_d) \mapsto g(x, \tau_d)$ that has the same analytical expression of (16) but for which we study the behavior of τ_d

$$g(x, \tau_d) = \beta \left(\lambda_0 - \frac{(\tau_d - \tau_c)(\varepsilon_c x + \varepsilon_d (1 - x))}{\alpha + \tau_d \theta (1 - x) + \tau_c \theta x} \right) - \ln \left(\frac{1}{x} - 1 \right). \tag{17}$$

Lemma 1. *Concerning function g defined in (16) we have the following properties:*

1. *$g(x) = 0$ has either a solution or three solutions on $(0, 1)$. Multiple solutions occurs only if*
$$\bar{\gamma} = \alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c \tau_d \theta - \varepsilon_d \tau_c \theta > 0.$$
2. *If $g(x) = 0$ has three solutions, it has a maximum point x_M and a minimum point $x_m \in \left(\frac{\theta}{\theta + \beta \varepsilon_c}, 1 \right)$ for which $x_M \leq x_m$.*

Proof. Proofs for each property are as follows:

1. Since
$$\lim_{x \to 0^+} g(x) = -\infty, \ \lim_{x \to 1^-} g(x) = +\infty, \tag{18}$$

from the Intermediate Values Theorem we have that $g(x) = 0$ always has at least a solution. We have

$$g'(x) = -\frac{\beta\bar{\gamma}(\tau_d - \tau_c)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2} + \frac{1}{x(1-x)} \tag{19}$$

from which we can rewrite $g'(x)$ as an algebraic fraction in which the denominator is strictly positive and the numerator provides the sign of $g'(x)$. In particular, we have $g'(x) = 0$ only if

$$-\beta(\tau_d - \tau_c)\bar{\gamma}x(1-x) + (\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2 = 0, \tag{20}$$

and, since the left hand side in (20) is a second degree polynomial, it can have at most 2 solutions, which means that g has at most 2 stationary points, and so $g(x) = 0$ has at most three solutions. Note that $g(x)$ is strictly increasing if

$$\bar{\gamma} = \alpha(\varepsilon_c - \varepsilon_d) + \theta(\varepsilon_c\tau_d - \varepsilon_d\tau_c) \leq 0 \tag{21}$$

as in that case the left hand side of (19) is the sum of a non negative terms and a strictly positive one, and hence $g'(x)$ is strictly positive. This also guarantees that if (20) has no solutions, we have that g is strictly increasing, while if it has two distinct solutions, since $g'(0^+) = +\infty$ and $g'(1^-) = +\infty$, they must fulfill $0 < x_M < x_m < 1$ and x_M and x_m must be a maximum and a minimum points, respectively. In fact, if (20) had a unique solution $\tilde{x}$ on $(0,1)$, since the left hand side of (20) would change its sign around $\tilde{x}$, we would have that $g(x)$ would be strictly increasing on $(0,\tilde{x})$ and strictly decreasing on $(\tilde{x},1)$, which would contradict $g'(1^-) = +\infty$. This allows concluding.

2. We have already shown the existence of $x_M < x_m$ in the last part of the proof of point 1. We rewrite the left hand side of (20) as

$$\gamma_0 + \gamma_1 x + \gamma_2 x^2 \tag{22}$$

where

$$\gamma_0 = (\alpha + \tau_d\theta)^2,\ \gamma_1 = -(\tau_d - \tau_c)(2\theta(\alpha + \tau_d\theta) + \beta\bar{\gamma}),\ \gamma_2 = (\tau_d - \tau_c)(\theta^2(\tau_d - \tau_c) + \beta\bar{\gamma}).$$

When multiple steady states exist we necessarily have $\bar{\gamma} > 0$, which guarantees that (22) represents a convex parabola, positive and decreasing at $x = 0$. It attains its vertex at

$$x_v = \frac{2\theta(\alpha + \tau_d\theta) + \beta\bar{\gamma}}{2(\theta^2(\tau_d - \tau_c) + \beta\bar{\gamma})}. \tag{23}$$

Note that

$$\frac{\partial x_v}{\partial \tau_d} = -\frac{\theta^2(\alpha+\tau_c\theta)(2\theta+\beta(\varepsilon_c+\varepsilon_d))}{2(\theta^2(\tau_d-\tau_c)+\beta\bar{\gamma})^2} < 0 \tag{24}$$

and

$$\lim_{\tau_d\to+\infty} x_v = 1 - \frac{\beta\varepsilon_c}{2(\theta+\beta\varepsilon_c)} > \frac{\theta}{\theta+\beta\varepsilon_c}. \tag{25}$$

Since $x_v \le x_m$ and $x_v > \frac{\theta}{\theta+\beta\varepsilon_c}$, we obtain the conclusion on x_m.

□

Lemma 2. *Concerning function $g(x,\tau_d)$ defined in (17) we have the following properties:*

1. *For each given $x \in (0,1)$, function $\tau_d \to g(x,\tau_d)$ pointwise decreases toward function*

$$g(x,+\infty) = \beta\left(\lambda_0 - \frac{\varepsilon_c x}{\theta(1-x)} - \frac{\varepsilon_d}{\theta}\right) - \ln\left(\frac{1}{x}-1\right) \tag{26}$$

2. *If there is $\tilde{\tau}_d > 0$ for which $g(x,\tau_d)=0$ has a unique solution $x(\tau_d)$ for all $\tau_d \in (\tilde{\tau}_d,+\infty)$, we have*

$$\lim_{\tau_d\to+\infty} x(\tau_d) = 1$$

3. *If there is $\tilde{\tau}_d > 0$ for which $x \mapsto g(x,\tau_d)$ has multiple solutions $x_1(\tau_d) < x_2(\tau_2) < x_3(\tau_d)$ for $\tau_d \in (\tilde{\tau}_d,+\infty)$, we have*

$$x_1(\tau_d) < \frac{\theta}{\theta+\beta\varepsilon_c},\ \lim_{\tau_d\to+\infty} x_3(\tau_d) = 1$$

Proof. A direct check shows that

$$\frac{\partial g(x,\tau_d)}{\partial \tau_d} = -\frac{\beta(\alpha+\tau_c\theta)(\varepsilon_d(1-x)+\varepsilon_c x)}{(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2} < 0,$$

so as τ_d increases, function g pointwise decreases. Recalling that if (21) holds true we have that, for any fixed τ_d, function $x \to g(x,\tau_d)$ is strictly increasing and noting that (21) is equivalent to

$$\tau_d \le \frac{\alpha\varepsilon_d - \alpha\varepsilon_c + \varepsilon_d\tau_c\theta}{\varepsilon_c\theta}$$

we have that function $x \to g(x,\tau_d)$ is strictly increasing if τ_d is a suitably small fixed value.

Setting $x \in (0,1)$ and computing the limit as $\tau_d \to +\infty$ we easily find that $g(x,\tau_d)$ pointwise converges toward function (26). Note that $g(0^+,+\infty) = -\infty$ and $g(1^-,+\infty) = -\infty$. Moreover, since

$$g'(x,+\infty) = \frac{-(\theta+\beta\varepsilon_c)x+\theta}{\theta x(1-x)^2}$$

we have that $g'(x,+\infty) = 0$ at $x_{M,+\infty} = \theta/(\beta\varepsilon_c+\theta)$, which is the maximum point of $g(x,+\infty)$. From (24) and (25), we have that $x_v > x_{M,+\infty}$. This guarantees that, when

$g(x)$ has two stationary points, $x_{M,+\infty} < x_v < x_m$ holds true. Moreover, we note that $x_m(\tau_d) \to 1$ as $\tau_d \to +\infty$, as otherwise we could find some $x > \tilde{x}$ for which, for suitably large τ_d, $g(x) < g(\tilde{x}, +\infty) < g(x_m)$, which is impossible since x_m is the global minimum point of $g(x)$ on $(x_M, 1)$. Moreover, there is $\tilde{\tau}_d > 0$ such that for $\tau_d \in (\tilde{\tau}_d, +\infty)$ we have $g(x_m(\tau_d), \tau_d) < 0$ and $g(x_m(\tau_d), +\infty) < 0$, so since $x \mapsto g(x, \tau_d)$ is strictly increasing on $(x_m(\tau_d), 1)$, for suitably large values of τ_d we have that there exists $x_3(\tau_d) \in (x_m(\tau_d), 1)$ at which $g(x_3(\tau_d), \tau_d)$, which means $x_3(\tau_d) \to 1$ as $\tau_d \to +\infty$ thanks to the Squeeze Theorem. This holds both if $x_3(\tau_d)$ is the unique solution to $g(x, \tau_d) = 0$ or not, and proves point 2 and the latter statement in point 3.

Moreover, in order to have multiple solutions as $\tau_{x_3(\tau_d)d} \to +\infty$, we must have $g(x_{M,+\infty}, +\infty) \geq 0$, as otherwise g, pointwise approaching $g(x, +\infty)$, must become negative on $(0, x_m(\tau_d))$ for suitably large values of τ_d, and hence for these values the unique solution to $g(x) = 0$ would be that on $(x_m(\tau_d), 1)$. Since $g(x_{M,+\infty}, \tau_d) > g(x_{M,+\infty}, +\infty) \geq 0$ and $g(0^+, \tau_d) = -\infty$, thanks to the Intermediate Values Theorem we have that $x_1(\tau_d) < x_{M,+\infty}$ (point 2 of Lemma 1 guarantees that the solution on $(0, x_{M,+\infty})$ is unique). This concludes the proof. $\square$

Proof of Proposition 1. We set $p_{t+1} = p_t = x$ in (5). A direct check shows that setting $p = 0$ does not provide a steady state, so we can remove $\max\{\}$ function and focus on positive steady values of p. If we solve (5) with respect to p, we find the expression of p^*. Recalling that $\alpha\theta \neq 0$ and $x \in (0, 1)$, we have that p is well-defined and positive. The behavior of p^* with respect to θ and τ_d can be obtained by computing the related derivatives. Finally, noting that

$$\frac{\partial p^*}{\partial x^*} = \frac{\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2} = \frac{-\alpha(\varepsilon_d - \varepsilon_c) + \theta\varepsilon_c\varepsilon_d\left(\frac{\tau_d}{\varepsilon_d} - \frac{\tau_c}{\varepsilon_c}\right)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2}$$

we can immediately conclude $\square$

Proof of Proposition 2. We set $x_{t+1} = x_t = x$ and $p_{t+1} = p_t = p$ in (4). If we solve the resulting latter equation of (4) with respect to p, we indeed find the expression of p^* obtained Proposition 1. Replacing p^* in the former equation of (4) we find the expression of x^* in (8). With simple algebraic manipulations, condition $x = f(x)$ for the existence of a steady state can be rewritten as $g(x) = 0$, where g is the function (16) studied in Lemma 1. Point 1 of Lemma 1 allows concluding about the multiplicity of steady states and provides condition (9), equivalent to $\bar{\gamma} > 0$. $\square$

Proposition 3. Setting $\theta = 0$, from point 1 of Lemma 1 we find $\bar{\gamma} = \alpha\varepsilon_c - \alpha\varepsilon_d < 0$ and hence $g(x) = 0$ always has a unique solution, which provides component x^* of a steady state $\mathbf{s}^*$. A simple geometrical argument show that x^* increases with τ_d. Noting that $p^* = \frac{\varepsilon_c x^* + \varepsilon_d(1-x^*)}{\alpha}$ is decreasing with respect to x^*,we can conclude. $\square$

Proposition 4. Setting $\varepsilon_c = \tau_c = 0$, from point 1 of Lemma 1 we find $\bar{\gamma} = -\alpha\varepsilon_d < 0$ and hence $g(x) = 0$ always has a unique solution, which provides component x^* of a steady state $\mathbf{s}^*$. A simple geometrical argument show that x^* increases with τ_d. Since $\partial p^*/\partial\tau_d < 0$ and, if a unique steady state exists, $\partial p^*/\partial x^* < 0$, we have $dp^*/d\tau_d < 0$ and we can conclude. □

We prove Propositions 5, 6, 7 all at once in the next proof.

Propositions 5, 6, 7. We recall that if $\mathbf{s}^* = (x^*, p^*)$ is a steady state to model (4) if and only if $g(x^*) = 0$ (and, consequently, p^* is given by (8)), where g is function (16) studied in Lemma 1. Let $x_{M,+\infty}$ be the maximum point of function $g(x, +\infty)$ defined in (26) in Lemma 2. We recall from the proof of Lemma 2 that when g has two stationary points, we have that $x_{M,+\infty} < x_v < x_m$ holds true and $x_m \to 1$ as $\tau_d \to +\infty$.

We start considering the case in which

$$g(x_{M,+\infty}, +\infty) = \frac{\beta\lambda_0\theta - \theta - \beta\varepsilon_d - \theta\ln\left(\frac{\beta\varepsilon_c}{\theta}\right)}{\theta} < 0. \tag{27}$$

Note that function $\theta \to \beta\lambda_0\theta - \theta - \beta\varepsilon_d - \theta\ln\left(\frac{\beta\varepsilon_c}{\theta}\right)$ is a function approaching $-\beta\varepsilon_d < 0$ as $\theta \to 0^+$, it is positively diverging as $\theta \to +\infty$ and its derivative is $\beta\lambda_0 - \ln\left(\frac{\beta\varepsilon_c}{\theta}\right)$, and hence it is decreasing on $(0, \beta\varepsilon_c e^{-\beta\lambda_0})$ and increasing on $(\beta\varepsilon_c e^{-\beta\lambda_0}, +\infty)$, so there is $\theta_b > 0$ such that condition (27) is fulfilled on $(0, \theta_b)$.

Let $\theta \in (0, \theta_b)$. We distinguish two situations, depending on whether there exists or not some τ_d for which $g(x) = 0$ has three solutions. To have three solutions to $g(x) = 0$ we necessarily need that (20) has two distinct solutions, and this occurs if its discriminant

$$\begin{aligned}\Delta \;=\;& \beta(\tau_d - \tau_c)\bar{\gamma} \cdot [\beta\varepsilon_c\theta\tau_d^2 - (4\alpha\theta + 4\tau_c\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d + \beta\varepsilon_c\tau_c\theta + \beta\varepsilon_d\tau_c\theta)\tau_d \\ & -4\alpha^2 - 4\alpha\tau_c\theta - \alpha\beta\varepsilon_c\tau_c + \alpha\beta\varepsilon_d\tau_c + \beta\varepsilon_d\tau_c^2\theta]\end{aligned}$$

is strictly positive. We recall that $\tau_d > \tau_c$ and that from point 1 of Lemma 1 we already know that to have three solutions, we can focus on those parameter configurations for which $\bar{\gamma} > 0$, as otherwise we have just one solution. So we study the sign of the factor within square brackets

$$\beta\varepsilon_c\theta\tau_d^2 - (4\alpha\theta + 4\tau_c\theta^2 - \beta(\alpha\varepsilon_c - \alpha\varepsilon_d - \varepsilon_c\tau_c\theta - \varepsilon_d\tau_c\theta))\tau_d - 4\alpha^2 - 4\alpha\tau_c\theta - \beta\tau_c(\alpha\varepsilon_c - \alpha\varepsilon_d - \varepsilon_d\tau_c\theta) \tag{28}$$

It is easy to see that (28) always vanishes for two values of τ_d, and that the smallest is always less than τ_c. The unique value $\bar{\tau}_d > \tau_c$ for which (28) vanishes is

$$\bar{\tau}_d = \frac{4\alpha\theta + 4\tau_c\theta^2 + (\alpha + \tau_c\theta)S - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d + \beta\varepsilon_c\tau_c\theta + \beta\varepsilon_d\tau_c\theta}{2\beta\varepsilon_c\theta},$$

in which we set

$$S = \sqrt{(\beta\varepsilon_d - \beta\varepsilon_c)^2 + 8\beta\varepsilon_c\theta + 8\beta\varepsilon_d\theta + 16\theta^2}. \tag{29}$$

At $\bar{\tau}_d$, we have that (20) have two coincident roots corresponding to

$$x_f = \frac{8\theta^2 + \beta^2\varepsilon_c^2 + (2\theta + \beta\varepsilon_c)S - \beta^2\varepsilon_c\varepsilon_d + 6\beta\varepsilon_c\theta + 2\beta\varepsilon_d\theta}{2(4\theta^2 + \beta^2\varepsilon_c^2 + (\theta + \beta\varepsilon_c)S - \beta^2\varepsilon_c\varepsilon_d + 3\beta\varepsilon_c\theta + \beta\varepsilon_d\theta)}, \tag{30}$$

Note that since $S > (\beta\varepsilon_d - \beta\varepsilon_c)^2$ (we have $\theta > 0$), we have $\beta^2\varepsilon_c^2 + \beta\varepsilon_c S - \beta^2\varepsilon_c\varepsilon_d > 0$, and the denominator of x_f is strictly positive.

We note that $x_f > x_{M,+\infty} = \frac{\theta}{\theta+\beta\varepsilon_c}$. In fact, such an inequality can be rewritten as

$$1 - \frac{\beta^2\varepsilon_c^2 + \beta\varepsilon_c S - \beta^2\varepsilon_c\varepsilon_d}{2(4\theta^2 + \beta^2\varepsilon_c^2 + (\theta + \beta\varepsilon_c)S - \beta^2\varepsilon_c\varepsilon_d + 3\beta\varepsilon_c\theta + \beta\varepsilon_d\theta)} > \frac{\theta}{\theta + \beta\varepsilon_c}$$

which after some algebraic manipulation becomes

$$2(4\theta^2 + \beta^2\varepsilon_c^2 + (\theta + \beta\varepsilon_c)S - \beta^2\varepsilon_c\varepsilon_d + 3\beta\varepsilon_c\theta + \beta\varepsilon_d\theta) > (\theta + \beta\varepsilon_c)(\beta\varepsilon_c + S - \beta\varepsilon_d)$$

and hence

$$(\theta + \beta\varepsilon_c)S - \beta^2\varepsilon_c\varepsilon_d + \beta^2\varepsilon_c^2 + 5\beta\varepsilon_c\theta + 3\beta\varepsilon_d\theta + 8\theta^2 > 0$$

which is true since we have already shown $\beta^2\varepsilon_c^2 + \beta\varepsilon_c S - \beta^2\varepsilon_c\varepsilon_d > 0$.

For $\tau_d \leq \bar{\tau}_d$ we have that g is strictly increasing, since either $\bar{\gamma} \leq 0$ (and from Lemma 1 this guarantees that g is strictly increasing) or $\bar{\gamma} > 0$ and (28) is negative and so $\Delta \leq 0$. So we look for possible multiple solutions to $g(x) = 0$ for $\tau_d > \bar{\tau}_d$, for which we have that g has a maximum and minimum point. If $g(x_f) > 0$, where x_f is defined by (30), since g pointwise decreases with respect to τ_d and $x_m \to 1$ as $\tau_d \to +\infty$, there is $\tau_{d,1}$ for which $g(x_m, \tau_{d,1}) = 0$ and $g(x) = 0$ has three solutions for some $\tau_d \geq \tau_{d,1}$.

If $g(x_f) \leq 0$, since g pointwise decreases with respect to τ_d, for $\tau_d > \bar{\tau}_d$ we have that $g(x, \tau_d) \leq g(x, \bar{\tau}_d)$ on $(0, x_f)$. Since if multiple solutions are present, from Lemma 2 we would need $x_1(\tau_d) < x_{M,+\infty} < x_f$, this is not possible since this would imply $g(x_1(\tau_d), \tau_d) = 0 < g(x_f) < 0$,which is impossible. So there is a unique solution on $(x_m, 1)$ for $\tau_d > \bar{\tau}_d$.

So we can discern between the scenarios in Proposition 5 and 6 depending on the sign of

$g(x_f)$. We have

$$
\begin{aligned}
g(x_f) = &-\frac{1}{\theta(4\theta+S+3\beta\varepsilon_c+\beta\varepsilon_d)} \\
&\cdot [(4\theta^2+\theta S+3\beta\varepsilon_c\theta+\beta\varepsilon_d\theta) \\
&\cdot \ln\left(\frac{\beta\varepsilon_c(S+\beta\varepsilon_c-\beta\varepsilon_d)}{8\theta^2+\beta^2\varepsilon_c^2+2\theta S-\beta^2\varepsilon_c\varepsilon_d+\beta\varepsilon_c S+6\beta\varepsilon_c\theta+2\beta\varepsilon_d\theta}\right) \\
&+8\theta^2\beta^2(\varepsilon_d^2-\varepsilon_c^2)+(2\theta+\beta\varepsilon_d+\beta\varepsilon_c)S \\
&+\beta\lambda_0\theta(-S-4\theta-3\beta\varepsilon_c-\beta\varepsilon_d)+2\beta\varepsilon_c\theta+6\beta\varepsilon_d\theta],
\end{aligned}
$$

for which we have $g(x_f) > 0$ if and only if

$$
\lambda_0 > \max\left\{\frac{S-\beta\varepsilon_c+\beta\varepsilon_d+2\theta\ln\left(\frac{\beta\varepsilon_c(S+\beta\varepsilon_c-\beta\varepsilon_d)}{8\theta^2+\beta^2\varepsilon_c^2+(2\theta+\beta\varepsilon_c)S-\beta^2\varepsilon_c\varepsilon_d+6\beta\varepsilon_c\theta+2\beta\varepsilon_d\theta}\right)}{2\beta\theta}, 0\right\} \tag{31}
$$

The non-null expression in the max of (31) is decreasing with respect to θ. In fact

$$
\begin{aligned}
\frac{S}{2\beta\theta} &= \sqrt{\frac{\beta^2\varepsilon_c^2-2\beta^2\varepsilon_c\varepsilon_d+\beta^2\varepsilon_d^2+8\beta\varepsilon_c\theta+8\beta\varepsilon_d\theta+16\theta^2}{4\beta^2\theta^2}} \\
&= \sqrt{\frac{\varepsilon_c^2-2\varepsilon_c\varepsilon_d+\varepsilon_d^2}{4\theta^2}+\frac{2\varepsilon_c+2\varepsilon_d}{\beta\theta}+\frac{4}{\beta^2}},
\end{aligned}
$$

and it is decreasing with respect to θ since the argument is decreasing with respect to θ. Similarly $(-\beta\varepsilon_c+\beta\varepsilon_d)/2\beta\theta$ is strictly decreasing with respect to θ. Finally, the last term is strictly decreasing with respect to θ if end only if the argument of ln() is strictly decreasing with respect to θ. The derivative of the argument of ln() with respect to θ is

$$
\begin{aligned}
&\frac{\beta\varepsilon_c}{(\beta^2\varepsilon_c^2-\varepsilon_d\beta^2\varepsilon_c+6\beta\varepsilon_c\theta+S\beta\varepsilon_c+2\varepsilon_d\beta\theta+8\theta^2+2S\theta)^2} \\
&\quad\cdot\left[2(S+\beta\varepsilon_c-\beta\varepsilon_d)(8\theta+S+3\beta\varepsilon_c+\beta\varepsilon_d)+\frac{16\theta(2\theta+\beta\varepsilon_c+\beta\varepsilon_d)(\beta\varepsilon_c+\beta\varepsilon_d+4\theta)}{S}\right]
\end{aligned}
$$

in which the sign if determined by the term within the square brackets. We show that it is negative. Rearranging this term and using $S^2 = (\beta\varepsilon_d-\beta\varepsilon_c)^2+8\beta\varepsilon_c\theta+8\beta\varepsilon_d\theta+16\theta^2$ we come to expression

$$
S(-\beta^2\varepsilon_c^2+\varepsilon_d\beta^2\varepsilon_c-4\beta\varepsilon_c\theta-4\theta^2)+2\beta^3\varepsilon_c^2\varepsilon_d-\beta^3\varepsilon_c^3-20\beta\varepsilon_c\theta^2-4\beta\varepsilon_d\theta^2-\beta^3\varepsilon_c\varepsilon_d^2-16\theta^3-8\beta^2\varepsilon_c^2\theta < 0
$$

If $-\beta^2\varepsilon_c^2+\varepsilon_d\beta^2\varepsilon_c-4\beta\varepsilon_c\theta-4\theta^2 \leq 0$ the previous inequality is indeed true. Conversely, if

$-\beta^2\varepsilon_c^2 + \varepsilon_d\beta^2\varepsilon_c - 4\beta\varepsilon_c\theta - 4\theta^2 > 0$ we can write

$$S < \frac{\beta^3\varepsilon_c^3 - 2\beta^3\varepsilon_c^2\varepsilon_d + \beta^3\varepsilon_c\varepsilon_d^2 + 8\beta^2\varepsilon_c^2\theta + 20\beta\varepsilon_c\theta^2 + 4\beta\varepsilon_d\theta^2 + 16\theta^3}{-\beta^2\varepsilon_c^2 + \varepsilon_d\beta^2\varepsilon_c - 4\beta\varepsilon_c\theta - 4\theta^2}$$

from which, taking the square of each member and moving everything to the left and side. we find

$$-\frac{16\beta^2\varepsilon_c\varepsilon_d\theta^2(2\theta + \beta\varepsilon_c + \beta\varepsilon_d)^2}{(\beta^2\varepsilon_c^2 - \varepsilon_d\beta^2\varepsilon_c + 4\beta\varepsilon_c\theta + 4\theta^2)^2} < 0$$

which is indeed negative.

This means that (31) is fulfilled on an interval $(\theta_a, +\infty)$ with $\theta_a > 0$ (since as $\theta \to 0^+$, it is never fulfilled).

To conclude the case of $\theta \in (0, \theta_b)$, we have that for $\theta \in (0, \min\{\theta_a, \theta_b\}]$ $g(x) = 0$ has a unique solution for any τ_d, while for $\theta \in (\theta_a, \theta_b)$ we have that $g(x) = 0$ has a three solutions for some τ_d. Let $\tau_c < \tau_{d,1} < \tau_{d,2} \leq +\infty$ be respectively the smallest and the largest value for which this occurs. Since from point 2 of Lemma 1 g is increasing on $(0, x_M)$ and on $(x_M, 1)$ and decreasing on (x_m, x_M), to have three steady states we must have $g(x_M) > 0$ and $g(x_m) \leq 0$. In particular, we must have $g(x_m) = 0$ for $\tau_d = \tau_{d,1}$. As τ_d increases, $g(x_M)$ must decrease until for $\tau_d = \tau_{d,2}$ we have $g(x_M) = 0$, so $\tau_{d,2} < +\infty$. For $\tau_{d,1} \leq \tau \leq \tau_{d,2}$ we then have three steady states (as $g(0^+) < 0$, $g(x_M) \geq 0$, $g(x_m) \leq 0$ and $g(1^-) > 0$), while for $\tau \geq \tau_{d,2}$ we again have a unique steady state since both $g(x_M)$ and $g(x_m)$ are negative, and the unique steady state belongs to interval $(x_m, 1)$, on which g is strictly increasing.

Let us consider the case in which (27) does not hold true, i.e. $\theta \in [\theta_b, +\infty)$. Let $\tau_{d,1}$ be the smallest value of τ_d for which $g(x_m, \tau_{d,1}) = 0$. This value must exist, as otherwise we would have that $g(x_m) > 0$ for any τ_d, but since $x_m \to 1$ as $\tau_d \to +\infty$, we necessarily have $g(x_m) < 0$ for suitably large values of τ_d, since $g(x_m, \tau_d) \to g(x_m, +\infty) < 0$. Moreover, since pointwise convergence is monotonic, if $\tau_d < \tau_{d,1}$ we have that g is positive on $(x_M, 1)$ (and hence it has a unique solution on $(0, x_M)$, on which it is strictly increasing from Lemma 1), while if $\tau_d > \tau_{d,1}$ we have that g is negative at x_m we have two other solutions to $g(x) = 0$ on (x_M, x_m) and on $(x_m, 1)$. Concluding, if (27) holds true we have that as τ_d increases we pass from a unique steady state to three steady states.

We necessarily must have $\theta_a \leq \theta_b$. If $\theta_b < \theta_a$, at $\theta = \theta_a$ at which $g(x_{M,+\infty}, +\infty) > 0$, and $g(x_f) = 0$. This is not possible because $g(x, \bar{\tau}_d)$ has to be strictly increasing, so $x_f < x_{M,+\infty}$, which is not possible since we proved that $x_f > x_{M,+\infty}$.

Now we prove the results about comparative statics of τ_d. From point 2 of Lemma 1 and point 1 of Lemma 2 we have that simple geometrical considerations allows concluding that x^* and (when exist) x_1^* and x_3^* are increasing with respect to τ_d. Let us now study what happens to the steady state pollution level. We start noting that if multiple steady states exists, from (9) and Proposition 1 we have that the steady state pollution level increases with the share of clear producers. Since $x_1^* < x_3^*$, this guarantees that $p_1^* < p_3^*$. We drop superscript $*$, but all the following computations are meant to be made at a steady state.

Moreover, we have

$$\frac{dp}{d\tau_d} = \frac{\partial p}{\partial \tau_d} + \frac{\partial p}{\partial x} \cdot \frac{dx}{d\tau_d} = \frac{\partial p}{\partial \tau_d} + \frac{\partial p}{\partial x} \cdot \left(-\frac{\frac{\partial (f(x)-x)}{\partial \tau_d}}{\frac{\partial (f(x)-x)}{\partial x}} \right) = \frac{\partial p}{\partial \tau_d} + \frac{\partial p}{\partial x} \cdot \left(-\frac{\frac{\partial f(x)}{\partial \tau_d}}{\frac{\partial (f(x)-x)}{\partial x}} \right) \tag{32}$$

where we use implicit differentiation of the equation $x = f(x)$ that implicitly defines x^* in (8). Note that

$$\frac{\partial (f(x)-x)}{\partial x} = \frac{\beta e^{\beta\left(\lambda_0 - \frac{(\tau_d-\tau_c)(\varepsilon_c x+\varepsilon_d(1-x))}{\alpha+\tau_d\theta(1-x)+\tau_c\theta x}\right)}(\tau_d-\tau_c)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{\left(e^{\beta\left(\lambda_0 - \frac{(\tau_d-\tau_c)(\varepsilon_c x+\varepsilon_d(1-x))}{\alpha+\tau_d\theta(1-x)+\tau_c\theta x}\right)}+1\right)^2(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2} - 1 \tag{33}$$

Moreover, in the next computations, since the derivatives are evaluated at the steady states, we will use identity

$$e^{\beta\left(\lambda_0 - \frac{(\tau_d-\tau_c)(\varepsilon_c x+\varepsilon_d(1-x))}{\alpha+\tau_d\theta(1-x)+\tau_c\theta x}\right)} = \frac{1}{x} - 1 \tag{34}$$

which can be easily obtained by rearranging (8). Using (34) in (33) we obtain

$$\frac{\partial (f(x)-x)}{\partial x} = \frac{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2} - 1. \tag{35}$$

We stress that the and, since we focus on the steady states $\mathbf{s}, \mathbf{s}_1$ and $\mathbf{s}_3$ at which $f(x)$ crosses the bisector line from above, the last expression is negative.

Moreover we have

$$\frac{\partial p}{\partial x} = \frac{\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta}{(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2} \tag{36}$$

and

$$\frac{\partial p}{\partial \tau_d} = -\frac{\theta(\varepsilon_c x+\varepsilon_d(1-x))(1-x)}{(\alpha+\tau_d\theta(1-x)+\tau_c\theta x)^2}$$

which used in (32) with (36) and (35) provides

$$\begin{aligned} \frac{dp}{d\tau_d} \quad = \quad & -\frac{(\varepsilon_c x+\varepsilon_d(1-x))(1-x)}{(\alpha+\tau_d\theta(1-x)+\tau_c\theta x)^2} \cdot \\ & \left[\frac{\beta(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)x(\alpha+\tau_c\theta)}{\left(\frac{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)}{(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2}-1\right)(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2} + \theta \right] \end{aligned}$$

The sign of the last derivative is determined by the opposite of the sign of the term within square brackets, and this can be rewritten as

$$\frac{(\alpha+\tau_d\theta(1-x)+\tau_c\theta x)((\tau_c\theta^2-\tau_d\theta^2-\alpha\beta\varepsilon_c+\alpha\beta\varepsilon_d-\beta\varepsilon_c\tau_d\theta+\beta\varepsilon_d\tau_c\theta)x+\tau_d\theta^2+\alpha\theta)}{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)-(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2}$$

where, recalling (35), we have that the denominator is negative, so the sign of $\frac{dp}{d\tau_d}$ is given by the sign of $-(\tau_c\theta^2 - \tau_d\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d - \beta\varepsilon_c\tau_d\theta + \beta\varepsilon_d\tau_c\theta)x - \tau_d\theta^2 - \alpha\theta$, i.e. by the sign of $-Ax - B$ where $A = \tau_c\theta^2 - \tau_d\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d - \beta\varepsilon_c\tau_d\theta + \beta\varepsilon_d\tau_c\theta$ and $B = \tau_d\theta^2 + \alpha\theta$. Since $B > 0$, we have that $Ax + B > 0$ for $x \in (0,1)$ if $A + B \geq 0$, i.e. if

$$\alpha\theta + \tau_c\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d - \beta\varepsilon_c\tau_d\theta + \beta\varepsilon_d\tau_c\theta \geq 0 \Leftrightarrow \tau_d \leq \frac{\alpha\theta + \tau_c\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d + \beta\varepsilon_d\tau_c\theta}{\beta\varepsilon_c\theta}$$

in which case we have $\frac{dp}{d\tau_d} \leq 0$, while if

$$\tau_d > \frac{\alpha\theta + \tau_c\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d + \beta\varepsilon_d\tau_c\theta}{\beta\varepsilon_c\theta}$$

we have $\frac{dp}{d\tau_d} < 0$ only if

$$x < x_d = \frac{\tau_d\theta^2 + \alpha\theta}{\tau_d\theta^2 - \tau_c\theta^2 + \alpha\beta\varepsilon_c - \alpha\beta\varepsilon_d + \beta\varepsilon_c\tau_d\theta - \beta\varepsilon_d\tau_c\theta} \tag{37}$$

Note that the right hand side is decreasing with respect to τ_d as

$$\frac{\partial x_d}{\partial \tau_d} = -\frac{\theta^2(\theta + \beta\varepsilon_d)(\alpha + \tau_c\theta)}{(\tau_c\theta^2 - \tau_d\theta^2 - \alpha\beta\varepsilon_c + \alpha\beta\varepsilon_d - \beta\varepsilon_c\tau_d\theta + \beta\varepsilon_d\tau_c\theta)^2}$$

Moreover, as $\tau_d \to +\infty$, it approaches

$$\frac{\theta}{\theta + \beta\varepsilon_c}$$

which, for $\varepsilon_c > 0$, is smaller than 1. Recalling point 2 and the latter limit in point 3 of Lemma 2 we have that there is always a steady state approaching 1 as $\tau_d \to +\infty$, so we necessarily have that $\frac{dp^*}{d\tau_d} > 0$ for suitably large values of τ_d. In general, for steady states for which $\frac{dx}{d\tau_d} > 0$, we can either have that if x approaches some $x_{+\infty} > \frac{\theta}{\theta+\beta\varepsilon_c}$, p initially decreases until (37) becomes an equality, then it increases. Conversely, if $x_{+\infty} \leq \frac{\theta}{\theta+\beta\varepsilon_c}$ we have that p^* decreases.

Note that when a couple of new steady state emerges, this occurs at $\tilde{x} > x_v$, where x_v is defined in (23), under the necessary condition $\bar{\gamma} > 0$ of point 1 of Lemma 1. Since

$$x_v - x_d = \frac{\beta\bar{\gamma}}{2((\tau_d - \tau_c)\theta^2 + \beta\bar{\gamma}} > 0$$

we have that when a new steady state appears, we have at it $\frac{dp}{d\tau_d} > 0$. This concludes the proof of Propositions 5, 6 and 7. $\square$

Proposition 8. We drop superscript $*$, but all the following computations are meant to be made at a steady state. We recall that the total derivative of p with respect to a given parameter has the same expression as in (32), in which τ_d has to be replaced by the parameter under consideration.

Comparative statics with respect to θ

Using (34) we have

$$\frac{\partial f(x)}{\partial \theta} = -\frac{\beta x(\tau_d - \tau_c)(1-x)(\varepsilon_d + \varepsilon_c x - \varepsilon_d x)(\tau_d + \tau_c x - \tau_d x)}{(\alpha + \tau_d \theta + \tau_c \theta x - \tau_d \theta x)^2} < 0,$$

so from simple geometrical considerations we have that when a unique steady state exists, we have that x^* decreases with θ, while when multiple steady state exist we have that x_1^* and x_3^* decrease with θ. We have

$$\frac{\partial p}{\partial \theta} = -\frac{(\varepsilon_c x + \varepsilon_d(1-x))(\tau_c x + \tau_d(1-x))}{(\alpha + \tau_d \theta(1-x) + \tau_c \theta x)^2},$$

which used in (32) with (36) and (35) provides

$$\frac{dp}{d\theta} = \frac{(\varepsilon_c x + \varepsilon_d(1-x))(\tau_c x + \tau_d(1-x))}{\beta x(\tau_d - \tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c \tau_d \theta - \varepsilon_d \tau_c \theta) - (\alpha + \tau_d \theta + \tau_c \theta x - \tau_d \theta x)^2} < 0$$

since, recalling (35), the denominator is negative, while the numerator is indeed positive.

Comparative statics with respect to α

Using (34) we have

$$\frac{\partial f(x)}{\partial \alpha} = -\frac{\beta x(\tau_d - \tau_c)(1-x)(\varepsilon_d(1-x) + \varepsilon_c x)}{(\alpha + \tau_d \theta + \tau_c \theta x - \tau_d \theta x)^2} < 0,$$

so from simple geometrical considerations we have that when a unique steady state exists, we have that x^* decreases with α, while when multiple steady state exist we have that x_1^* and x_3^* decrease with α. We have

$$\frac{\partial p}{\partial \alpha} = -\frac{\varepsilon_d(1-x) + \varepsilon_c x}{(\alpha + \tau_d \theta + \tau_c \theta x - \tau_d \theta x)^2},$$

which used in (32) with (36) and (35) provides

$$\frac{dp}{d\alpha} = \frac{\varepsilon_c x + \varepsilon_d(1-x)}{\beta x(\tau_d - \tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c \tau_d \theta - \varepsilon_d \tau_c \theta) - (\alpha + \tau_d \theta + \tau_c \theta x - \tau_d \theta x)^2} < 0$$

since, recalling (35), the denominator is negative, while the numerator is indeed positive.

Comparative statics with respect to ε_d

Using (34) we have

$$\frac{\partial f(x)}{\partial \varepsilon_d} = \frac{\beta x(\tau_d - \tau_c)(1-x)^2}{\alpha + \tau_d \theta(1-x) + \tau_c \theta x} > 0,$$

so from simple geometrical considerations we have that when a unique steady state exists, we have that x^* increases with $\alpha\varepsilon_d$, while when multiple steady state exist we have that x_1^* and x_3^* increase with ε_d. We have

$$\frac{\partial p}{\partial \varepsilon_d} = \frac{1-x}{\alpha + \tau_d\theta(1-x) + \tau_c\theta x},$$

which used in (32) with (36) and (35) provides

$$\frac{dp}{d\varepsilon_d} = \frac{-(1-x)(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)}{\beta x(\tau_d - \tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta) - (\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2} > 0$$

since, recalling (35), the denominator is negative, while the numerator is indeed positive.

Comparative statics with respect to ε_c

Using (34) we have

$$\frac{\partial f(x)}{\partial \varepsilon_c} = \frac{\beta x^2(\tau_d - \tau_c)(1-x)}{\alpha + \tau_d\theta(1-x) + \tau_c\theta x} > 0,$$

so from simple geometrical considerations we have that when a unique steady state exists, we have that x^* increases with ε_c, while when multiple steady state exist we have that x_1^* and x_3^* increase with ε_c. We have

$$\frac{\partial p}{\partial \varepsilon_c} = \frac{x}{\alpha + \tau_d\theta(1-x) + \tau_c\theta x},$$

which used in (32) with (36) and (35) provides

$$\frac{dp}{d\varepsilon_c} = \frac{-x(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)}{\beta x(\tau_d - \tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta) - (\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2} > 0$$

since, recalling (35), the denominator is negative, while the numerator is indeed positive.

Comparative statics with respect to τ_c

Using (34) we have

$$\frac{\partial f(x)}{\partial \tau_c} = -\frac{\beta x(\alpha + \tau_d\theta)(1-x)(\varepsilon_d(1-x) + \varepsilon_c x)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2} < 0,$$

so from simple geometrical considerations we have that when a unique steady state exists, we have that x^* increases as τ_c decreases, while when multiple steady state exist we have that x_1^* and x_3^* increase as τ_c decreases. We have

$$\frac{\partial p}{\partial \tau_c} = -\frac{\theta x(\varepsilon_c x + \varepsilon_d(1-x))}{(\alpha + \tau_d\theta(1-x) + \tau_c\theta x)^2}$$

which used in (32) with (36) and (35) provides

$$\frac{dp}{d\tau_c} = -\frac{x(\varepsilon_c x+\varepsilon_d(1-x))}{(\alpha+\tau_d\theta(1-x)+\tau_c\theta x)^2}\cdot \left[-\frac{\beta(\alpha+\tau_d\theta)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)}{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)-(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2}+\theta\right]$$

The latter factor can be rewritten as

$$-(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x) \cdot\frac{(\tau_c\theta^2-\tau_d\theta^2-\alpha\beta\varepsilon_c+\alpha\beta\varepsilon_d-\beta\varepsilon_c\tau_d\theta+\beta\varepsilon_d\tau_c\theta)x+\alpha\theta+\tau_d\theta^2+\alpha\beta\varepsilon_c-\alpha\beta\varepsilon_d+\beta\varepsilon_c\tau_d\theta-\beta\varepsilon_d\tau_c\theta}{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)-(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2}$$

so after some computation we have

$$\frac{dp}{d\tau_c} = \frac{(\varepsilon_c x+\varepsilon_d(1-x))(1-x)}{(\alpha+\tau_d\theta(1-x)+\tau_c\theta x)}\cdot \cdot\frac{-Ax-B}{\beta x(\tau_d-\tau_c)(1-x)(\alpha\varepsilon_c-\alpha\varepsilon_d+\varepsilon_c\tau_d\theta-\varepsilon_d\tau_c\theta)-(\alpha+\tau_d\theta+\tau_c\theta x-\tau_d\theta x)^2}$$

where we set $A=-(\tau_c\theta^2-\tau_d\theta^2-\alpha\beta\varepsilon_c+\alpha\beta\varepsilon_d-\beta\varepsilon_c\tau_d\theta+\beta\varepsilon_d\tau_c\theta)$ and $B=-(\alpha\theta+\tau_d\theta^2+\alpha\beta\varepsilon_c-\alpha\beta\varepsilon_d+\beta\varepsilon_c\tau_d\theta-\beta\varepsilon_d\tau_c\theta)$ and whose sign, recalling (35), is given by the sign of $Ax+B$ on $(0,1)$.

Since for $x=1$ we have $A+B=-\tau_c\theta^2-\alpha\theta<0$, we have two possibilities.

Case 1) If $B\le 0$, we have $Ax+B<0$ on $(0,1)$

Case 2) If $B>0$ we have $Ax+B>0$ on $(0,\tilde{x})$ and $Ax+B<0$ on $(\tilde{x},1)$, for some $\tilde{x}\in(0,1)$

We note that $B>0$ is equivalent to

$$\tau_c>\tau_{c,B}=\frac{\alpha\theta+\tau_d\theta^2+\alpha\beta\varepsilon_c-\alpha\beta\varepsilon_d+\beta\varepsilon_c\tau_d\theta}{\beta\varepsilon_d\theta}=\tau_d+\frac{(\alpha+\tau_d\theta)(\theta+\beta\varepsilon_c-\beta\varepsilon_d)}{\beta\varepsilon_d\theta}$$

which is possible only for $\theta+\beta\varepsilon_c-\beta\varepsilon_d<0$

We recall that as τ_c increases, x, x_1 and x_3 decrease. In what follows we identify by p and x the components of the steady state under investigation.

Let $\tau_{c,B}\le 0$, so we necessarily are in case 2), and hence p is decreasing.

Let $\tau_{c,B}>0$. On $(0,\tau_{c,B})$ we have $B\le 0$ (case 1), so p is decreasing. On $(\tau_{c,B},\tau_d)$ we have $B>0$ (case 2), so the behavior depends on whether $x=\tilde{x}$ or $x>\tilde{x}$ for $\tau_c=\tau_{c,B}$. Note that it is not possible to immediately have $x<\tilde{x}$, since this would imply a strictly positive derivative for p, which, however, is continuous negative and negative on $(0,\tau_{c,B})$. So if $x=\tilde{x}$, we have that have that p is decreasing also on $(\tau_{c,B},\tau_d)$, in the latter one it can be initially decreasing and then increasing or decreasing. This concludes the proof. □

Since stability conditions in Propositions 10-12 are particular cases of Propositions 13, we start proving this last one.

Proof of Proposition 13. The Jacobian matrix of System (4) is

$$J = \begin{pmatrix} 0 & \frac{\beta e^{\beta(\lambda_0 + p\tau_c - p\tau_d)}(\tau_d - \tau_c)}{(e^{\beta(\lambda_0 + p\tau_c - p\tau_d)}+1)^2} \\ \varepsilon_c - \varepsilon_d - p\tau_c\theta + p\tau_d\theta & \tau_d\theta(x-1) - \alpha - \tau_c\theta x + 1 \end{pmatrix}$$

Recalling steady state condition (34) for x^*, we have that evaluating J at a steady state we find

$$J^* = \begin{pmatrix} 0 & \beta x^*(\tau_d - \tau_c)(1 - x^*) \\ \varepsilon_c - \varepsilon_d - p^*\tau_c\theta + p^*\tau_d\theta & -\tau_d\theta(1 - x^*) - \alpha - \tau_c\theta x^* + 1 \end{pmatrix}$$

(from now on we drop superscript * for x and p). Replacing the steady state expression of p in terms of x we find

$$J^* = \begin{pmatrix} 0 & \beta x(\tau_c - \tau_d)(x-1) \\ \frac{\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x} & \tau_d\theta(x-1) - \alpha - \tau_c\theta x + 1 \end{pmatrix}$$

$$\mathrm{tr}(J^*) = -\tau_d\theta(1-x) - \alpha - \tau_c\theta x + 1$$

and

$$\det(J^*) \quad = \quad -\frac{\beta x(\tau_c - \tau_d)(x-1)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x}$$

We recall that for a two dimensional discrete dynamical system stability is guaranteed provided that

$$\begin{cases} 1 - \mathrm{tr}(J^*) + \det(J^*) > 0 \\ 1 + \mathrm{tr}(J^*) + \det(J^*) > 0 \\ 1 - \det(J^*) > 0 \end{cases}$$

and that when two conditions hold true while the remaining one is violated we can have the occurrence of a flip bifurcation (if it is the second condition that is violated) and of a Neimark-Sacker bifurcation (if it is the third condition that is violated). The first condition is related to a saddle-node bifurcation, corresponding to the emergence/disappearance of a new couple of steady states.

After some computations, we can obtain (15).

Let us introduce $c_1(x(\tau_d), \tau_d)$ defined by the right hand side of the first condition in (15). We have

$$\frac{dc_1}{d\tau_d} = \frac{\partial c_1}{\partial \tau_d} + \frac{\partial c_1}{\partial x} \cdot \frac{\partial x}{\partial \tau_d}$$

We have

$$\frac{\partial c_1}{\partial \tau_d} = \frac{\beta x(x-1)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x} - \theta(x-1)$$
$$- \frac{\beta\varepsilon_c\theta x(\tau_c - \tau_d)(x-1)}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x} - \frac{\beta\theta x(\tau_c - \tau_d)(x-1)^2(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2}$$

and evaluating it at the point at which $c_1(x(\tau_d), \tau_d) = 0$ we find

$$\frac{(\alpha + \tau_c\theta)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - 2\varepsilon_d\tau_c\theta + \varepsilon_d\tau_d\theta - \varepsilon_c\tau_c\theta x + \varepsilon_c\tau_d\theta x + \varepsilon_d\tau_c\theta x - \varepsilon_d\tau_d\theta x)}{(\tau_c - \tau_d)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}$$

Moreover

$$\frac{\partial c_1}{\partial x} = \tau_c\theta - \tau_d\theta - \frac{\beta(\tau_c - \tau_d)(x-1)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x}$$
$$-\frac{\beta x(\tau_c - \tau_d)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x} + \frac{\beta\theta x(\tau_c - \tau_d)^2(x-1)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2}$$

and evaluating it at the point at which $c_1(x(\tau_d), \tau_d) = 0$ we find

$$\frac{2\alpha x - \alpha - \tau_d\theta + \tau_c\theta x + \tau_d\theta x}{x(1-x)}$$

Note that the first condition in (15) can be rewritten as

$$(\alpha - \tau_d\theta x + \tau_d\theta + \tau_c\theta x)\left[\left(-\frac{x(\tau_d - \tau_c)(1-x)(\alpha\varepsilon_c - \alpha\varepsilon_d + \varepsilon_c\tau_d\theta - \varepsilon_d\tau_c\theta)}{(\alpha + \tau_d\theta + \tau_c\theta x - \tau_d\theta x)^2}\right)\beta + 1\right] > 0$$

which, recalling (35), is

$$-(\alpha - \tau_d\theta x + \tau_d\theta + \tau_c\theta x)\frac{\partial(f(x) - x)}{x} > 0,$$

where $f(x)$ is the function defined in by the equation $x = f(x)$ that implicitly defines x^* in (8).

If we consider $\mathbf{s}, \mathbf{s}_1$ and $\mathbf{s}_3$, we have that $f(x)$ crosses the bisector line from above, we have that $\frac{\partial(f(x)-x)}{x} < 0$, and hence the condition is fulfilled. Conversely, at $\mathbf{s}_2$, we have that $f(x)$ crosses the bisector line from below, so $\frac{\partial(f(x)-x)}{x} < 0$, and hence the condition is not fulfilled. □

Proof of Proposition 9. Setting $\alpha = 1$ and $\theta = 0$ in (15) we immediately find (12). □

Proof of Proposition 10. Setting $\varepsilon_c = \tau_c = 0$ in the first condition in (15) we find

$$\alpha + \tau_d\theta(1-x) + \frac{\alpha\beta\varepsilon_d\tau_d x(1-x)}{\alpha + \tau_d\theta(1-x)}$$

which is always fulfilled, while setting $\varepsilon_c = \tau_c = 0$ in the second and third conditions provides (13). If we set $\varepsilon_c = \tau_c = 0$ in the latter equality in (8), we find

$$x^* = \frac{1}{\mathrm{e}^{\beta\left(\lambda_0 - \frac{\varepsilon_d\tau_d(1-x^*)}{\alpha - \tau_d\theta(1-x^*)}\right)} + 1}$$

in which at $x^*=1$ the right hand side becomes $\frac{1}{e^{\beta\lambda_0+1}}$, which means that as $\tau_d \to +\infty$, we can not have $x^* \to 1$ (we neither can have $x^* \to 0$,since $x^*(\tau_d)$ is increasing). This means that since

$$\lim_{\tau_d\to+\infty} 2-\tau_d\theta(1-x)-\alpha+\frac{\alpha\beta\varepsilon_d\tau_d x(1-x)}{\alpha+\tau_d\theta(1-x)} = \lim_{\tau_d\to+\infty} 2-\tau_d\theta(1-x)-\alpha+\frac{\alpha\beta\varepsilon_d x}{\theta} = -\infty$$

the former condition in (13) is violated for suitably large values of τ_d. We note that as $\tau_d \to +\infty$ the latter condition in (13) may hold true or not, since

$$\lim_{\tau_d\to+\infty} 1-\frac{\alpha\beta\varepsilon_d x^*}{\theta}$$

can be positive or negative depending on $\alpha\beta\varepsilon_d x^*/\theta$.

If $\tau_d=\tau_c=0$, both conditions in (13) are indeed fulfilled. □

Proof of Proposition 11. Setting $\tau_d=\tau_c$ we have that the first and the third conditions in (15) reduces to $\alpha+\tau_c\theta>0$ and $1>0$, respectively, and hence are fulfilled. The second condition reduces to $1-\tau_d\theta-\alpha$, so stability is guaranteed for $\tau_c\theta+\alpha<2$.

Let us consider $\tau_d \to +\infty$. We have that $\tau_d(1-x)$ is solution to

$$\tau_d(1-x)=\tau_d\left(1-\frac{1}{e^{\beta\left(\lambda_0-\frac{(\tau_d-\tau_c)(\varepsilon_c x+\varepsilon_d(1-x))}{\alpha+\tau_d\theta(1-x)+\tau_c\theta x}\right)}+1}\right)$$

Let $y=\tau_d(1-x)$, so that the previous equality can be rewritten as

$$1-\frac{y}{\tau_d}=\frac{1}{e^{\beta\left(\lambda_0-\frac{\tau_c\left(\varepsilon_c\left(\frac{y}{\tau_d}-1\right)-\frac{\varepsilon_d y}{\tau_d}\right)}{\alpha+\theta y-\tau_c\theta\left(\frac{y}{\tau_d}-1\right)}+\frac{\tau_d\left(\varepsilon_c\left(\frac{y}{\tau_d}-1\right)-\frac{\varepsilon_d y}{\tau_d}\right)}{\alpha+\theta y-\tau_c\theta\left(\frac{y}{\tau_d}-1\right)}\right)}+1}=f(y) \tag{38}$$

Note that since the change of variable is linear, there are as many solutions to the last equation as steady states x. Moreover, if at a steady state component x does not approach 1, we have that the corresponding $y=\tau_d(1-x)$ diverges for $\tau_d\to+\infty$, so just if $x\to1$ is the unique solution to $1-\frac{y}{\tau_d}=f(y)$ that may be bounded as $\tau_d\to+\infty$. This also guarantees that if x does not approach 1, the second condition in (15) is violated for suitably large values of τ_d. In fact, we can rewrite it as

$$-\frac{\theta(1-x)(\theta(1-x)+\beta\varepsilon_c x)\tau_d^2}{\theta(1-x)\tau_d+\alpha+\tau_c\theta x}+\xi_1\tau_d+\xi_0$$

where ξ_0 and ξ_1 depends on all the parameters but τ_d, we have that the first addend is dominating as $\tau_d\to+\infty$ and hence the left hand side in the second condition in (15) eventually becomes definitively negative.

Now we consider the case of $x \to 1$. We have $f(0) < 1$ and

$$f\left(\frac{1}{\tau_d}\right) = \frac{1}{e^{\frac{\beta\left(\varepsilon_c\tau_d - \varepsilon_c\tau_c + \varepsilon_d\tau_c - \varepsilon_d\tau_d - \varepsilon_c\tau_d^3 + \alpha\lambda_0\tau_d^2 + \varepsilon_c\tau_c\tau_d^2 - \lambda_0\tau_c\theta + \lambda_0\tau_d\theta + \lambda_0\tau_c\tau_d^2\theta\right)}{\tau_d\theta - \tau_c\theta + \alpha\tau_d^2 + \tau_c\tau_d^2\theta}} + 1}$$

which, as $\tau_d \to +\infty$, can be asymptotically approximated by

$$f\left(\frac{1}{\tau_d}\right) \sim \frac{1}{e^{\beta\left(\frac{-\varepsilon_c\tau_d}{\alpha+\tau_c\theta}\right)} + 1}$$

which means that $f(1/\tau_d)$ pointwise converges toward $y = 1$ with an exponential speed, differently from $1 - \frac{y}{\tau_d}$ which has a polynomial speed in approaching $y = 1$. This means that there exists $\tilde{\tau}_d$ such that for $\tau_d > \tilde{\tau}_d$ we obtain $f(1/\tau_d) > 1 - 1/\tau_d$. Hence, for $\tau_d > \tilde{\tau}_d$ there is a solution to (38) on $(0, 1/\tau_d)$, which proves that the unique bounded solution to $1 - \frac{y}{\tau_d} = f(y)$ approaches $y = 0$. Repeating the proof for $y = \tau_d^2(1-x)$ we obtain the same conclusion.

This means that as $\tau_d \to +\infty$, since $x \to 1$, $\tau_d(1-x)$ and $\tau_d^2(1-x)$ vanish, we have that the first and the last conditions in (15) reduce to $\alpha + \tau_c\theta > 0$ and $1 > 0$, and hence hold true, while the second one reduces to $2 - \tau_c\theta - \alpha > 0$, which allows concluding. $\square$

Proof of Proposition 12. Setting $\theta = 0$, this conditions simplify into

$$\begin{cases} \alpha + \beta x(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c)(1-x) > 0 \\ 2 - \alpha + \beta x(1-x)(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c) > 0 \\ 1 - \beta x(1-x)(\varepsilon_d - \varepsilon_c)(\tau_d - \tau_c) > 0 \end{cases}$$

in which the first and the second conditions are always fulfilled. $\square$

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