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Strategy-proof preference aggregation and the anonymity-neutrality tradeoff*

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Abstract

Consider a setting in which individual strict preferences need to be aggregated into a social strict preference relation. For two alternatives and an odd number of agents, it follows from May's Theorem that the majority aggregation rule is the only one satisfying anonymity, neutrality and strategy-proofness (SP). For more than two alternatives, anonymity and neutrality are incompatible for many instances and we explore this tradeoff for strategy-proof rules. The notion of SP that we employ is Kemeny-SP (K-SP), which is based on the Kemeny distance between social orderings and strengthens previously used concepts in an intuitive manner. Dropping anonymity and keeping neutrality, we identify and analyze the first known nontrivial family of K-SP rules, namely semi-dictator rules. For two agents, semi-dictator rules are characterized by local unanimity, neutrality and K-SP. For an arbitrary number of agents, we generalize semi-dictator rules to allow for committees and show that they retain their desirable properties. Dropping neutrality and keeping anonymity, we establish possibility results for three alternatives. We provide a computer-aided solution to the existence of a locally unanimous, anonymous and K-SP rule for two agents and four alternatives. Finally, we show that there is no K-SP and anonymous rule which always chooses one of the agents' preferences.

Keywords: Preference aggregation, strategy-proofness, anonymity, neutrality, Kemeny

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distance, semi-dictator rule

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1 Introduction

We study the problem of aggregating individual preferences into one preference representing society. Applications range from elections where voters with individual preferences choose a representative of the society, to instances in which a group of agents, such as committees, governing bodies or consortia, have to choose a group preference, to macroeconomic models that analyze the choices of a representative agent of the society.[1] In all of the above situations the societal preference must posess the same properties as individual preferences and agents would like it to be "as similar as possible" to their own preference.

A more specific application of this framework can be found in the academic job market, where departments often need to fill an open position without knowing whether any given applicant will accept their offer. One solution to this problem is to specify a choice from any conceivable set of available candidates. However, this idea is not practical when the number of candidates exceeds even a modest threshold. An alternative, and preferable, approach is to determine an ordering of the candidates and choose the best among those who are available.

In our paper, we address this problem and study methods of selecting a collective (or social) ordering of the alternatives when individual preferences over them are themselves orderings. We refer to such procedures as *aggregation rules* or simply *rules*. Formally, this is the classical framework of Arrovian preference aggregation, pioneered in Arrow [3].

There are three canonical properties that aggregation rules aspire to satisfy: fairness, efficiency, and incentive-compatibility or, as is more commonly referred to in this literature, *strategy-proofness*. The fairness notions of anonymity and neutrality are intuitively appealing and enjoy widespread applications. A rule satisfies *anonymity* if it is invariant to a reshuffling of agents' identities, and thus treats agents equally when determining the collective outcome. By contrast, a rule satisfies *neutrality* if it does not systematically favor one alternative over another, which is a salient concern when the alternatives represent morally relevant entities.

When there are just two alternatives and an odd number of agents, May's Theorem [24] establishes that majority rule is the only rule satisfying anonymity, neutrality and strategy-proofness.[2] However, when there are more than two alternatives we show that, for many instances, there exists no rule that can simultaneously satisfy both anonymity and neutrality.

[1] The macroeconomics literature generally does not address the question of how to obtain the representative agent's preference from individual preferences.

[2] For a general number of agents, May's characterization encompasses for multi-valued rules and substitutes strategy-proofness with an intuitive monotonicity requirement known as positive responsiveness.

This result is closely related to a similar finding by Moulin [26] regarding choice rules (that only choose a winning alternative and not an entire ordering). Evidently, obtaining May-like possibility results for strategy-proofness involves dropping either anonymity or neutrality. This is the path we pursue in the current paper.

The efficiency standard we impose is *local unanimity* (where society's preference shall respect any unanimous preference over two alternatives), an analogue of weak Pareto due to Arrow [3] that has been extensively used in the recent literature on preference aggregation. As far as non-manipulability is concerned, we adopt *Kemeny strategy-proofness* (K-SP) by employing the intuitive and axiomatically-founded notion of Kemeny distance [20, 14]. This significantly strengthens previously used incomplete concepts based on the betweeness relation [12, 18, 4, 5]. Prior to our work there was no known nontrivial rule satisfying K-SP for all problem instances.

When we dispense with anonymity but keep neutrality, we propose and analyze the first known nontrivial (i.e., non-dictatorial and non-constant) family of K-SP rules, *semi-dictator* rules. As their name suggests, these rules grant outsize influence to a single agent but stop short of being full dictatorships. They do so by incorporating voting by committee (Barbera et al. [10]) in a way that meaningfully restricts the power of the semi-dictator without violating K-SP. When there are just two agents, semi-dictator rules are characterized by local unanimity, neutrality and Kemeny strategy-proofness. For an arbitrary number of agents, we show that semi-dictator rules retain their desirable properties.

When we dispense with neutrality but keep anonymity, we establish possibility results for the case of three alternatives. In particular, we show that certain subfamilies of Condorcet-Kemeny [12] and fixed-benchmark [5] rules satisfy anonymity, local unanimity, and K-SP. These results do not carry over when the number of alternatives exceeds three. To explore the case of four alternatives and two agents, we frame the rule-existence problem as an integer program. Consequently, we are able to identify computationally a locally unanimous, anonymous and K-SP rule. This rule is, in some sense, similar to semi-dictator rules in that it grants special status to a unique "losing alternative". This alternative is always placed as low as possible in the social ordering, subject to respecting local unanimity, and the rule treats all other alternatives in a balanced (though non-neutral) fashion. Finally, we show that, for many instances, there is no anonymous and K-SP rule which always selects one of the agents' preferences. This preference selection property is related to the requirement of "peak selection" often used in one-public goods problems (Moulin[25] and others). We suspect that stronger impossibility results involving anonymity, K-SP and various notions of efficiency are likely to hold.

Related Work. Our paper is relevant to two strands of the broader social-theoretic literature. The first regards May's Theorem and its various extensions when the number of alternatives exceeds two, whereas the second deals with the nontrivial issue of how to model strategy-proofness in Arrovian aggregation.

We begin with the relevant literature on May's Theorem in multi-alternative environments. We note that, unlike our paper, all the references we discuss deal with *choice* rules, as opposed to *aggregation* rules. That is, rules which select a winning alternative, not an ordering of alternatives. Goodin and List [17] allowed for multi-valued rules and extended May's result to the setting in which agents cast single-alternative votes among a set of more than two alternatives. In particular, they showed that majority rule is characterized by anonymity, neutrality and positive responsiveness in this richer environment. Dasgupta and Maskin [15] focused on single-valued rules and showed that majority rule uniquely satisfies anonymity, neutrality, Pareto efficiency and an independence property known as the Chernoff condition over the largest possible class of problem instances. Working with domain restrictions in which choice rules are single-valued, Alemante et al. [1] showed that Condorcet, plurality, approval voting, and maximin rules satisfy anonymity, neutrality, and a certain monotonicity property. Conversely, Horan et al. [19] allowed for multi-valued rules and focused on the domain restriction of problems admitting strict Condorcet winners. In this setting, they characterized the rule selecting the Condorcet winner(s) with anonymity, neutrality, positive responsiveness, and an independence property they refer to as Nash independence.

Regarding the literature on strategy-proof preference aggregation, Bossert and Storcken [13] were the first to study incentive-compatibility in the Arrovian setting. Working within the Kemeny framework, they established an impossibility result involving the much stronger property of group K-SP, ontoness, and a relatively esoteric invariance property of extrema independence. More recently, Bossert and Sprumont [12] proposed the notion of *betweeness* strategy-proofness (Btw-SP), according to which misreporting cannot lead to an outcome that is between the one obtained under truthful reporting and the agent's own preferences. This property amounts to requiring that the truthful social ordering not be unambiguously dominated by the one produced under misreporting and is thus a necessary, but weak standard of non-manipulability. Bossert and Sprumont [12] identified a number of rules that satisfy Btw-SP, and axiomatized a few of them on the basis of Btw-SP and other properties. Building on these results, Athanasoglou [4] demonstrated that all rules identified in [12] violate K-SP. Harless [18] and Athanasoglou [5] investigated the interplay of Btw-SP with various solidarity properties.

Since Arrovian aggregation with Kemeny preferences admits a graph-theoretic interpretation, our work shares some parallels with the literature on single-peaked preferences and strategyproof facility location (Moulin [25], Barbera et al. [9], Schummer and Vohra [29], Aziz et al. [6]). In these settings, so-called generalized median (or phantom) voter mechanisms are often characterized with strategyproofness and efficiency criteria. While related to voting-by-committees, these sorts of median voter mechanisms are not well-defined in the Arrovian context and thus not directly relevant.

The paper is organized as follows. Section 2 introduces the model, explores the incompatibility of anonymity and neutrality, states May's Theorem for two alternatives and defines our notions of efficiency and strategy-proofness. Section 3 drops anonymity and explores the possibilities of neutrality and our basic properties of local unanimity and K-SP. It characterizes semi-dictator rules for two agents, and generalizes this class to arbitrary numbers of agents and alternatives while maintaining its properties. Section 4 drops neutrality and explores the possibilities of anonymity and our basic properties. For three alternatives, we define two families of rules which satisfy our basic properties and anonymity. For four alternatives and two agents we provide the computer-aided solution for the existence of a rule satisfying our basic properties and anonymity. Furthermore, for three alternatives and three agents we provide a characterization of median rules with tie-breaking by strengthening local unanimity to preference selection. Section 5 concludes. The Appendix contains all proofs omitted from the main text.

2 Model

Let $A = \{a_1, \ldots, a_m\}$ denote a finite set of $m \geq 2$ alternatives and $N = \{1, ..., n\}$ a finite set of $n \geq 2$ agents. Let $\mathcal{R}$ denote the set of **orderings** of A (i.e. complete, reflexive, antisymmetric, and transitive binary relations). Each agent $i \in N$ has a preference relation $R_i \in \mathcal{R}$ over A. We interchangeably write aR_ib and $(a, b) \in R_i$ to denote that agent i finds alternative a at least as good as alternative b. For each $B \subset A$, let $\mathcal{R}(B)$ denote the set of orderings over B. For all $R \in \mathcal{R}$, the ordering $-R$ is defined such that for all $a, b \in A$ with $a \neq b$, $(a, b) \in R$ if and only if $(b, a) \in -R$.

A **(preference) profile** $R_N = (R_1, ..., R_n)$ is an n-tuple of orderings, representing the preferences of all agents in N. The set of preference profiles is denoted by $\mathcal{R}^N$. Given $R \in \mathcal{R}$ and $B \subset A$, let $R|_B$ denote the restriction of R to B, and $R_N|_B = (R_j|_B)_{j\in N}$. If $a \in B$ and $a\ R_i\ b$ for all $b \in B$, then we say that a is the *most preferred* alternative of R_i in B.

For convenience, we often denote an ordering by listing the alternatives from left to right in increasing rank (where the first ranked alternative is the most preferred one in A). Thus, if we write $R = a_1a_2...a_m$, then alternative a_1 is ranked first, a_2 second, and so on. When the ordering of only the first t positions of R is relevant, we write $a_1...a_t...$ to mean that the ordering of the remaining positions can be arbitrary.

A **rule** is a function $f : \mathcal{R}^N \to \mathcal{R}$. A rule f is *dictatorial* if there exists $i \in N$ such that for all $R_N \in \mathcal{R}^N$ we have $f(R_N) = R_i$. A rule f is *constant* if $f(R_N) = f(R'_N)$ for all $R_N, R'_N \in \mathcal{R}^N$. A **choice rule** is a function $\varphi : \mathcal{R}^N \to A$.

2.1 Anonymity and neutrality

How might rules incorporate a concern for fairness? Two properties, which are intuitively appealing, are anonymity and neutrality.

Anonymity requires that a rule be invariant to reshuffling agent identities. Formally, let $\sigma : N \mapsto N$ denote a permutation of N. For each profile $R_N \in \mathcal{R}^N$, let $R_{\sigma(N)} \equiv (R_{\sigma(1)}, ..., R_{\sigma(n)})$ denote the profile where agents have been relabeled according to σ.

Anonymity. For each $R_N \in \mathcal{R}^N$ and each permutation σ of N, $f(R_N) = f(R_{\sigma(N)})$.

Thus, anonymity ensures that the identity of agents does not affect the outcome of the rule. It excludes dictatorial rules from consideration.

A second fairness property focuses on the identity of alternatives (and not agents) and requires that the rule does not systematically favor one alternative over another. Formally, let $\pi : A \mapsto A$ be a permutation of A. For each $R \in \mathcal{R}$, let $\pi R \in \mathcal{R}$ be the ordering such that for all $a, b \in A$, $\pi(a)\ \pi R\ \pi(b)$ if and only if $a\ R\ b$. For each $R_N \in \mathcal{R}^N$, let $\pi R_N \equiv (\pi R_1, ..., \pi R_n)$.

Neutrality. For each $R_N \in \mathcal{R}^N$ and each permutation π of A,

$$f(\pi R_N) = \pi f(R_N).$$

Neutrality ensures that all alternatives are treated equally. If it holds, the rule cannot discriminate either for or against a particular alternative. It excludes constant rules from consideration.

While anonymity and neutrality are seemingly mild requirements, they are generally incompatible.[3] The existence of an anonymous and neutral rule will depend on the values of n and m, the number of agents and alternatives. Theorem 1 provides a necessary and sufficient condition to this effect. The proofs of all results are relegated to the Appendix.

Theorem 1 *An anonymous and neutral rule exists if and only if all prime factors of n are strictly greater than m.*

[3] Note that no such incompatibility exists when rules are not *strict*, i.e. when their image is expanded from $\mathcal{R}$ to the set of subsets of $\mathcal{R}$ (see, e.g., Young and Levenglick [30]).

Thus, anonymity and neutrality are compatible for a relatively small set of problem instances. The proof of Theorem 1 is a straightforward adaptation of an earlier result by Moulin [26] that applies to choice rules (Theorem 1 of [26]). Nevertheless, it is worth noting (albeit unsurprising) that the incompatibility of neutrality and anonymity is more acute for aggregation rules than it is for choice rules.

2.2 Two alternatives: May's Theorem

Theorem 1 establishes that for the majority of problem instances, it is impossible to satisfy both anonymity and neutrality. When $m = 2$, however, this incompatibility becomes less stark and May's Theorem [24], a classic result in social choice, provides an important possibility result.

To state May's Theorem, we need to formally define strategy-proofness. Given $R_N \in \mathcal{R}^N$ and $R_i' \in \mathcal{R}$, the notation (R_i', R_{-i}) denotes the profile that is identical to R_N except for the preferences of agent i that are equal to R_i'. Minimal strategy-proofness (Min-SP) ensures that it is impossible to obtain an outcome that is exactly identical to one's preferences, unless truthtful preference revelation yields the same result.

Minimal strategy-proofness (Min-SP). There do not exist $R_N \in \mathcal{R}^N, i \in N$, $R_i' \in \mathcal{R}$ such that $f(R_i', R_{-i}) = R_i \neq f(R_N)$.

Min-SP is the absolute weakest possible standard of strategy-proofness in the preference aggregation framework. However, when $m = 2$ there is no difference between Min-SP and stronger notions of non-manipulability.[4] For two alternatives and an odd number of agents, the majority rule chooses for any profile the relation which is possessed by a majority of agents.

Theorem 2 *(May's Theorem [24]). Let $|A| = m = 2$ and $|N| = n$ be odd. Then majority rule is the only rule satisfying by anonymity, neutrality and Min-SP.*

When $m = 2$ and n is even, anonymity and neutrality are incompatible by Theorem 1. The classical version of May's Theorem deals with this issue by allowing for multi-valued rules, in which case the anonymity-neutrality tradeoff vanishes.[5] By contrast, we insist on single-valued rules and we examine how May's Theorem is modified when we drop either anonymity or neutrality. To this end, we define the following two families of rules.

[4]For two alternatives, any choice function selecting for each profile a unique alternative corresponds to the rule choosing for this profile the strict relation where the selected alternative is preferred over the non-selected alternative.

[5]Indeed, May's Theorem is originally formulated as a characterization of majority rule on the full domain with anonymity, neutrality, and a monotonicity property referred to as *positive responsiveness*.

Definition 1 *Let $A = \{a, b\}$. A rule f is a quota-majority rule if there exists an integer $q_a \in \{0, 1, ..., n\}$ such that, for all $R_N \in \mathcal{R}^N$ we have $f(R_N) = ab$ if $|\{i \in N : R_i = ab\}| \geq q_a$ and $f(R_N) = ba$ otherwise.*

Definition 2 *Let $A = \{a, b\}$. A rule f is a collegial-majority rule if there exists a (possibly empty) set $\mathcal{T} \subseteq 2^N$ such that (i) $\cap_{T \in \mathcal{T}} T \neq \emptyset$ if $\mathcal{T} \neq \emptyset$ and (ii) $T \backslash T' \neq \emptyset$ and $T' \backslash T \neq \emptyset$ for all distinct $T, T' \in \mathcal{T}$, and for all $R_N \in \mathcal{R}^N$ we have $f(R_N) = ab$ if $T \subseteq \{i \in N : R_i = ab\}$ for some $T \in \mathcal{T}$ and $f(R_N) = ba$ otherwise.*

Proposition 1 demonstrates how May's Theorem is modified when either neutrality or anonymity are dropped from the list of requirements a rule should satisfy.

Proposition 1 *Let $|A| = m = 2$.*

(i) (Perry and Powers [27]) Quota-majority rules are the only rules satisfying anonymity and Min-SP.

(ii) Collegial-majority rules are the only rules satisfying neutrality and Min-SP.[6]

2.3 Efficiency and strategy-proofness

In the preference-aggregation framework, combining anonymity or neutrality together with Min-SP imposes very weak constraints on acceptable rules. For example, constant rules choosing always the same preference are anonymous and Min-SP, and dictatorial rules are neutral and Min-SP.

Therefore, efficiency requirements need to be imposed. We formulate three efficiency properties, in increasing order of strength. The first is self-explanatory.

Unanimity. For all $R_N \in \mathcal{R}^N$ and all $R \in \mathcal{R}$, if $R_i = R$ for all $i \in N$ then $f(R_N) = R$.

Given orderings $R, R', R'' \in \mathcal{R}$, we say that R' is **between** R and R'' and write $R' \in [R, R'']$, if *all* pairs of alternatives that belong to *both* R'' and R *also* belong to R'.

Definition 3 *Given orderings $R, R', R'' \in \mathcal{R}$, R' is between R and R'' (denoted by $R' \in [R, R'']$) if $R \cap R'' \subseteq R'$.*

It is rational to posit that if $R' \neq R''$ and $R' \in [R_i, R'']$, then agent i with preferences R_i has *unambiguous* preference for R' over R''. For every ordering $R \in \mathcal{R}$, this binary relation on

[6] Here (ii) is a special case of Theorem 3.3 in Austen-Smith and Banks [7].

$\mathcal{R}$ is reflexive, transitive, anti-symmetric but not complete.[7] We refer to it as the **betweeness extension** applied to R.

Efficiency. There do not exist $R_N \in \mathcal{R}^N$ and $R' \in \mathcal{R}$ such that $R' \in [R_i, f(R_N)]$ for all $i \in N$ and $R' \neq f(R_N)$.

Local Unanimity. For all $R_N \in \mathcal{R}^N$, $\bigcap_{i \in N} R_i \subseteq f(R_N)$.

A rule satisfies efficiency if it selects an ordering such that there exists no other which all agents find unambiguously better. By contrast, local unanimity applies to preference profiles in which there is unanimous agreement over individual binary comparisons. When such unanimous consensus is present, local unanimity requires the rule to follow it. First discussed by Arrow [3], local unanimity implies efficiency (Footnote 11 in Harless [18]) but not the other way around. A few previous papers have used the term "strong efficiency" to refer to local unanimity [18, 5].

We now address the vulnerability of a rule to strategic manipulation in a way that strengthens Min-SP. The first concept we introduce draws from the betweeness relation.

Betweeness strategy-proofness (Btw-SP). There do not exist $R_N \in \mathcal{R}^N, i \in N$, $R'_i \in \mathcal{R}$ such that $f(R'_i, R_{-i}) \in [R_i, f(R_N)]$ and $f(R'_i, R_{-i}) \neq f(R_N)$.

A rule is Btw-SP if, by misreporting one's preferences, it is not possible to obtain an outcome that is unambiguously better than the outcome under truthfulness. Though stronger than Min-SP, this property still provides a relatively weak notion of non-manipulability. Beginning with the work of Bossert and Sprumont [12], various rules have been found to satisfy Btw-SP, and it has formed the basis of various characterizations [12, 5, 18].

While Btw-SP is a useful benchmark for strategy-proofness, the incompleteness of the betweeness relation diminishes its impact. Indeed, if two orderings are incomparable, then we cannot say whether a preference misreport is profitable or not. Therefore, we search for a way to capture preferences over orderings that is consistent with betweeness when the latter produces clear results, but that also yields a complete relation. To this end, we follow a two-stage approach. First, we determine a way to measure the distance between two orderings. Second, we employ this concept of distance to propose a way of ranking orderings in $\mathcal{R}$.

To guide the first part of our exercise, we rquire three basic requirements that a distance function on $\mathcal{R}$ should satisfy: (i) metric conditions, (ii) consistency with betweeness[8]; and (iii) invariance to relabeling of the alternatives. Improving on the classic result of Kemeny

[7] It is incomplete because, given any $R \in \mathcal{R}$, there will exist multiple pairs of orderings R', R'' that satisfy $R \cap R'' \nsubseteq R'$ and $R \cap R' \nsubseteq R'$.

[8] In the sense that if $R' \in [R, R'']$, then the distance from R to R'' is equal to the sum of the distance or R to R' and the distance of R' to R''.

and Snell [21], Can and Storcken [14] showed that there is only one function satisfying these three requirements: the Kemeny distance [20], a well-known metric in the space of orderings.[9] The formal definition follows.

Given two orderings $R, R' \in \mathcal{R}$, let $D(R, R') = (R \setminus R') \cup (R' \setminus R)$. The **Kemeny distance** between R and R', denoted by $\delta(R, R')$, is defined as $\delta(R, R') = \frac{|D(R,R')|}{2}$. In words, $\delta(R, R')$ is the number of unordered alternative pairs on whose relative ranking the two orderings disagree. For example, if $R = abc$ and $R' = cab$, then $R \setminus R' = \{(a, c), (b, c)\}$, $R' \setminus R = \{(c, a), (c, b)\}$ and $\delta(R, R') = 2$.

With this definition in mind, every ordering $R \in \mathcal{R}$ induces a complete, reflexive and transitive binary relation on $\mathcal{R}$, the **Kemeny extension** applied to R, whereby orderings are ranked on the basis of their Kemeny distance to R. The smaller this distance, the more preferred is the ordering. This binary relation stipulates that two Kemeny-equidistant orderings from R are indifferent for an agent with preference R.

Given any $R \in \mathcal{R}$, it is easy to verify that for all $R', R'' \in \mathcal{R}$ such that $R' \neq R''$, if $R' \in [R, R'']$, then $\delta(R, R') < \delta(R, R'')$. Thus, the Kemeny extension of a preference preserves the betweeness relation.

We now introduce the incentive-compatibility property based on the Kemeny extension.

Kemeny strategy-proofness (K-SP). There do not exist $R_N \in \mathcal{R}^N, i \in N$, $R'_i \in \mathcal{R}$ such that $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R_N))$.

A rule is K-SP if no misrepresentation yields an outcome which is more preferred according to the Kemeny extension (applied to the deviating agent's preference) than the one obtained under truthfulness.[10] In other words, K-SP ensures that by misreporting, no agent can obtain an outcome that is closer to his true preference according to the Kemeny distance.

Since the Kemeny extension preserves the betweeness extension, K-SP implies Btw-SP. In fact, it strengthens the latter property significantly: none of the known nontrivial Btw- SP rules satisfy it, unless the number of alternatives is restricted to three (see Athanasoglou [4] and Section 4). Along related lines, Bossert and Storcken [13] established an impossibility result involving the much stronger coalitional version of K-SP, ontoness, and a relatively esoteric invariance property to which they refer as extrema independence.

Verifying whether K-SP is satisfied by a given rule can be difficult, as comparing orderings on the basis of their Kemeny distance from a certain benchmark is not easy. Computer simulations are often needed to generate counterexamples, even for problem instances of small size [4]. For this reason, when investigating a rule's K-SP, it would be helpful to restrict

[9] For completeness, we provide their characterization in the Online Appendix.

[10] In what follows, and in a slight abuse of grammar, we use the acronym K-SP to denote both "Kemeny strategy-proofness" as well as "Kemeny strategy-proof".

the set of preference misrepresentations that need to be compared to truthful reporting.

One way of achieving this goal is by showing that small deviations from truthfulness are sufficient to adjudicate the rule's K-SP. Along these lines, a number of recent papers have focused on identifying necessary and sufficient conditions for the equivalence between global and local measures of strategy-proofness [28, 22, 23]. Since the results of those papers do not readily apply to the Arrovian aggregation framework with Kemeny-based preferences, we explore the local-global equivalence directly.

Before proceeding, we specify what we mean by local measures of strategy-proofness in our setting.

Local Kemeny strategyproofness (Local K-SP). There do not exist $R_N \in \mathcal{R}^N, i \in N$, $R'_i \in \mathcal{R}$ such that $\delta(R_i, R'_i) = 1$ and $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R_N))$.

Thus, a rule is Local K-SP if by misreporting the order of a unique adjacent alternative pair, it is not possible to obtain an outcome that is closer in Kemeny distance to one's true preferences. Clearly, K-SP implies Local K-SP. The following result establishes that the opposite holds as well, provided the rule also satisfies Min-SP, the weakest possible measure of global non-manipulability. Note that Local K-SP does not imply Min-SP.[11]

Proposition 2 *If a rule satisfies Local K-SP and Min-SP, then it satisfies K-SP.*

Proposition 2 will be useful in establishing the K-SP of the rules we introduce in the next section.

3 Keep neutrality - Drop anonymity

In this section we explore locally unanimous and K-SP rules that satisfy neutrality but fail anonymity. We are able to establish a full characterization when the number of agents is two.

3.1 Two agents

We focus on the two-agent case and define a family of rules that forms the cornerstone of this section.

Definition 4 *A* **two-agent semi-dictator rule** *is parameterized by the following two inputs:*

[11] We provide an example in the Online Appendix. We conjecture, but so far have been unable to prove, that local K-SP combined with some mild efficiency property implies Min-SP. This would mean that, given mild efficiency requirements, local K-SP is equivalent to K-SP. Such a result would more closely align with the contribution of Kumar et al. [23].

(i) A **semi-dictator** $i \in N \in \{1, 2\}$*.*

(ii) A **position set** $P \subset \{1, 2, ..., m-1\}$ *satisfying for all distinct* $p, p' \in P$, $|p - p'| > 2$.

Let $R_N \in \mathcal{R}^N$*. Without loss of generality, suppose that the semi-dictator* i *has preference* $R_i = a_1 a_2 ... a_m$*. Let* $f_k^{(i,P)}(R_N)$ *the* k*th-ranked alternative in the ordering* $f^{(i,P)}(R_N)$*, where* $k = 1, 2, ..., m$*. The semi-dictator rule is defined as follows (where* $N = \{i, j\}$*):*

$$f_k^{(i,P)}(R_N) = \begin{cases} a_{k-1}, & \text{if } k-1 \in P \text{ and } a_k R_j a_{k-1} \\ a_{k+1}, & \text{if } k \in P \text{ and } a_{k+1} R_j a_k \\ a_k, & \text{otherwise.} \end{cases} \tag{1}$$

for all $k = 1, 2, ..., m$*.*

A two-agent semi-dictator rule $f^{(i,P)}$ produces an ordering that is identical to the preferences of the semi-dictator i except possibly at the alternatives occupying ranks $\{p, p+1\}$ where $p \in P$. In particular, given the semi-dictator's preferences $R_i = a_1 a_2 .... a_m$, for every position $p \in P$, alternatives a_p and a_{p+1} will be assigned rank either p or $p+1$, in accordance with agent j's preferences.

Figure 1 illustrates a two-agent semi-dictator rule when $m = 14$, semi-dictator 1, $P = \{4, 7, 13\}$ and $R_1 = a_1 a_2 .... a_{14}$.

It is important that any two distinct positions in the set P have distance greater than two as otherwise the two-agent semi-dictator rule might be manipulated by the semi-dictator. We illustrate this below for five alternatives where either $P = \{1, 3\}$ or $P = \{1, 4\}$.

Example 1 Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $N = \{1, 2\}$.

On the one hand, if $i = 1$ and $P = \{1, 3\}$, then the semi-dictator rule is not K-SP: let $R_1 = a_1 a_2 a_3 a_4 a_5$ and $R_2 = a_2 a_1 a_4 a_3 a_5$; then $f^{(1,P)}(R_1, R_2) = a_2 a_1 a_4 a_3 a_5$ and $\delta(R_1, f^{(1,P)}(R_1, R_2)) = 2$; for $R'_1 = a_1 a_3 a_2 a_4 a_5$ we have $f^{(1,P)}(R'_1, R_2) = R'_1$ and $\delta(R_1, R'_1) = 1$, and agent 1 profitably manipulates from (R_1, R_2) via R'_1. However, agent 2 cannot profitably manipulate $f^{(1,P)}$,[12] it is sufficient for the agent, who is not the semi-dictator, that any two distinct positions in P differ by at least two.

On the other hand, if $i = 1$ and $P = \{1, 4\}$, then the semi-dictator rule is K-SP. As above it follows that agent 2 cannot profitably manipulate. For agent 1, let $R_1 = a_1 a_2 a_3 a_4 a_5$ and R_2

[12] This is true in general and can be easily seen with the help of Proposition 2. First, agent 2 cannot profit from exchanging the positions of adjacent alternatives in his true ranking: if R'_2 is such that $\delta(R_2, R'_2) = 1$, then either $f^{(1,P)}(R_1, R_2) = f^{(1,P)}(R_1, R'_2)$ or $\delta(f^{(1,P)}(R_1, R_2), f^{(1,P)}(R_1, R'_2)) = 1$ and they differ exactly for the two alternatives where R_2 and R'_2 differ, which is not profitable for agent 2. Second, agent 2 cannot manipulate such that his true ranking is chosen: if R'_2 is such that $f^{(1,P)}(R_1, R'_2) = R_2$, then for any $p \in P$ (where $R_1 = a_1 \cdots a_m$), we have $R'_2|_{\{a_p, a_{p+1}\}} = R_2|_{\{a_p, a_{p+1}\}}$ which then implies $f^{(1,P)}(R_1, R'_2) = f^{(1,P)}(R_1, R_2)$, a contradiction.

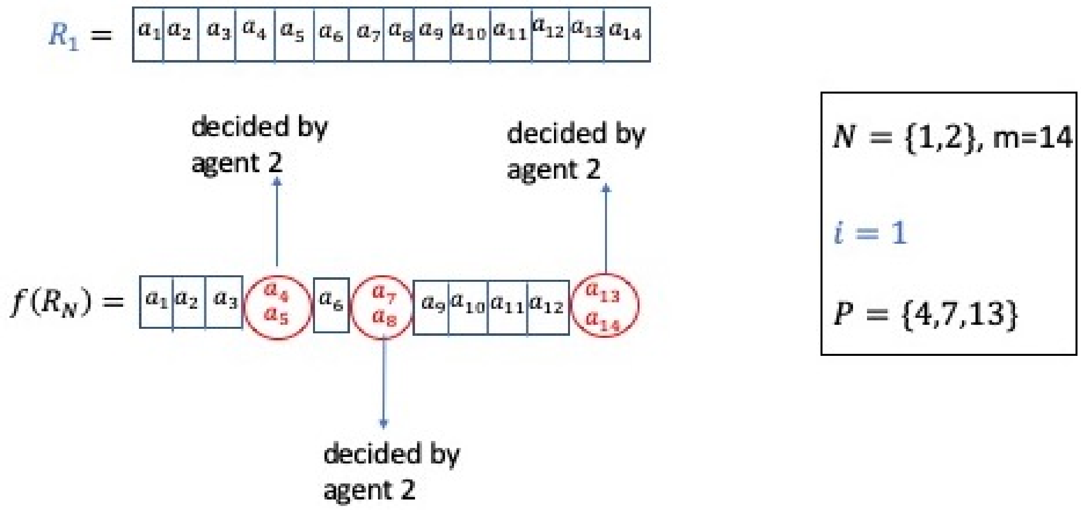

Figure 1: An illustration of a two-agent semi-dictator rule $f = f^{(i,P)}$ with $i = 1,\ P = \{4, 7, 13\}$ and $R_1 = a_1a_2....a_{14}$. The rule applied to profile R_N produces a social ordering that is identical to R_i, except possibly at ranks (4,5), (7,8) and (13,14) where the relative order of adjacent alternatives is determined by agent 2's preferences R_2. For example, if agent 2 prefers a_5 to a_4, a_7 to a_8 and a_{14} to a_{13}, then $f(R_N) = a_1a_2a_3[a_5a_4]a_6[a_7a_8]a_9a_{10}a_{11}a_{12}[a_{14}a_{13}]$, where the brackets are added for emphasis.

be arbitrary. Then by definition, $\delta(R_1, f^{(1,P)}(R_1, R_2)) \leq 2$. If $\delta(R_1, f^{(1,P)}(R_1, R_2)) \leq 1$, then it is easy to see that agent 1 cannot manipulate. Thus, let $\delta(R_1, f^{(1,P)}(R_1, R_2)) = 2$. Now agent 1 cannot manipulate by pushing a_1 or a_2 to fourth or fifth position in his deviation R'_1 as then $\delta(R_1, f^{(1,P)}(R'_1, R_2)) \geq 2$. Similarly agent 1 cannot manipulate by pushing a_4 or a_5 to first or second position in his deviation. Now in the deviation R'_1 either the first two ranked alternatives are $\{a_1, a_2\}$ or the last two ranked alternatives are $\{a_4, a_5\}$. Then when a_3 is ranked third we have $f^{(1,P)}(R'_1, R_2) = f^{(1,P)}(R_1, R_2)$, when a_3 is ranked first or second we have $f^{(1,P)}(R'_1, R_2)|_{\{a_4,a_5\}} = f^{(1,P)}(R_1, R_2)|_{\{a_4,a_5\}}$ and $\delta(R_1, f^{(1,P)}(R'_1, R_2)) \geq 2$, and when a_3 is ranked fourth or fifth we have $f^{(1,P)}(R'_1, R_2)|_{\{a_1,a_2\}} = f^{(1,P)}(R_1, R_2)|_{\{a_1,a_2\}}$ and $\delta(R_1, f^{(1,P)}(R'_1, R_2)) \geq 2$. The detailed argument can be found in the proof of Theorem 4.

Our first main contribution is the characterization of two-agent semi-dictator rules with local unanimity, neutrality and K-SP.

Theorem 3 *Let $N = \{1, 2\}$ and $|A| = m \geq 3$. A rule f satisfies local unanimity, neutrality and K-SP if and only if f is a two-agent semi-dictator rule.*

The proof of Theorem 3 proceeds by establishing the characterization for $m = 3$ and

$m = 4$, and then tackles the case $m \geq 4$ by induction. The argument for $m = 4$ is particularly involved as it requires the careful examination of many different sub-cases.

3.2 More Than Two Agents

Our second main contribution is to define the general class of semi-dictator rules and to show that they satisfy the properties of Theorem 3. In doing so, we also identify the first nontrivial K-SP rule for an arbitrary number of agents and alternatives. Unlike the two-agent case, a characterization of semi-dictator rules for general n and m remains elusive.

We begin by defining the concept of a committee, which plays a central role in the analysis. Committees and extensions thereof (e.g., left-right coalition systems) have been studied extensively in a variety of models of social choice (Barbera [8]).

Definition 5 *A **committee** is a non-empty collection $\mathcal{C}$ of subsets of N satisfying the following two conditions:*

(1) for all $C, C' \subset N$, we have $C \in \mathcal{C} \,\&\, C \subset C' \Rightarrow C' \in \mathcal{C}$ and

(2) for all $C \subset N$, we have $C \in \mathcal{C} \Leftrightarrow N \setminus C \notin \mathcal{C}$.

Committees serve the following function in semi-dictator rules, which is reminiscent to the "voting by committee" procedure of Barbera et al. [10]. Given any pair $\{a, b\} \subset A$, a committee $\mathcal{C}$ indicates the set of the so-called "winning coalitions" when deciding the order of alternatives a and b. In particular, if the preference profile in question is such that the set of agents preferring a to b is contained in $\mathcal{C}$, then the semi-dictator rule will rank a above b. This procedure is well-defined since agents have complete preferences over alternatives and the definition of committees ensures that $C \in \mathcal{C}$ iff $N \setminus C \notin \mathcal{C}$.

Clearly, when $m = 2$, the voting by committees procedure is K-SP. The challenge is to design an aggregation rule that integrates voting by committee when $m \geq 3$ in a way that does not lead to violations of transitivity and K-SP. This is exactly what semi-dictator rules accomplish.

Definition 6 *A* **semi-dictator rule** *is parameterized by the following three inputs:*

(i) A **semi-dictator** *$i \in N$.*

(ii) A **position set** *$P \subset \{1, 2, ..., m-1\}$ satisfying for all distinct $p, p' \in P$, $|p - p'| > 2$.*

(iii) For each position $p \in P$ a **committee** *$\mathcal{C}_p$ on $N \setminus \{i\}$. Let $\mathcal{C} \equiv \mathcal{C}_p|_{p \in P}$ denote the corresponding family of committees.*

Let $R_N \in \mathcal{R}^N$. Without loss of generality, suppose that the semi-dictator i has preferences $R_i = a_1 a_2 ... a_m$. Let $f_k^{(i,P,\mathcal{C})}(R_N)$ denote the kth-ranked alternative in the ordering $f^{(i,P,\mathcal{C})}(R_N)$, where $k = 1, 2, ..., m$. The semi-dictator rule is defined as follows:

$$f_k^{(i,P,\mathcal{C})}(R_N) = \begin{cases} a_{k-1}, & \text{if } k-1 \in P \text{ and } \{j \in N \setminus \{i\} : \ a_k \ R_j \ a_{k-1}\} \in \mathcal{C}_{k-1} \\ a_{k+1}, & \text{if } k \in P \text{ and } \{j \in N \setminus \{i\} : \ a_{k+1} \ R_j \ a_k\} \in \mathcal{C}_k \\ a_k, & \text{otherwise.} \end{cases} \tag{2}$$

for all $k = 1, 2, ..., m$.

A semi-dictator rule $f^{(i,P,\mathcal{C})}$ when applied to a profile R_N produces an ordering that is identical to the preferences of the semi-dictator i except possibly at the alternatives occupying ranks $\{p, p+1\}$ where $p \in P$. Given the semi-dictator's preferences $R_i = a_1 a_2 .... a_m$, for every position $p \in P$, alternatives a_p and a_{p+1} will be assigned rank either p or $p+1$. If the set of agents preferring a_p to a_{p+1} in profile R_N (i.e., the set $\{j \in N \setminus \{i\} : \ a_p \ R_j \ a_{p+1}\}$) belongs to the committee $\mathcal{C}_p$, then alternative a_p is assigned rank p and a_{p+1} rank $p+1$, consistent to the semi-dictator's preferences; if not, a_{p+1} is assigned rank p and a_p rank $p+1$, in contrast to the semi-dictator's preference. This procedure is well-defined because there is always a gap between pairs of adjacent alternatives whose order is decided by committee –this is guaranteed by the requirement that if $p, p' \in P$ such that $p \neq p'$, then $|p - p'| > 2$. Furthermore, analogous to Example 1 it can be seen that the gap has to be greater than two as otherwise the semi-dictator might be able to manipulate the rule.

Figure 2 illustrates a semi-dictator rule when $m = 14$, semi-dictator 1 with $R_1 = a_1 a_2 .... a_{14}$ and $P = \{4, 7, 13\}$. The structure of the committees $\mathcal{C}_4, \mathcal{C}_7, \mathcal{C}_{13}$ is specified in the figure and its caption.

While semi-dictator rules are neutral, they are obviously not anonymous. A way of improving their fairness from the point of view of the agents is by maximizing the number of alternative pairs to be decided by committees. Along these lines, the number of alternatives pairs whose relative order is decided by committee can range from 0 (when the semi-dictator is in fact a dictator) to $\lfloor \frac{m+1}{3} \rfloor$. If we wanted to constrain the semi-dictator's power as much as possible ex-ante, we would choose a semi-dictator rule with $|P| = \lfloor \frac{m+1}{3} \rfloor$.

We now turn to the efficiency and incentive properties of semi-dictator rules. The main result we establish is the K-SP of all semi-dictator rules. As mentioned earlier, Proposition 2 allows us to simplify the proof by focusing only on adjacent deviations from truthful reporting.

Theorem 4 *Semi-dictator rules satisfy local unanimity, neutrality and K-SP.*

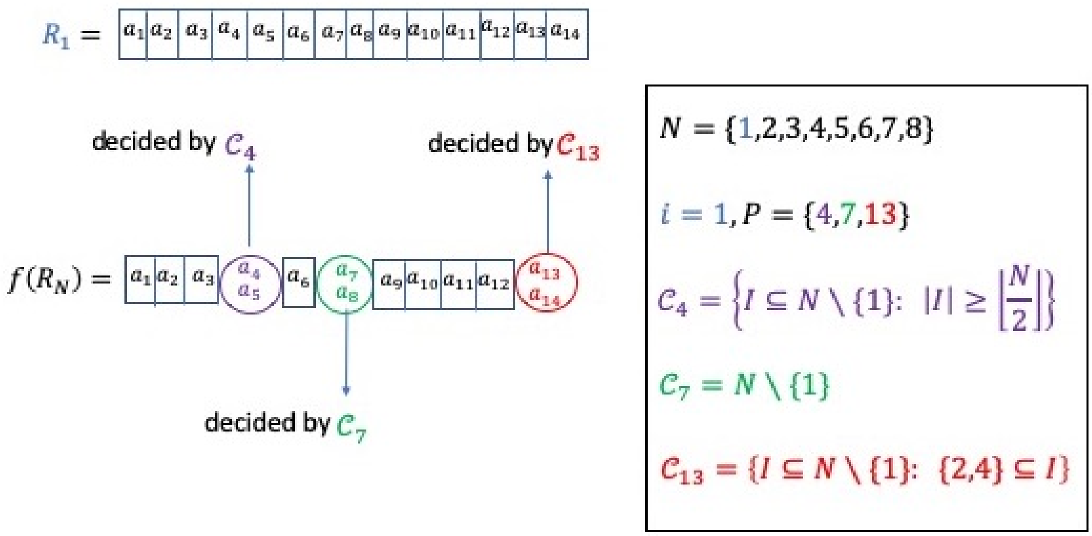

Figure 2: An illustration of a semi-dictator rule $f = f^{(i,P,\mathcal{C})}$ with $i = 1$, $P = \{4, 7, 13\}$ and $R_1 = a_1a_2....a_{14}$. The rule applied to profile R_N produces a social ordering that is identical to R_1, except possibly at ranks (4,5), (7,8) and (13,14) where the relative order of adjacent alternatives: (i) $\{a_4, a_5\}$ is determined by majority rule with ties broken against the semi-dictator; (ii)$\{a_7, a_8\}$ follows the semi-dictator's wishes unless all other agents rank a_8 before a_7; and (iii) $\{a_{13}, a_{14}\}$ goes against the semi-dictator's wishes as long as both agents 2 and 4 prefer a_{14} to a_{13}.

Remark 1. It is worth noting that semi-dictator rules can be generalized to allow for committees that depend not only on the position set P, but also on the alternative pairs whose order the committee determines. In other words, we could define semi-dictator rules where we introduce for each position $p \in P$ and unordered pair of alternatives $\{a, b\} \subset A$ a **committee** $\mathcal{C}_p(\{a, b\})$. The proof of K-SP, detailed in Theorem 4, carries over to this more general setting. Of course, if we extend semi-dictator rules in this manner they will fail to be neutral.

4 Keep anonymity - Drop neutrality

In this section we explore locally unanimous and K-SP rules that satisfy anonymity but fail neutrality. In contrast to the previous sections, we are not able to find a family of rules that satisfies these properties on the full domain. Instead, we establish possibility results for the two special cases where there are three alternatives or there are four alternatives and two agents. In addition, we show that anonymity, K-SP and a property with antecedents in the literature we refer to as *preference selection*, are incompatible for many instances. We suspect, but have been unable to prove, that there exists no locally unanimous, anonymous and K-SP rule for general n and m.

4.1 Existence

4.1.1 Three alternatives

We begin by defining a set of orderings of the elements of $\mathcal{R}$ that will prove useful later on. Note that for all $R \in \mathcal{R}$, the ordering $-R$ is defined so that for all $a, b \in A$ such that $a \neq b$, $(a, b) \in R$ if and only if $(b, a) \in -R$.

Definition 7 *An ordering $\succeq$ of $\mathcal{R}$ is* **regular** *if, for all $R \in \mathcal{R}^N$, whenever R is ranked first by $\succeq$, then for all R', R'' different than R and $-R$ and such that $R'' \in [R', R]$, we cannot have both $-R \succeq R'$ and $-R \succeq R''$.*

For example, if $\succeq$ is regular and ranks *abc* first, then we cannot have both $cba \succeq bac$ and $cba \succeq bca$, and we also cannot have both $cba \succeq acb$ and $cba \succeq cab$. An example of an ordering $\succeq$ that is regular is one that ranks orderings on the basis of their Kemeny distance from a benchmark R (the smaller the distance, the higher the rank), with ties broken arbitrarily. We may call such an ordering *Kemeny-consistent*.

We proceed by defining two families of rules that are known in the literature and play an important role in this section.

Definition 8 *Let $\succeq$ be an ordering on $\mathcal{R}$. For all $R_N \in \mathcal{R}^N$, let*

$$K(R_N) = \arg\min_{R \in \mathcal{R}} \sum_{i \in N} \delta(R, R_i). \tag{3}$$

*The $\succeq$-***Condorcet-Kemeny rule** *is defined as the aggregation rule which assigns to each $R_N \in \mathcal{R}^N$ the ordering belonging to $K(R_N)$ ranked highest according to $\succeq$.*

Definition 9 *Let $\succeq$ be an ordering on $\mathcal{R}$. Rule f is the* **fixed-benchmark rule**[13] *associated with $\succeq$ if, for all $R_N \in \mathcal{R}^N$,*

$$f(R_N) = R \text{ where } R \supseteq \bigcap_{i \in N} R_i \text{ and } R \succeq R' \text{ for all } R' \in \mathcal{R} \text{ such that } R' \supseteq \bigcap_{i \in N} R_i. \tag{4}$$

[13] The literature refers to those rules as fixed-order status-quo rules. As we will use the term of status-quo later, we will refer to them here as "fixed-benchmark rules".

Condorcet-Kemeny and fixed-benchmark rules are locally unanimous, anonymous and Btw-SP [12, 5]. Since they use an exogenous ordering $\succeq$ on $\mathcal{R}$ to break ties, they violates neutrality.[14]

Proposition 3 demonstrates that, when $m = 3$, any $\succeq$-Condorcet-Kemeny and any $\succeq$-fixed-benchmark rule will satisfy K-SP if and only if the ordering $\succeq$ is regular.

Proposition 3 *Let $|A| = m = 3$.*

(1) The $\succeq$-Condorcet-Kemeny rule satisfies K-SP if and only if $\succeq$ is regular.

(2) The $\succeq$-fixed-benchmark rule satisfies K-SP if and only if $\succeq$ is regular.

Unfortunately, Proposition 3 does not extend to four or more alternatives. This was already known for Condorcet-Kemeny rules, as Athanasoglou [4] showed that all such rules will fail K-SP for $m \geq 4$ and $n \geq 5$. As for fixed-benchmark rules, we show why all of them will fail K-SP for $m \geq 4$ and $n = 12$. Without loss of generality, suppose $m = 4$ (as for $m > 4$ we let all agents rank $m - 4$ alternatives at the bottom identically) and suppose f is a $\succeq$-fixed-benchmark rule such that $\succeq$ ranks *abcd* first. Consider the profile R_N with 12 agents where each agent has a different ordering and $\bigcap_{i\in N} R_i = (d, a)$.[15] Then there exists exactly one agent j such that $\delta(R_j, f(R_N)) = 5$, e.g., if $f(R_N) = dabc$, then this agent j has preference $R_j = cbda$. Now, if agent j deviates to $R'_j = abcd$, then $f(R'_j, R_{-j}) = abcd$ and $\delta(R_j, abcd) = 4$, a violation of K-SP. A similar argument works for any other $\succeq$-fixed-benchmark rule.

4.1.2 Four alternatives and two agents

Below we focus on the case of four alternatives and two agents.

We begin by showing that even in such environments, both families of rules considered in Proposition 3 fail K-SP. We restrict attention to Kemeny-consistent orderings $\succeq$ but suspect a similar reasoning to hold for any other $\succeq$ that are regular without being Kemeny-consistent. Suppose f is a $\succeq$-Condorcet-Kemeny rule or a fixed-order status quo rule where $\succeq$ is Kemeny-consistent with *abcd* as the highest-ranked ordering (as we'll see both rules yield identical outcomes in the following example). Consider the profile $(R_1, R_2) = (cbda, dabc)$, and suppose $dabc \succeq bcda$ so that $f(R_1, R_2) = dabc$[16] and $\delta(R_1, f(R_1, R_2)) = 5$. Then the

[14]To be precise, fixed-benchmark rules are parameterized with a special kind of *partial order* on $\mathcal{R}$ that is referred to as *conclusive* (Athanasoglou [5]). To avoid uninteresting complications, we focus on fixed-benchmark rules which employ a full linear ordering $\succeq$.

[15]Note that there are 6 orderings ranking d first, there are 4 orderings ranking d second and above a, and there are 2 orderings ranking d third and above a.

[16]If $bcda \succeq dabc$, then we repeat a similar reasoning for the profile $(dacb, bcda)$.

deviation $R_1' = cbad$ yields $f(R_1', R_2) = abcd$, leading to $\delta(R_1, f(R_1', R_2)) = 4$ and a violation of K-SP. Relabeling alternatives, we conclude that for all Kemeny-consistent orderings $\succeq$ we can construct a two-agent problem where K-SP is violated for both types of rules.

The failure of the rules of Proposition 3 means that we have to search elsewhere for possible locally unanimous, anonymous and K-SP rules.

Theorem 5 *Let $N = \{1, 2\}$ and $|A| = m = 4$. There exists a rule satisfying local unanimity, anonymity and K-SP.*

We established Theorem 5 by framing the existence of an anonymous, locally unanimous and K-SP rule as an *integer program* and obtained a computational solution on Matlab. All details are available in the Online Appendix where all 576 profiles are listed, and both the integer program and its implementation in Matlab are described.

The calculated family of rules satisfying the desired properties have the following characteristics:

1. A *losing alternative* (say a) is identified and placed as low as possible in the society's ranking subject to respecting local unanimity. For example, for any $R \in \mathcal{R}$, $f(R, -R)$ always places the losing alternative a at the bottom.

2. All other alternatives are treated symmetrically in the sense that they have identical rank-frequency vectors as detailed below.

If we evaluate the rule at all possible ($(4!)^2 = 576$) profiles, we obtain the following rank-frequency matrix:

	1	2	3	4
a	36	84	156	300
b	180	164	140	92
c	180	164	140	92
d	180	164	140	92

Table 1: Cell $[x, k]$ indicates the number of profiles in which alternative x is ranked kth by the rule. Here alternative a is the losing alternative.

The calculated rule has the following features. When both agents rank the same alternative at the bottom, then by local unanimity this alternative is ranked at the bottom by the social ordering. Now when considering the subdomain where both agents rank the same non-losing alternative at the bottom, say d, then the rule restricted to the other three alternatives is a rule satisfying local unanimity, anonymity and K-SP. It turns out that those rules are "fixed-status-quo rules with tie-breaking". This also applies to the subdomains where both agents rank the same non-losing alternative at the top (and by local unanimity this alternative is at the top of the social ordering). We define below such rules.

Example 2 Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Fix a status-quo ordering R_0, say $R_0 = cba$. The R_0-fixed-status-quo rule with tie-breaking f makes the following choice for any profile $R_N = (R_1, R_2) \in \mathcal{R}^N$:

(i) if R_0 is locally unanimous for R_N, then $f(R_N) = R_0$;

(ii) if R_0 is not locally unanimous for R_N and $R_1 \in [R_0, R_2]$, then $f(R_N) = R_1$;

(iii) if R_0 is not locally unanimous for R_N and $R_2 \in [R_0, R_1]$, then $f(R_N) = R_2$; and

(iv) otherwise, we have either $[R_1 = acb \,\&\, R_2 = bac]$ or $[R_1 = bac \,\&\, R_2 = acb]$ and set $f(R_N) = bac$.[17]

It is obvious that f satisfies local unanimity and anonymity, and one can also check K-SP.

Indeed, the rule described in Example 2 is the fixed-status-quo rule with tie-breaking when both agents rank the non-losing alternative d at the bottom. Let $\underline{R}_0^d$ denote the fixed-status-quo ordering when both agents rank d at the bottom, and $\overline{R}_0^d$ denote the fixed-status-quo ordering when both agents rank d at the top. Then for the rule we found it holds that $\underline{R}_0^d = bca$ and $\overline{R}_0^d = cba$, i.e. a is ranked as low as possible by $\underline{R}_0^d$ and $\overline{R}_0^d$ and the order of b and c is reversed for those two fixed-status-quo orderings. This pattern is confirmed as it holds for any non-losing alternative and we detail all the fixed-status-quo orderings when both agents rank a non-losing alternative at the bottom or at the top (see also Figure 3).

$$\begin{array}{ll} \underline{R}_0^b = dca & \overline{R}_0^b = cda \\ \underline{R}_0^c = bda & \overline{R}_0^c = dba \\ \underline{R}_0^d = cba & \overline{R}_0^d = bca \end{array} . \tag{5}$$

Note also the following: the non-losing alternatives are ranked in a cycle by the fixed-status-quo orderings where a non-losing alternative is ranked at the bottom as $\underline{R}_0^b$ ranks c above d, $\underline{R}_0^c$ ranks d above b, and $\underline{R}_0^d$ ranks b above c. Now the reverse holds for the fixed-status-quo orderings where a non-losing alternative is ranked at the top.

[17] More precisely, R_i is chosen if the third ranked alternative of R_i and the first ranked alternative of R_0 coincide.

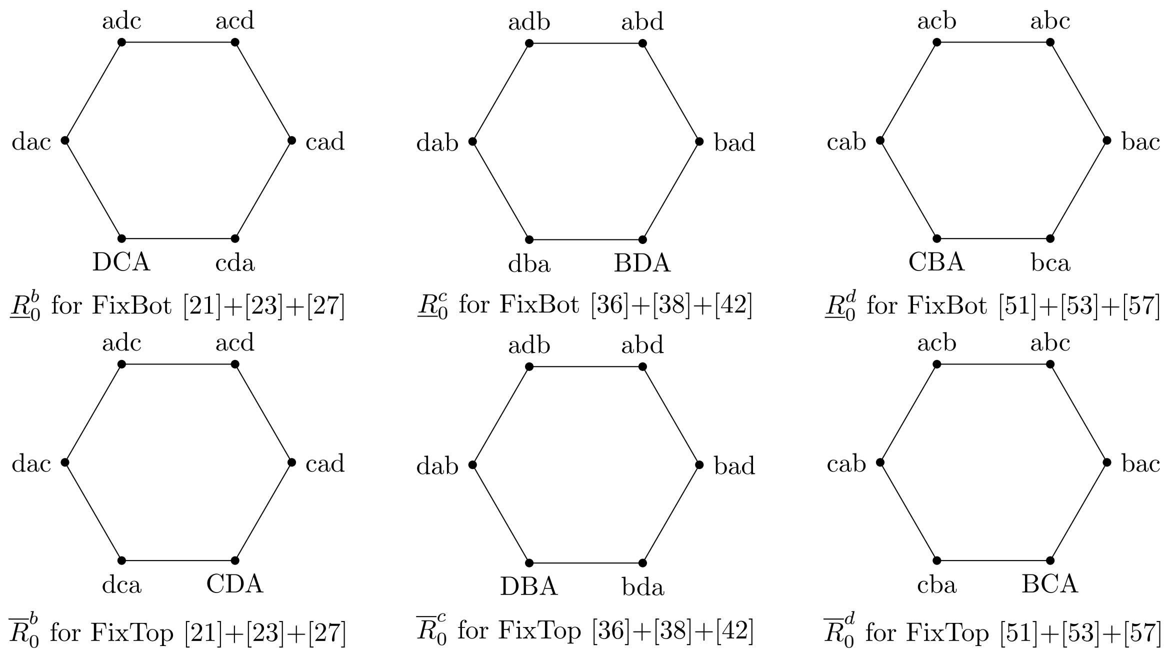

Figure 3: Fixed-status-quo orderings (indicated in capital letters) when both agents rank a non-losing alternative at the bottom or at the top. The numbers refer to the excel file and the column FixBottom/FixTopProfiles.

Conversely, a "rotating-status-quo rule with tie-breaking" appears when both agents rank the losing alternative a at the bottom. Then the rule makes the following choices:

$$f(bcda, dcba) = cdba, \;\; f(bdca, cdba) = dbca \text{ and } f(cbda, dbca) = bcda. \tag{6}$$

Now the same applies to the subdomain where both agents rank the losing alternative at the top (and by local unanimity this alternative is at the top of the social ordering). That is, we again have a "rotating-status-quo rule with tie-breaking", with the difference that the rotating status quo is the opposite of the one where the losing alternative is ranked at the bottom, i.e.

$$f(abcd, adcb) = abdc, \;\; f(abdc, acdb) = acbd \text{ and } f(acbd, adbc) = adcb. \tag{7}$$

We indicate the outputs of (6) and (7) in Figure 4. The output of the rule and its complete formal definition is given in the Online Appendix (but can also be derived from the previous results using local unanimity and K-SP).[18] We also give the formal definition of the rule

[18]For instance, if $R_1 = dcba$, then (i) for $R_2 = dbca$ by local unanimity we have $f(dcba, dbca) \in \{dbca, dcba\}$, and by K-SP and $f(bdca, cdba) = dbca$ we obtain $f(dbca, dcba) = dbca$, (ii) for $R_2 = bdca$ we

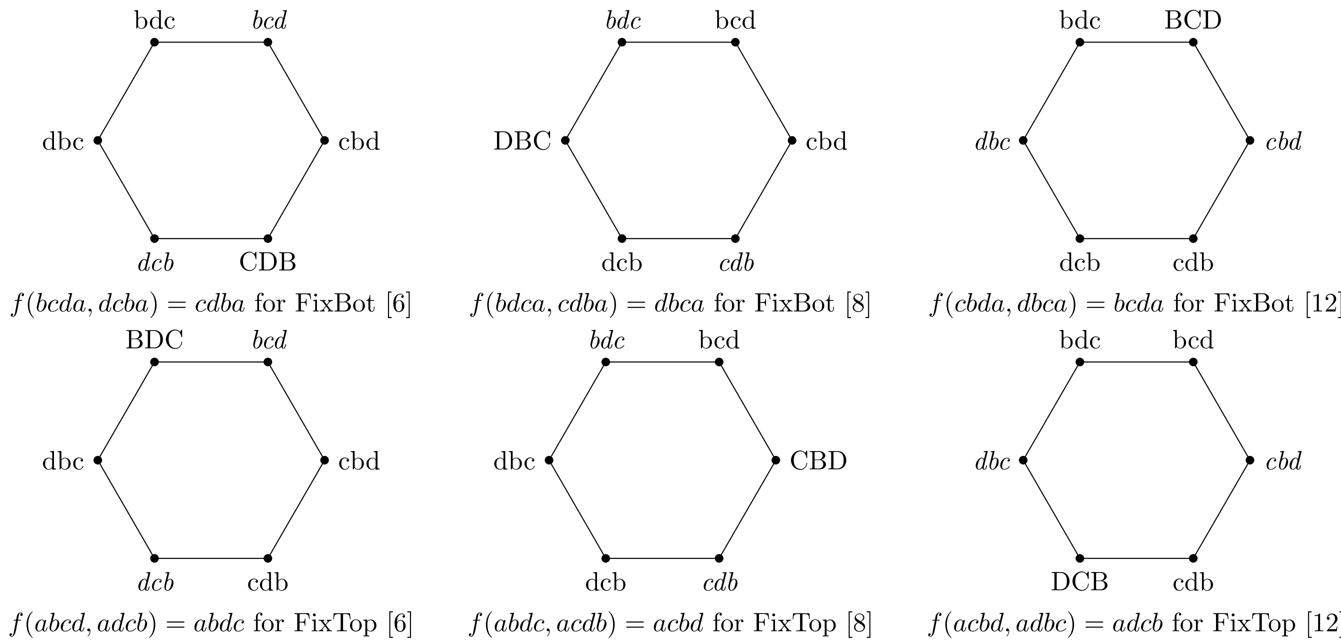

Figure 4: Rotating-status-quo rules with tie-breaking where both agents rank a at the bottom or at the top. The outcomes are indicated in capital letters while italic letters are used for preferences, and the numbers refer to the table in the excel file.

below. For any profile $R_N \in \mathcal{R}^N$,

- if $R_1 \cap R_2 \cap \{(a,b),(a,c),(a,d)\} = \emptyset$, then $f(R) = f(R_1|_{\{b,c,d\}}a, R_2|_{\{b,c,d\}}a)$ (i.e. the outcome of the rule is the same as for the profile where a is pushed to the bottom for both agents and the rotating-status-quo rule with tie-breaking given by (6) is used);

- if $R_1 \cap R_2 \cap \{(a,b),(a,c),(a,d)\} = \{(a,x)\}$ (where $x \in \{b,c,d\}$), then $f(R) = f(R_1|_{A\setminus\{x\}}x, R_2|_{A\setminus\{x\}}x)$ (i.e. the outcome of the rule is the same as for the profile where x is pushed to the bottom for both agents and the $\underline{R}_0^x$-fixed-status-quo rule given by (5) is used);

- if $R_1 \cap R_2 \cap \{(a,b),(a,c),(a,d)\} = \{(a,y),(a,z)\}$ (where $\{b,c,d\} = \{x,y,z\}$), then $f(R) = f(xR_1|_{A\setminus\{x\}}, xR_2|_{A\setminus\{x\}})$ (i.e. the outcome of the rule is the same as for the profile where x is pushed to the top for both agents and the $\overline{R}_0^x$-fixed-status-quo rule given by (5) is used); and

- if $R_1 \cap R_2 \cap \{(a,b),(a,c),(a,d)\} = \{(a,b),(a,c),(a,d)\}$, then both agents rank a first

have $f(dcba, bdca) = dbca$ from K-SP and $f(bdca, cdba) = dbca$, and (iii) for any $R_2 \neq bdca, dbca$, $f(R_1, R_2)$ is determined by K-SP and $f(dcba, bcda) = cdba$.

and $f(R)$ is the outcome of the rotating-status-quo rule with tie-breaking given by (7) where both agents rank a first.

The rule f satisfies local unanimity, anonymity and K-SP (for two agents and four alternatives).

4.2 Preference selection

In May's theorem, the rule always chooses a preference of one of the agents. The same holds for the median voter rule choosing always the peak of one of the agents (and by Black's [11] theorem in symmetric single-peaked environments the median voter's preference is the majority relation[19]). The property below adapts the one of "peak selection" (often used in one-public goods problems) to our context.

Preference selection: For all $R_N \in \mathcal{R}^N$, we have $f(R_N) \in \{R_1, \ldots, R_n\}$.

Preference selection also corresponds to the fact that always a member of the society shall be chosen to represent the social preference. Again this is in the vein of macroeconomics where the representative consumer shall be a member of the society. Note that preference selection implies local unanimity and efficiency.

Next we present two impossibility results for preference selection in conjunction with anonymity and K-SP. In other words, if always some agent is chosen to represent society, then either anonymity or K-SP is violated.

Theorem 6 *Let $m \geq 3$ and $|N|$ be even. There exists no rule satisfying preference selection, anonymity and K-SP.*

Theorem 7 *Let $m \geq 4$ and $|N| = 3k$. There exists no rule satisfying preference selection, anonymity and K-SP.*

Note that the above two theorems exclude situations where there are both three alternatives and three agents. As it turns out, preference selection and K-SP together with either neutrality or anonymity characterize families of rules which are reminiscent to Black's median rules. In other words, then it is possible to choose a representative agent while assuring K-SP and either neutrality or anonymity.

For three alternatives and three agents we denote by $\triangle$ the triangle profiles where agents' preferences are K-equidistant from each other (e.g. (abc, cab, bca)). Note that

[19] Ehlers and Storcken [16] characterized the class of Arrovian aggregation rules in single-peaked environments.

there are 12 triangular profiles and neutrality divides them into two sets of 6 profiles, i.e. $\triangle = \triangle' + \triangle''$ where $\triangle'$ contains (abc, cab, bca) and all profiles obtained from it by permuting alternatives, and where $\triangle''$ contains (abc, bca, cab) and all profiles obtained from it by permuting alternatives. Similarly anonymity divides $\triangle$ into two sets of 6 profiles, i.e. $\triangle = \hat{\triangle}' + \hat{\triangle}''$ where $\hat{\triangle}'$ contains (abc, cab, bca) and all profiles obtained from it by permuting agents, and where $\hat{\triangle}''$ contains (acb, bac, cab) and all profiles obtained from it by permuting agents.

Proposition 4 *Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$.*

(i) A non-dictatorial rule f satisfies preference selection, neutrality and K-SP if and only if f is a median rule with agent-based tie-breaking (i.e. there exist $i, j \in N$ (where $i = j$ is possible)) such that for all $R_N \in \triangle'$, $f(R_N) = R_i$, for all $R_N \in \triangle''$, $f(R_N) = R_j$, and for all $R_N \in \mathcal{R}^N \backslash \triangle$, the median is chosen, i.e. $f(R_N) = R_i$ if $R_i \in [R_j, R_k]$ (where $N = \{i, j, k\}$) and $[R_j, R_k]$ is the unique shortest path from R_j to R_k to which R_i belongs to).

(ii) A rule f satisfies preference selection, anonymity and K-SP if and only if f is a median rule with preference-based tie-breaking (i.e. there exist $\hat{R}_0' \in \{abc, cab, bca\}$ and $\hat{R}_0'' \in \{acb, bac, cab\}$) such that for all $R_N \in \hat{\triangle}'$, $f(R_N) = \hat{R}_0'$, for all $R_N \in \hat{\triangle}''$, $f(R) = \hat{R}_0''$, and for all $R_N \in \mathcal{R}^N \backslash \triangle$, the median is chosen, i.e. $f(R_N) = R_i$ if $R_i \in [R_j, R_k]$ (where $N = \{i, j, k\}$) and $[R_j, R_k]$ is the unique shortest path from R_j to R_k to which R_i belongs to).

Somewhat surprisingly in (i) and (ii) above, we have neutrality versus anonymity *and at the same time* agent-based tie-breaking versus preference-based tie-breaking.

5 Conclusion

We have considered the aggregation of individual preferences into one social preference. Here the set of alternatives can be public (like public goods), private (like consumption bundles) or a mix of both of them. This is due to the fact that any agent prefers social preferences which are closer to his own preference. Applications range from electing an individual to represent the society to choosing the representative consumer of the economy.

In such settings fairness is important. We explored the tradeoff between anonymity (where agents are treated equally) and neutrality (where alternatives are treated equally), two fundamental properties which are generally impossible to jointly satisfy. As basic requirements we consider local unanimity, an analogue of Pareto efficiency, and Kemeny strategy-proofness

(K-SP), whereby any agent prefers preferences which are closer to his own in terms of Kemeny distance. Dispensing with anonymity and keeping neutrality, we proposed semi-dictator rules and showed they are the first non-trivial family to satisfy local unanimity, neutrality and K-SP. Furthermore, for two agents these properties characterize semi-dictator rules. Dispensing with neutrality and keeping anonymity, we found a computer-aided solution to the existence of a rule satisfying local unanimity, anonymity and K-SP when there are two agents and four alternatives. For three alternatives we provided two families of rules satisfying the desired properties. Finally, for three alternatives and three agents, we characterized median rules with tie-breaking via preference selection, K-SP and either neutrality or anonymity.

When choosing the society's preference, we must decide how to resolve the anonymity-neutrality fairness tradeoff. If the society finds equal treatment of agents to be more important than equal treatment of alternatives, then neutrality should be dropped and anonymity maintained; otherwise, the opposite should occur. Our results help to clarify the consequences of such a judgment call as regards the design of efficient and strategy-proof aggregation rules.

Appendix

Below we provide all proofs of our results in the main text.

Proof of Theorem 1: The proof is a straightforward adaptation of the argument of Theorem 1 in Moulin [26].

First we prove necessity. Suppose that $n = k \cdot p$ for some integers k and p such that $p \leq m$. We will show that no anonymous and neutral rule exists. Consider the profile R_N satisfying (where bold fonts are added for clarity):

$$\begin{aligned}
R_1 = R_{p+1} = R_{2p+1} = \ldots.R_{(k-1)\cdot p+1} &= \boldsymbol{a_1 a_2 \ldots. a_p}\ a_{p+1}\ldots a_m \\
R_2 = R_{p+2} = R_{2p+2} = \ldots.R_{(k-1)\cdot p+2} &= \boldsymbol{a_2 \ldots. a_{p-1} a_1}\ a_{p+1}\ldots a_m \\
R_3 = R_{p+3} = R_{2p+3} = \ldots.R_{(k-1)\cdot p+3} &= \boldsymbol{a_3 \ldots. a_p a_1 a_2}\ a_{p+1}\ldots a_m \\
&\vdots \\
R_p = R_{p+p} = R_{2p+p} = \ldots.R_{(k-1)\cdot p+p} &= \boldsymbol{a_p a_1 \ldots. a_{p-1}}\ a_{p+1}\ldots a_m.
\end{aligned}$$

Now define the permutation $\pi : A \mapsto A$ as follows:[20]

[20] Here $\mod_p(k+1)$ denotes the number modular to p, i.e. $\mod_p(p+1) = p$ and $\mod_p(k+1) = k+1$ if $k < p$.

$$\pi(k) = \begin{cases} \mod{}_p(k+1) & \text{if } k \in \{1, 2, ..., p\} \\ k & \text{otherwise.} \end{cases}$$

Simple algebra yields $f(\pi R_N) = f(R_{\sigma(N)})$, where the permutation $\sigma : N \mapsto N$ is given by: for all $l \in \{0, \ldots, k-1\}$ and all $i \in \{1, \ldots, p\}$,

$$\sigma(lp + i) = lp + \mod{}_p(i+1).$$

Anonymity requires $f(R_{\sigma(N)}) = f(R_N)$ whereas neutrality requires $f(\pi R_N) = \pi f(R_N) \neq f(R_N)$. This contradicts $f(\pi R_N) = f(R_{\sigma(N)})$.

We now prove sufficiency. Suppose every prime factor of n is greater than m. This means that it is not possible to write $n = k \cdot p$ for some integers k, p such that $p \leq m$. We proceed by displaying a rule that is anonymous and neutral. Given a profile R_N and $a \in A$, define the quantity

$$l(R_N, a) = |\{i \in N : \ a \text{ ranked last by agent } i \text{ in profile } R_N\}|,$$

i.e., the number of agents who rank a last in profile R_N. In addition, given a profile R_N and $B \subseteq A$, define the

$$L(R_N, B) = \{a \in B : \ a = \arg\max_{b \in B} l(R_N|_B, b)\},$$

i.e., the set of alternatives attaining the maximum of function $l(R_N|_B, \cdot)$ over set B.

Suppose there exists $B^* \subseteq A$ with $|B^*| > 1$ with the property that, for all $a \in B^*$, there exists the same number k^* of agents ranking a last in R_N. This implies that $n = k^* \cdot |B'|$, which, since $|B'| \leq m$, contradicts the stated hypothesis on n and m. Hence, for all $B \subseteq A$, we have:

$$L(R_N, B) \subseteq B \ \text{ and } \ \{L(R_N, B) = B \Leftrightarrow |B| = 1\}.$$

As a result, given any $B \subseteq A$, the decreasing sequence

$$B_0 = B,$$
$$B_t = B_{t-1} \setminus L(R_N, B_{t-1}), \ t = 1, 2, ...$$

will converge to a singleton for some $t \in \{1, 2, ..., m\}$. We call this alternative $a^*(B)$.

Given a rule f and a profile R_N, let $f_k(R_N)$ denote the kth ranked alternative in $f(R_N)$. Now, define the aggregation rule f^* as the output of the following algorithm:

Input: R_N

1. Initialize $A_0 = A$.
2. For $k = 1, 2, ...m$
   (a) Set $f^*_k(R_N) = a^*(A_{k-1}) \equiv a_k$.
   (b) Set $A_k = A_{k-1} \setminus a_k$.

Output: $f^*(R_N) = a_1 a_2 ... a_m$

The above algorithm is well-defined and terminates at $k = m$, since at every k the alternative $a^*(A_{k-1})$ is well-defined. The rule f^* is anonymous and neutral. □

Proof of Proposition 2: Suppose rule f is locally K-SP and Min-SP but not K-SP. Then there exists an agent i and profile $R_N = (R_i, R_{-i})$ such that $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R_N))$, for some R'_i satisfying $\delta(R_i, R'_i) > 1$.

Denote $f(R'_i, R_{-i}) = R^*$. By Min-SP , $f(R^*, R_{-i}) = R^*$. Suppose $\delta(R_i, R^*) = T - 1$ and consider a shortest path between R_i and R^*, which we denote $\{R_i = R^1, R^2, ..., R^T = R^*\}$. To avoid cumbersome notation, let $x^t \equiv f(R^t, R_{-i})$ for all $t = 1, 2, ..., T$. By assumption, we have $x^1 = f(R_N)$ and $x^T = R^*$.

We will show by backwards induction that $\delta(R^t, x^t) \leq \delta(R^t, R^*)$ for all t. The base case $t = T$ follows trivially because $x^T = R^*$. Suppose $\delta(R^k, x^k) = \delta(R^k, R^*)$ for all $k = t, t+1, ..., T$. By local L-SP at (R^{t-1}, R_{-i}) and the induction hypothesis applied to $k = t$:

$$\delta(R^{t-1}, x^{t-1}) \leq \delta(R^{t-1}, x^t) \leq 1 + \delta(R^t, x^t) \leq 1 + \delta(R^t, R^*) = \delta(R^{t-1}, R^*).$$

Thus, the induction step is complete, implying that $\delta(R^t, x^t) \leq \delta(R^t, R^*)$ for all $t = 1, 2, ..., T$. When applied to $t = 1$ this yields $\delta(R_i, f(R_N)) \leq \delta(R_i, R^*)$, which is a contradiction to $f(R'_i, R_{-i}) = R^*$ and $\delta(R_i, f(R'_i, R_{-i})) < \delta(R_i, f(R_N))$. □

Proof of Theorem 3:

Note that the (if) direction is a special case of Theorem 4 which we show later.

The proof of the (only if) direction proceeds in three steps.

1. First, we prove the characterization when $m = 3$.
2. Then, we use Step 1 to prove the characterization for $m = 4$.
3. Using Step 2 as a base case, we prove the characterization for $m \geq 4$ by induction.

Step 1: The case $m = 3$.

Suppose $N = \{1, 2\}$ and $A = \{a, b, c\}$.

For the (only if) direction, let f satisfy the properties. Focusing on agent 1, we consider the following three rules.

(a) For all $R_N = (R_1, R_2) \in \mathcal{R}^N$, $f^0(R_N) = R_1$; or

(b) For all R_N with $R_1 = a_1a_2a_3$, $f^1(R_N) = (a_1 R_2|_{\{a_2,a_3\}})$ (where $R_2|_{\{a_2,a_3\}}$ denotes the restriction of R_2 to a_2 and a_3); or

(c) For all R_N with $R_1 = a_1a_2a_3$, $f^2(R_N) = (R_2|_{\{a_1,a_2\}} a_3)$ (where $R_2|_{\{a_1,a_2\}}$ denotes the restriction of R_2 to a_1 and a_2).

Denote the corresponding rules where agent 2 plays the role of agent 1 and vice versa by g^0, g^1 and g^2.

Recall that a profile R_N is opposite if $R_2 = -R_1$. Consider $f(abc, cba)$, and without loss of generality, let $f(abc, cba) \in \{acb, abc, bac\}$. If this is not the case, then focus on agent 2 and rules g^0, g^1, g^2 and apply similar reasoning. We distinguish between three cases.

1. $f(abc, cba) = abc$. Then by neutrality, for any opposite profile R_N, agent 1's preference is chosen, i.e. $f(R, -R) = R = f^0(R_N)$ for all $R \in \mathcal{R}$. Since $f(R_1, -R_1) = R_1$ for any choice of R_1, K-SP applied to agent 2 implies that, i.e., $f(R_1, R_2) = R_1 = f^0(R_1, R_2)$ for all $R_1, R_2 \in \mathcal{R}$.

2. $f(abc, cba) = acb$. Then by neutrality, for any opposite profile R_N we have $f(R_N) = f^1(R_N)$. If $R_N = (R_1, R_2)$ is not opposite, then consider $f(R_1, -R_1) = f^1(R_1, -R_1)$ and $f(R_2, -R_2) = f^1(R_2, -R_2)$. Now if $R_2 \neq -R_1$ is on the half circle that links R_1 to $-R_1$ which includes $f^1(R_1, -R_1)$, then by local unanimity we have $f(R_1, R_2) \in \{R_1, f^1(R_1, -R_1), R_2\}$. K-SP applied to agent 2 at profile $(R_1, -R_1)$ yields $f(R_1, R_2) \neq R_2$. Similarly, K-SP applied to agent 2 at profile (R_1, R_2) implies $f(R_1, R_2) \neq R_1$. Hence we conclude $f(R_1, R_2) = f^1(R_1, R_2)$.

   Otherwise, $R_2 \neq -R_1$ is not on the half circle containing $f^1(R_1, -R_1)$. For clarity, and without loss of generality (due to neutrality) suppose $R_1 = abc$, so that $f(abc, cba) = acb$ and $R_2 \in \{bac, bca\}$. If $R_2 = bca$, then by neutrality, $f(-R_2, R_2) = abc$, and by K-SP, $f(R_N) = abc = f^1(R_N)$, the desired conclusion. If $R_2 = bac$, then from the previous fact, $f(acb, bca) = abc$. By K-SP and local unanimity, $f(acb, R_2) = abc$, and using again K-SP and local unanimity, we obtain $f(R_N) = abc = f^1(R_N)$, the desired conclusion.

3. $f(abc, cba) = bac$. Note that in this case $f(abc, cba) = bac = f^2(abc, cba)$. Then using similar arguments as in Case 2 it follows that $f(R_N) = f^2(R_N)$ for all R_N.

$\square$

Step 2: The case $m = 4$**.**

Let $A = \{a, b, c, d\}$. For the (only if) direction, let f satisfy the properties. Let $\underline{f}$ denote the rule where f is restricted to the domain where both agents rank at the bottom the same alternative. By local unanimity, also $\underline{f}$ ranks at the bottom this alternative. Thus, $\underline{f}$ is a two agents-three alternatives rule. Furthermore, neutrality implies that the same type of rule is chosen when the two agents rank the same alternative at the bottom. Similarly we denote by $\overline{f}$ the rule where f is restricted to the domain where both agents rank at the top the same alternative. We consider three cases: (I) $\underline{f} = f^1$, (II) $\underline{f} = f^0$ and (III) $\underline{f} = f^2$ (as $\underline{f} \in \{g^0, g^1, g^2\}$ is analogous to the one by switching the roles of agent 1 and agent 2).

(I) $\underline{f} = f^1$:

Suppose that $\underline{f}$ is of type f^1. We show that f must be a semi-dictator rule with semi dictator 1 and agent 2 chooses the preference in f of the second and third alternatives of 1's preference.

First, we show that $f_1(R_N) = top(R_1)$ for all R_N. Suppose not, i.e. $f_1(R_N) \neq top(R_1)$. Then by K-SP, $f(R_1, f(R_N)) = f(R_N)$. Let $R_1' : top(R_1)f(R_N)|_{A\setminus\{top(R_1)\}}$. Then $R_1' \in [R_1, f(R_N)]$ and by K-SP, $f(R_1', f(R_N)) = f(R_N)$. If R_1' and $f(R_N)$ rank the same alternative at the bottom, then this is a contradiction to $\underline{f}(R_1', f(R_N)) = f^1(R_1', f(R_N))$ and

$$f_1(R_1', f(R_N)) = f_1(R_N) \neq top(R_1) = top(R_1').$$

Thus, $bot(f(R_N)) = top(R_1)$. Let $R_2' : f_1(R_N)f_2(R_N)f_4(R_N)f_3(R_N)$. Note that $f(R_1', f(R_N))$ is of K-distance one to R_2'. Thus, $f(R_1', R_2')$ is of distance at most one to R_2', which implies $f_1(R_1', R_2') \neq top(R_1')$, a contradiction because $f_4(R_1', f(R_N)) = f_4(R_N) = top(R_1) = top(R_1')$, $bot(R_1') = bot(R_2')$, $f(R_1', R_2') = \underline{f}(R_1', R_2')$ and $\underline{f} = f^1$.

Second, we show that $f_2(R_N) \neq bot(R_1)$ for all R_N. Suppose that $f_2(R_N) = bot(R_1) \neq bot(R_2)$ (as otherwise we have a contradiction to the fact that $\underline{f}$ is of type f^1). Let $R_1 : abcd$. By K-SP, $f(R_1, f(R_N)) = f(R_N)$ and $f(R_N) \in \{adbc, adcb\}$. Thus, without loss of generality, we may suppose $R_2 = f(R_N)$.

Case 1: $f(R_N) : adcb$.

Then $\delta(R_1, f(R_N)) = 3$. Let $R_1' : bacd$. Then from $top(f(R_1', f(R_N))) = top(R_1') = b$ and by local unanimity, $f(R_1', f(R_N)) \in \{badc, bacd\}$. But then $\delta(R_1, f(R_1', f(R_N))) \leq 2$, a contradiction to K-SP.

Case 2: $f(R_N) : adbc$.

Let $\hat{R}_1 : acbd$ and $\hat{R}_2 : adcb$. Note that $\hat{R}_N = R_N^{b\leftrightarrow c}$ and by neutrality, $f(\hat{R}_N) = adcb$.

Consider $(\hat{R}_1, f(R_N))$. Then $\delta(f(R_N), f(\hat{R}_N)) = 1$ and by K-SP (as agent 2 could deviate from $(\hat{R}_1, f(R_N))$ to $\hat{R}_N = (\hat{R}_1, \hat{R}_2)$), $\delta(f(R_N), f(\hat{R}_1, f(R_N))) \leq 1$. Thus, $f(\hat{R}_1, f(R_N)) \in \{adbc, dabc, abdc, adcb\}$. Then

(i) $f(\hat{R}_1, f(R_N)) = adbc$ implies that $\overline{f} = g^0$ and $f(R_1, adcb) = adcb$, which yields a contradiction as in Case 1;

(ii) $f(\hat{R}_1, f(R_N)) = dabc$ contradicts the fact $f_1(\hat{R}_1, f(R_N)) = top(\hat{R}_1) = a$;

(iii) $f(\hat{R}_1, f(R_N)) = abdc$ implies that $\overline{f}$ is of type g^2 which implies for $\bar{R}_1 : abcd$ and $\bar{R}_2 : acbd$ we have both $f(\bar{R}_N) = \overline{f}(\bar{R}_N) = g^2(\bar{R}_N) = abcd$ and $f(\bar{R}_N) = \underline{f}(\bar{R}_N) = f^1(\bar{R}_N) = acbd$, a contradiction; and

(iv) $f(\hat{R}_1, f(R_N)) = adcb$ implies that $\overline{f}$ is of type g^1.

For (iv) we derive a contradiction in three steps. In the first step we show that $f_2(R_N) = top(R_2|_{A\setminus\{top(R_1)\}})$ for any profile R_N. In the second step then we show that either agent 1 always chooses the third alternative in $f(R_N)$ or agent 2 always chooses the third alternative. In the third step we show that f violates K-SP (and therefore, (iv) cannot occur).

In the first step we show that $f_2(R_N) = top(R_2|_{A\setminus\{top(R_1)\}})$ for any profile R_N. Suppose $f_2(R_N) \neq top(R_2|_{A\setminus\{top(R_1)\}})$. By K-SP and neutrality, without loss of generality, we may suppose $R_1 = f(R_N) = abcd$. By $\overline{f} = g^1$, we have $a = top(R_1) \neq top(R_2)$ and $top(R_2) = top(R_2|_{A\setminus\{top(R_1)\}}) \neq a, b$ (as $b = f_2(R_N) \neq top(R_2|_{A\setminus\{top(R_1)\}})$). Similarly, by $\underline{f} = f^1$, we must have $bot(R_2) \neq d = bot(R_1)$. We distinguish two subcases ($top(R_2) = c$ or $top(R_2) = d$): if $top(R_2) = c$, then for $R_2' : cabd$ we have $R_2' \in [R_2, f(R_N)]$ (as $R_1 = f(R_N) = abcd$) and $f(R_1, R_2') = f(R_N) = abcd$ which is a contradiction as $\underline{f} = f^1$ and $f(R_1, R_2') = \underline{f}(R_1, R_2') = acbd$; and if $top(R_2) = d$, then for $R_2' : dabc$ we have $R_2' \in [R_2, f(R_N)]$, $f(R_1, R_2') = f(R_N) = abcd$ and $\delta(R_2', f(R_N)) = 3$ which yields a contradiction to K-SP as for $R_2'' : adbc$ we have (from $\overline{f} = g^1$) $f(R_1, R_2'') = \overline{f}(R_1, R_2'') = adbc$ and $\delta(R_2', f(R_1, R_2'')) = 1$.

For the second step, consider $R_1' : abcd$, $R_2' : abdc$ and $R_N' = (R_1', R_2')$. By local unanimity, $f(R_N') = R_1'$ or $f(R_N') = R_2'$. We show that if $f(R_N') = R_1'$, then agent 1 always chooses the third alternative, and if $f(R_N') = R_2'$, then agent 2 always chooses the third alternative. Without loss of generality, let $f(R_N') = R_1'$. Let R_N be an arbitrary profile. By the above, we have $f_1(R_N) = top(R_1)$ and $f_2(R_N) = top(R_2|_{A\setminus\{top(R_1)\}})$. Suppose $f_3(R_N) \neq top(R_1|_{A\setminus\{f_1(R_N), f_2(R_N)\}})$. Then by K-SP, $f(R_1, f(R_N)) = f(R_N)$. Let $R_1' = f_1(R_N)f_2(R_N)f_4(R_N)f_3(R_N)$. Now by neutrality and our assumption, $f(R_1', f(R_N)) = R_1'$. Note that $f_4(R_N) = top(R_1|_{A\setminus\{f_1(R_N), f_2(R_N)\}})$, and $\delta(R_1, R_1') < \delta(R_1, f(R_N))$, which is a contradiction to K-SP.

In the third step we show that f violates K-SP. Let R_1 : $abcd$ and R_2 : $adcb$. If $f(R'_N) = R'_1$, then by the above, $f(R_N)$: $adbc$. Let $\hat{R}_1$: $bacd$. Then $f(\hat{R}_1, R_2) = bacd$, $\delta(R_1, f(\hat{R}_1, R_2)) = 1 < \delta(R_1, f(R_N))$, which is a contradiction to K-SP. If $f(R'_N) = R'_2$, then as above it can be shown agent 2 always chooses the third alternative in $f(R_N)$. But then consider R_1 : $abcd$ and R_2 : $adcb$. Then $f(R_N) = adcb$ but for $\hat{R}_1$: $bacd$ we have $f(\hat{R}_1, R_2)$: $badc$, $\delta(R_1, f(\hat{R}_1, R_2)) = 2 < 3 = \delta(R_1, f(R_N))$. Thus, (iv) also leads to a contradiction.

We have shown $f_1(R_N) = top(R_1)$ and $f_2(R_N) \neq bot(R_1)$ for all R_N. Now if $bot(R_2) = bot(R_1)$, then we have $f(R_N) = \underline{f}(R_N)$ where $f_1(R_N) = top(R_1)$ and $f_4(R_N) = bot(R_1)$, and by $\underline{f} = f^1$, R_2 decides the ranking over the second and third alternative in R_1 (which is the desired conclusion).

Before treating the remaining case $bot(R_2) \neq bot(R_1)$, as an intermediary step, we show $\overline{f} = f^2$. Consider R_1 : $abcd$ and R_2 : $acbd$. Then $f(R_N) = \underline{f}(R_N) = acbd$. As also $f(R_N) = \overline{f}(R_N)$, we obtain $\overline{f} \neq f^0, f^1, g^2$. Thus, $\overline{f} \in \{f^2, g^0, g^1\}$. If $\overline{f} \in \{g^0, g^1\}$, then for $R_1 : abcd$ and $R_2 : adcb$ we have $f_2(R_N) = d$, a contradiction to $f_2(R_N) \neq bot(R_1)$. Hence, we obtain $\overline{f} = f^2$.

Finally, suppose $bot(R_1) \neq bot(R_2)$. We show $f_4(R_N) = bot(R_1)$. Suppose $f_4(R_N) \neq bot(R_1)$ and by neutrality, without loss of generality, let $R_1 = abcd$. Then $f_1(R_N) = a$ and by $f_2(R_N) \neq bot(R_1)$, $f_3(R_N) = d$. Thus, $f(R_N) \in \{abdc, acdb\}$.

If $f(R_N) = acdb$, then by K-SP, $f(R_1, f(R_N)) = acdb \neq \overline{f}(R_1, f(R_N)) = f^2(R_1, f(R_N))$, which is a contradiction to the above.

Hence, $f(R_N) = abdc$. Then by cP_1d and local unanimity, dP_2c. As $top(R_2) \neq a$, we have then $top(R_2) \in \{b, d\}$. If $top(R_2) = d$, then dP_2b. By K-SP, $f(f(R_N), R_2) = f(R_N) = abdc$. But now for $R'_2 : adbc$ we have (from $\underline{f} = f^1$) $f(f(R_N), R'_2) = \underline{f}(f(R_N), R'_2) = adbc$ which is a contradiction to K-SP as

$$\delta(f(R_2, f(f(R_N), R'_2)) = \delta(R_2, adbc) < \delta(R_2, abdc) = \delta(R_2, f(R_N)) = \delta(R_2, f(f(R_N), R_2)),$$

where the inequality follows from dP_2b.

If $top(R_2) = b$, then from local unanimity, $f(R_N) = abdc$ and cP_1d we obtain dP_2c. Now for $R'_2 : bdca$ we have $f_1(R_N) = a = f_1(R_1, R'_2)$ and by K-SP, $f(R_1, R'_2) = f(R_N) = abdc$ (where agent 2 is misreporting). But then by K-SP, $f(R_1, abdc) = abdc \neq abcd = f^2(R_1, abdc) = \overline{f}(R_1, abdc)$, a contradiction.

Hence, we have shown that for any profile R_N, $f_1(R_N) = top(R_1)$ and $f_4(R_N) = bot(R_1)$. As $\underline{f} = f^1$ it follows that agent 2 chooses the preference in $f(R_N)$ of the second and third alternatives of R_1 (as by K-SP we may suppose $top(R_2) = f_1(R_N)$ and $bot(R_2) = f_4(R_N)$).

(II) $\underline{f} = f^0$.

We show that for any profile R_N, $f(R_N)$ restricted to its first three alternatives coincides with R_1 restricted to these alternatives. Again by K-SP, without loss of generality, let $R_2 = f(R_N)$. If $f(R_N)|_{A\setminus\{f_4(R_N)\}} \neq R_1|_{A\setminus\{f_4(R_N)\}}$, then let $R'_1 : R_1|_{A\setminus\{f_4(R_N)\}} f_4(R_N)$ and then $f(R'_1, R_2) = \underline{f}(R'_1, R_2) = f^0(R'_1, R_2) = R'_1$ which is a contradiction to K-SP as by $f_4(R_N) = f_4(R'_1, R_2)$ we have $\delta(R_1, f(R'_1, R_2)) < \delta(R_1, f(R_N))$.

Hence, for any R_N,

$$f(R_N) : R_1|_{A\setminus\{f_4(R_N)\}} f_4(R_N). \tag{8}$$

Next we show $f_1(R_N) = top(R_1)$. If $f_1(R_N) \neq top(R_1)$, then by the previous fact, $f_4(R_N) = top(R_1)$ and $f_3(R_N) = bot(R_1)$. Let $R'_2 : f_1(R_N)f_2(R_N)f_4(R_N)f_3(R_N)$. As $\delta(R'_2, f(R_N)) = 1$, $f(R_1, R'_2)$ must be of K-distance one or zero to R'_2 which implies $top(R_1) \neq f_1(R_1, R'_2)$ and by (8), $f_4(R_1, R'_2) = top(R_1)$ and $f(R_1, R'_2) = f(R_N)$. Then $\delta(R_1, f(R_1, R'_2)) = 3$. Let $R'_1 : R_1|_{A\setminus\{f_3(R_N)\}} f_3(R_N)$. Then $f(R'_1, R'_2) = \underline{f}(R'_1, R'_2) = f^0(R'_1, R'_2) = R'_1$ and $\delta(R_1, f(R'_1, R'_2)) < 3 = \delta(R_1, f(R_N)) = \delta(R_1, f(R_1, R'_2))$, a contradiction to K-SP. We further show $f_2(R_N) = top(R_1|_{A\setminus\{top(R_1)\}})$. If $f_2(R_N) \neq top(R_1|_{A\setminus\{top(R_1)\}})$, then for $R_1 = abcd$ we obtain from $f_1(R_N) = top(R_1)$ and (8) that $f(R_N) = acdb$. Again let $R_2 = f(R_N)$. Then $\overline{f} \in \{g^0, g^2\}$. By considering the profile $\hat{R}_N = (abcd, acbd)$ we then get a contradiction to $\overline{f} = g^0$ as $f(\hat{R}_N) = \underline{f}(\hat{R}_N) = f^0(\hat{R}_N) = \hat{R}_1$ and $f(\hat{R}_N) = \overline{f}(\hat{R}_N) = g^0(\hat{R}_N) = \hat{R}_2$. Hence, $\overline{f} = g^2$. Recall that $f(R_N) = acdb = R_2$ and $\delta(R_1, f(R_N)) = 2$. Let $R'_1 : bacd$. But then $f_1(R'_1, R_2) = top(R'_1) = b$. By local unanimity then $f(R'_1, R_2) = bacd$ and $\delta(R_1, f(R'_1, R_2)) = 1$, a contradiction to K-SP.

We have shown that for any profile R_N, $f_1(R_N) = top(R_1)$ and $f_2(R_N) = top(R_1|_{A\setminus\{top(R_1)\}}$. Now considering $R'_1 = abcd$ and $R'_2 = abdc$ we have $f(R'_N) \in \{abcd, abdc\}$. If $f(R'_N) = abcd$, then R_1 always decides the ranking of the third and fourth alternative in $f(R_N)$: for any profile R_N by neutrality we may suppose $R_1 = abcd$; then $f_1(R_N) = a$ and $f_2(R_N) = b$; again by K-SP we may choose $R_2 = f(R_N)$ and obtain $f(R_N) = f(R_1, f(R_N))$; hence, $(R_1, f(R_N)) = R'$ and we must have $f(R_N) = abcd = R_1$, the desired conclusion. If $f(R'_N) = abdc$, then similarly it can be shown that R_2 always decides the ranking of the third and fourth alternative in $f(R_N)$.

(III) $\underline{f} = f^2$.

We show that for any profile R_N, $\{f_3(R_N), f_4(R_N)\}$ consists of the third and fourth ranked alternative in R_1 and R_2 decides the ranking of the first two alternatives in $f(R_N)$.

First, we show $f_1(R_N) \neq bot(R_1)$. Let $R_1 : abcd$ and suppose $f_1(R_N) = d$ and $R_2 = f(R_N)$ (by K-SP). By $\underline{f} = f^2$, we must have $bot(R_2) \neq d$. Let $R'_1 : R_1|_{A\setminus\{f_4(R_N)\}} f_4(R_N)$. Then $f(R'_1, R_2) = \underline{f}(R'_1, R_2)$ which implies $f_3(R'_1, R_2) = d$ and $f_4(R'_1, R_2) = f_4(R_N)$. But then by

$f_1(R_N) = d$, $\delta(R_1, f(R_N)) > \delta(R_1, f(R'_1, R_2)$, a contradiction to K-SP.

Second, we show $f_2(R_N) \neq bot(R_1)$. Let $R_1 : abcd$ and suppose $f_2(R_N) = d$ and $R_2 = f(R_N)$ (by K-SP). By $\underline{f} = f^2$, we must have $bot(R_2) \neq d$. Let $R'_1 : R_1|_{A\setminus\{f_4(R_N)\}} f_4(R_N)$. Then $f(R'_1, R_2) = \underline{f}(R'_1, R_2)$ which implies $f_3(R'_1, R_2) = d$ and $f_4(R'_1, R_2) = f_4(R_N)$. But then by $f_1(R_N) = d$, $f(R_N) = R_2$ and thus, $f(R_N)|_{A\setminus\{d, f_4(R_N)\}} = f(R'_1, R_2)|_{A\setminus\{d, f_4(R_N)\}}$, we obtain $\delta(R_1, f(R_N)) > \delta(R_1, f(R'_1, R_2)$, a contradiction to K-SP.

Thus, we have shown $bot(R_1) \in \{f_3(R_N), f_4(R_N)\}$. We show that the third alternative in R_1 cannot be ranked first or second in $f(R_N)$, i.e. for $R_1 : abcd$ we have $\{c, d\} = \{f_3(R_N), f_4(R_N)\}$ (as $bot(R_1) \neq f_1(R_N), f_2(R_N)$). Again, by K-SP, without loss of generality, $R_2 = f(R_N)$. If $bot(R_2) = d = bot(R_1)$, then this is obvious from $\underline{f} = f^2$. Thus, let $bot(R_2) \neq d$. If $bot(R_2) = c$, then this follows from local unanimity as $d \in \{f_3(R_N), f_4(R_N)\}$. Thus, let $bot(R_2) \in \{a, b\}$.

Third, we show $f_1(R_N) \neq c$. If $f_1(R_N) = c$, then from $f_4(R_N) = bot(R_2) \neq c, d$ and $bot(R_1) = d \in \{f_3(R_N), f_4(R_N)\}$ we obtain $f(R_N) \in \{cadb, cbda\}$. Let $R'_2 : R_2|_{\{a,b,c\}} d$. Then $f(R_1, R'_2) = \underline{f}(R_1, R'_2) = R_2|_{\{a,b\}} cd$. But then $\delta(R'_2, f(R_N)) = 1 < \delta(R'_2, f(R_1, R'_2))$ (as $top(R'_2) = c$), a contradiction to K-SP.

Fourth, we show $f_2(R_N) \neq c$ in four steps: in the first step we show $\overline{f} = g^2$; in the second step we show $bot(R_2) \notin \{f_1(R_N), f_2(R_N)\}$; in the third setp we show that for any profile the third ranked alternative in R_2 can never be chosen first in $f(R_N)$; and in the fourth step we obtain a contradiction by showing that for certain profiles no alternative can be ranked first (using the first three steps).

In showing the first step, $f_2(R_N) = c$ implies $f(R_N) \in \{acdb, bcda\}$ (as $d = bot(R_1) \neq bot(R_2) = f_4(R_N)$ and $d \in \{f_1(R_N), f_2(R_N)\}$). If $f(R_N) = acdb$, then $\overline{f} = g^0$ or $\overline{f} = g^2$. For $\overline{f} = g^0$, let $R'_1 = abdc$ and then $f(R'_1, R_2) = \overline{f}(R'_1, R_2) = g^0(R'_1, R_2) = acdb$ which is a contradiction as $c = bot(R'_1) \notin \{f_1(R'_1, R_2), f_2(R'_1, R_2)\}$. Thus, $f(R_N) = acdb$ implies $\overline{f} = g^2$. We show that $f(R_N) = bcda$ also implies $\overline{f} = g^2$: note that $\delta(R_1, f(R_N)) = 3$ which implies by K-SP for $R'_1 = bacd$ that $f(R'_1, R_2) = bcda$ (as $f_1(R'_1, R_2) = b$, $f(R'_1, R_2)$ must rank c above d by local unanimity, and d must be ranked before a by K-SP as otherwise agent 1 profitably misreports from $f(R_N) = bcda$); but then $\overline{f} = g^0$ or $\overline{f} = g^2$ and again $\overline{f} = g^0$ yields a contradiction as above. Thus, $\overline{f} = g^2$.

In the second step we show that $bot(R_2) \notin \{f_1(R_N), f_2(R_N)\}$ for any profile R_N. Again suppose $f_1(R_N) = bot(R_2)$ and $f(R_N) = R_1 = abcd$. Thus, $bot(R_2) = a$. For $R'_2 : R_2|_{\{b,c\}} ad$ we have (from $\underline{f} = f^2$) $f(R_1, R'_2) = bacd$ and $\delta(R_2, f(R_1, R'_2)) < \delta(R_2, f(R_N))$, a contradiction to K-SP. Next suppose $f_2(R_N) = bot(R_2)$ and $f(R_N) = R_1 = abcd$. Thus, $bot(R_2) = b$. For $R'_2 : aR_2|_{\{c,d\}} b$ we have (from $\underline{f} = g^2$) $f(R_1, R'_2) = acdb$ and $\delta(R_2, f(R_1, R'_2)) < \delta(R_2, f(R_N))$, a contradiction to K-SP.

In the third step we show that the third ranked alternative in R_2 can never be chosen first by $f(R_N)$. Suppose $R_2 = abcd$, $f_1(R_N) = c$ and $R_1 = f(R_N)$ (by K-SP). But then $bot(R_1) \neq bot(R_2)$ as otherwise $f(R_N) = \underline{f}(R_N)$ and by $\underline{f} = f^2$, R_2 decides the ranking of the first two alternatives in $f(R_N)$ and $f_1(R_N) \neq c$. Thus, from $f_1(R_N) \neq d$ and $R_1 = f(R_N)$, we have $f(R_N) \in \{cbda, cadb\}$. If $f(R_N) = cbda$, then for $R_2' = bcda$ we have $f(R_1, R_2') = \underline{f}(R_1, R_2') = f^2(R_1, R_2') = bcda$ and $\delta(R_2, f(R_1, R_2')) < \delta(R_2, f(R_N))$, a contradiction to K-SP. If $f(R_N) = cadb$, then for $R_2' = acdb$ we have $f(R_1, R_2') = \underline{f}(R_1, R_2') = f^2(R_1, R_2') = acdb$ and $\delta(R_2, f(R_1, R_2')) < \delta(R_2, f(R_N))$, a contradiction to K-SP.

In the fourth step we now obtain a contradiction for the profile R_N where $R_1 = abcd$ and $R_2 = dcba$ as by $bot(R_1) \neq bot(R_2)$ and $bot(R_1), bot(R_2) \in \{f_3(R_N), f_4(R_N)\}$ we have $\{a, d\} = \{f_3(R_N), f_4(R_N)\}$. On the other hand the third ranked alternative in R_1 is c and the third ranked alternative in R_2 is b and $f_1(R_N) \in \{b, c\}$ which is a contradiction as then no alternative can be ranked first in $f(R_N)$ as the third ranked alternative in R_1 and in R_2 are never ranked first by $f(R_N)$. This concludes the proof that $f_2(R_N)$ cannot be the third ranked alternative in R_1 for any profile R_N.

We have shown that $\{f_3(R_N), f_4(R_N)\}$ always consists of the third and fourth ranked alternatives in R_1. We next show that R_2 determines the ranking of the first two alternatives in $f(R_N)$. Again, by K-SP, without loss of generality, $R_1 = f(R_N)$. If $f(R_N)|_{\{f_1(R_N), f_2(R_N)\}} \neq R_2|_{\{f_1(R_N), f_2(R_N)\}}$, then let $R_2' : R_2|_{\{f_1(R_N), f_2(R_N)\}} f_3(R_N) f_4(R_N)$ and then $f(R_1, R_2') = \underline{f}(R_1, R_2') = f^2(R_1, R_2') = R_2'$, a contradiction to K-SP. Thus, R_2 always determines the ranking of the first two alternatives. Now considering $R_1 = abcd$ and $R_2 = abdc$ we have $f(R_N) \in \{abcd, abdc\}$. If $f(R_N) = abcd$, then it can be shown analogously that R_1 always decides the ranking of the third and fourth alternative in $f(R_N)$, and if $f(R_N) = abdc$, then it can be shown analogously that R_2 always decides the ranking of the third and fourth alternative in $f(R_N)$. □

Note that using the same argument as in Example 1, it follows that agent 2 cannot decide both the ranking of the first and second alternative and the ranking of the third and fourth alternative. This finishes the proof of Theorem 3 for four alternatives.

Step 3: Induction on m.

Now by induction suppose that Theorem 3 is true for $k \geq 4$ alternatives (where $m = k$). Let $A = \{a_1, \ldots, a_{k+1}\}$. Let f satisfy the properties. Let $\underline{f}$ denote the rule where f is restricted to the domain where both agents rank at the bottom the same alternative, i.e. $\underline{\mathcal{R}^N} = \{R \in \mathcal{R}^N : bot(R_1) = bot(R_2)\}$ and $\underline{f} = f|\underline{\mathcal{R}^N}$. By local unanimity, for any $R_N \in \underline{\mathcal{R}^N}$, $\underline{f}$ ranks at the bottom the alternative $bot(R_1) = bot(R_2)$. Thus, by the induction hypothesis and neutrality, $\underline{f}$ is a semi-dictator rule. Similarly we denote by $\overline{f}$ the rule where f is restricted to the domain where both agents rank at the top the same alternative, and again by the induction hypothesis, $\overline{f}$ is a semi-dictator rule. Note that in a semi-dictator rule with semi-dictator i, for any profile R_N any alternative in R_i can move at most one position up or at most one position down in the chosen ranking.

First, we show that $\underline{f}$ and $\overline{f}$ have the same semi-dictator. Suppose not, say agent 1 is the semi-dictator of $\underline{f}$ and agent 2 is the semi-dictator of $\overline{f}$. Consider $R_1 : a_1 \ldots a_{k+1}$ and $R_2 : a_1 \ldots a_{k-3}a_k a_{k-1}a_{k-2}a_{k+1}$. Note that we have $f(R_N) = \underline{f}(R_N) = \overline{f}(R_N)$. Thus, by local unanimity and the fact that both a_k and a_{k-2} can move at most one position up in $f(R_N)$, we have $f(R_N) \in \{a_1 \ldots a_{k-1}a_{k-2}a_k a_{k+1}, a_1 \ldots a_{k-1}a_k a_{k-2}a_{k+1}\}$. But then either a_k moves down two positions from R_2 to $f(R_N)$ or a_{k-2} moves two positions down from R_1 to $f(R_N)$, a contradiction.

Hence, $\underline{f}$ and $\overline{f}$ have the same semi-dictator say agent 1. But then the positions p where agent 2 is decisive with $1 < p < k$ coincide in $\underline{f}$ and $\overline{f}$ (by considering profiles R_N where $top(R_1) = top(R_2)$ and $bot(R_1) = bot(R_2)$ and $f(R_N) = \underline{f}(R_N) = \overline{f}(R_N)$ noting $p + 1 \leq k$ by $p < k$). Using the same argument as above, it also follows that if $p = 1$ is a position in $\underline{f}$ where agent 2 is decisive, then position 2 does not belong to $\overline{f}$.

Now consider any profile R_N. By neutrality, we may suppose $R_1 : a_1 \ldots a_{k+1}$. By K-SP again we may suppose $R_2 = f(R_N)$ (because otherwise $R_2 \neq f(R_N)$ and $f(R_N) = f(R_1, f(R_N))$ by K-SP). If $f_{k+1}(R_N) = a_{k+1} = bot(R_1)$, then $bot(R_1) = bot(R_2)$ as $R_2 = f(R_N)$ and $f(R_N) = \underline{f}(R_N)$; and we are done by the induction hypothesis. Otherwise ($f_{k+1}(R_N) \neq a_{k+1}$) we show $f_k(R_N) = a_{k+1}$. If not, i.e. $f_l(R) = a_{k+1}$ with $l < k$, then let $R_1' : f(R_N)|_{A\setminus\{a_{k+1}\}}a_{k+1}$. Then $R_1' \in [R_1, f(R_N)]$ and by K-SP and local unanimity, $f(R_1', R_2) = f(R_N)$. If $f_1(R_N) = top(R_1')$, then this is a contradiction to the fact that agent 1 is the semi-dictator of $\overline{f}$ and a_{k+1} moves more than one position up in $f(R_N)$. If $f_1(R_N) \neq top(R_1')$, then $f_1(R_N) = a_{k+1}$ and by $k \geq 4$ we obtain a contradiction to K-SP when we exchange in R_1' the positions of the last two alternatives, i.e. for $R_1'' : f(R_N)|_{A\setminus\{a_{k+1}, f_{k+1}(R_N)\}}a_{k+1}f_{k+1}(R_N)$ we obtain $f(R_1'', R_2) = \underline{f}(R_1'', R_2)$.

Hence, $f_{k+1}(R_N) \neq a_{k+1}$ implies $f_k(R_N) = a_{k+1}$. Similarly, by considering $\overline{f}$ this also implies $f_{k+1}(R_N) = a_k$ (as otherwise $f_{k+1}(R_N) = a_l$ for $l < k$ and a_l moves more than one position down from R_1 to $f(R_N)$). But then we are done as now we consider $R_2 : a_1 \ldots a_{k-1}a_{k+1}a_k$. If $f(R_N) = R_2$, then k is a position in f where agent 2 is decisive and otherwise not. Furthermore, note that the ranking of $f(R_N)$ over $\{a_1, \ldots, a_{k-1}\}$ is decided by $\underline{f}$.

Note that using the same argument as in Example 1, it follows that any two positions where agent 2 decides the ranking must have distance greater than two. □

Proof of Theorem 4:

Consider a semi-dictator rule $f = f^{(i,P,\mathcal{C})}$. That local unanimity and neutrality are satisfied is obvious from Definitions 5 and 6, so we focus on proving K-SP. Without loss of generality assume $i = 1$ and $R_1 = a_1 a_2 ... a_m$. By Proposition 2 it is sufficient to prove that f is Min-SP and Locally K-SP.

As before with local unanimity, Min-SP follows immediately from Definitions 5 and 6. We turn to proving Local K-SP.

Consider a profile R_N and suppose agent j changes his preferences to R'_j, where $\delta(R_j, R'_j) = 1$. If $j \neq 1$, then by the definition of semi-dictator rules and committees, it is clear that this agent cannot profit from misreporting. In fact, $\delta(R_j, f(R'_j, R_{-j})) - \delta(R_j, f(R_N)) \in \{0, 1\}$.

Thus, suppose $j = 1$, meaning that the semi-dictator is the agent who misreports. Suppose R'_1 is identical to R_1 except that the order of two adjacent alternatives a_k and a_{k+1} is flipped for some $k = 1, 2, ..., m-1$. We distinguish between four cases:

1. $k-1 \notin P$, $k \notin P$ and $k+1 \notin P$. In this case, $f_l(R_N) = f_l(R'_1, R_{-1})$ for all $l \notin \{k, k+1\}$. Since $k-1, k$ and $k+1$ do not belong to P, it is immediate that $f_k(R_N) = a_k, f_{k+1}(R_N) = a_{k+1}$ and $f_k(R'_1, R_{-1}) = a_{k+1}, f_{k+1}(R'_1, R_{-1}) = a_k$. This implies $\delta(R_1, f(R'_1, R_{-1})) - \delta(R_1, f(R_N)) = 1$.

2. $k \in P$. In this case, $f_l(R_N) = f_l(R'_1, R_{-1})$ for all $l \notin \{k, k+1\}$. In both profiles R_N and (R'_1, R_{-1}) the relative order of adjacent alternatives (a_k, a_{k+1}) is decided by committee $\mathcal{C}_k$. Eq. (2) and the definition of committees imply that agent 1's misreport can never be profitable.

3. $k+1 \in P$. In this case, $f_l(R_N) = f_l(R'_1, R_{-1})$ for all $l \notin \{k, k+1, k+2\}$. So let us focus on those three ranks and the alternatives that occupy them. As $k \notin P$, we have $f_k(R_N) = a_k$ and $f_k(R'_1, R_{-1}) = a_{k+1}$. Subsequently, we focus on alternatives occupying ranks $k+1$ and $k+2$. In profile R_N, the rule f determines the relative order of adjacent alternatives (a_{k+1}, a_{k+2}) via the committee $\mathcal{C}_{k+1}$ whereas in profile (R'_1, R_{-1}), the rule f determines the relative order of (a_k, a_{k+2}) via the committee $\mathcal{C}_{k+1}$. The most profitable misreport occurs when committee $\mathcal{C}_{k+1}$ ranks a_{k+2} before a_{k+1} (meaning $f_{k+1}(R_N) = a_{k+2}$, $f_{k+2}(R_N) = a_{k+1}$) and committee $\mathcal{C}_{k+1}$ ranks a_k before a_{k+2} (meaning $f_{k+1}(R'_1, R_{-1}) = a_k$, $f_{k+2}(R'_1, R_{-1}) = a_{k+2}$). In that case, truthful reporting results in the triplet $a_k a_{k+2} a_{k+1}$ occupying ranks $k, k+1, k+2$, whereas misreporting in the triplet $a_{k+1} a_k a_{k+2}$ in those same ranks. Putting together the various possible outcomes of committees $\mathcal{C}_{k+1}$ and $\mathcal{C}_{k+1}$ implies $\delta(R_1, f(R'_1, R_{-1})) - \delta(R_1, f(R_N)) \in \{0, 1, 2\}$.

4. $k-1 \in P$. In this case, $f_l(R_N) = f_l(R'_1, R_{-1})$ for all $l \notin \{k-1, k, k+1\}$. So let us focus on those three ranks and the alternatives that occupy them. As $k+1 \notin P$, we

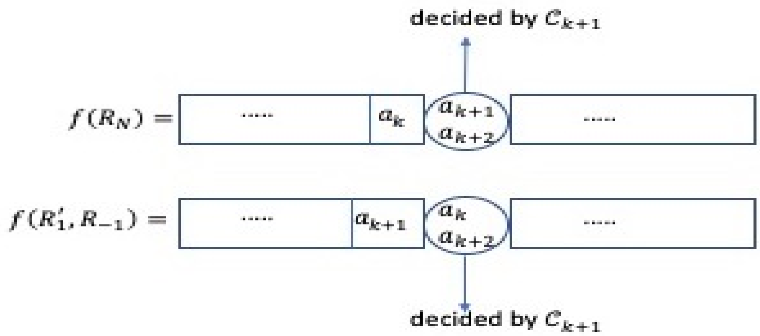

Figure 5: An illustration of Case 3. Orderings $f(R_N)$ and $f(R_1', R_{-1})$ are identical at all ranks $l \notin \{k, k+1, k+2\}$. Committee $\mathcal{C}_{k+1}$ determines the alternatives occupying ranks $k+1$ and $k+2$.

have $f_{k+1}(R_N) = a_{k+1}$ and $f_k(R_1', R_{-1}) = a_k$. Subsequently, we focus on alternatives occupying ranks $k-1$ and k. In profile R_N, the rule f determines the relative order of adjacent alternatives (a_{k-1}, a_k) via the committee $\mathcal{C}_{k-1}$ whereas in profile (R_1', R_{-1}), the rule f determines the relative order of (a_{k-1}, a_{k+1}) via the committee $\mathcal{C}_{k-1}$. The most profitable misreport occurs when committee $\mathcal{C}_{k-1}$ ranks a_k before a_{k-1} (meaning $f_{k-1}(R_N) = a_k$, $f_k(R_N) = a_{k-1}$) and committee $\mathcal{C}_{k-1}$ ranks a_{k-1} before a_{k+1} (meaning $f_{k-1}(R_1', R_{-1}) = a_{k-1}$, $f_k(R_1', R_{-1}) = a_{k+1}$). In that case, truthful reporting results in the triplet $a_k a_{k-1} a_k$ occupying ranks $k-1, k, k+1$, whereas misreporting in the triplet $a_{k-1} a_{k+1} a_k$ in those same ranks. Putting together the various possible outcomes of committees $\mathcal{C}_{k-1}$ and $\mathcal{C}_{k-1}$ implies $\delta(R_1, f(R_1', R_{-1})) - \delta(R_1, f(R_N)) \in \{0, 1, 2\}$.

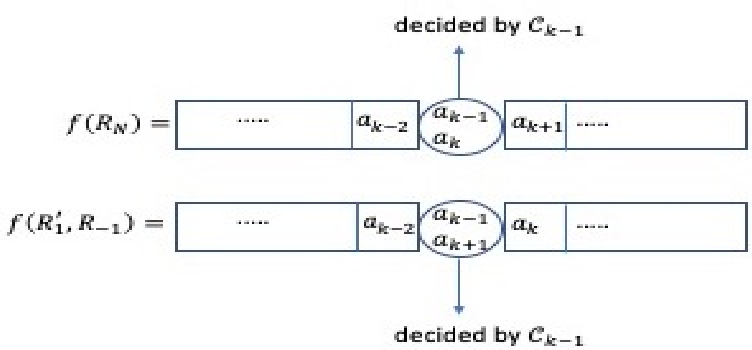

Figure 6: An illustration of Case 4. Orderings $f(R_N)$ and $f(R_1', R_{-1})$ are identical at all ranks $l \notin \{k-1, k, k+1\}$. Committee $\mathcal{C}_{k-1}$ determines the alternatives occupying ranks $k-1$ and k.

$\square$

Proof of Proposition 3:

We begin with part (1). Suppose g is a Condorcet-Kemeny rule with ordering $\succeq$. Let $A = \{a, b, c\}$ and $R_N \in \mathcal{R}^N$. We will first argue that if K-SP is violated, then the ordering $\succeq$ must fail regularity.

Suppose, without loss of generality, that agent i's preferences are given by $R_i = abc$ and that there exists $R'_i \in \mathcal{R}$ such that $\delta(R_i, g(R_N)) > \delta(R_i, g(R'_i, R_{-i}))$. We distinguish between 4 cases:

(i) $\delta(R_i, g(R_N)) = 0$. But since $0 \leq \delta(R_i, R)$ for all $R \in \mathcal{R}$, we immediately reach a contradiction.

(ii) $\delta(R_i, g(R_N)) = 1$. Then, we must have $\delta(R_i, g(R'_i, R_{-i})) = 0$. Hence, $R_i = g(R'_i, R_{-i})$. This implies that rule g is not Btw-SP which contradicts Proposition 5 in [12].

(iii) $\delta(R_i, g(R_N)) = 3$. Then, we must have $\delta(R_i, g(R'_i, R_{-i})) < 3$. Let $\tilde{R}_i$ denote the ordering which is exactly the opposite of R_i (which reverses the direction of all binary comparisons). Then, it must be the case that $g(R_N) = \tilde{R}_i$ and $g(R'_i, R_{-i}) \neq \tilde{R}_i$. This again contradicts the Btw-SP of g.

(iv) $\delta(R_i, g(R_N)) = 2$. This is the only nontrivial case and we address it in what follows.

To violate K-SP we must have $\delta(R_i, g(R'_i, R_{-i})) < 2$. Suppose, first, that $\delta(R_i, g(R'_i, R_{-i})) = 0$. Repeating the argument of case (ii), we arrive at a contradiction.

Thus, we must have $\delta(R_i, g(R'_i, R_{-i})) = 1$. Now, $\delta(R_i, g(R_N)) = 2$ implies that we must have either $g(R_N) = cab$ or $g(R_N) = bca$. Suppose that $g(R_N) = cab$ (the proof for case $g(R_N) = bca$ is similar). Then, to avoid violating Btw-SP we must have $g(R'_i, R_{-i}) = bac$. We will argue how this cannot happen unless the ordering $\succeq$ violates regularity.

Given profile R_N, define the 3×3 matrix E, where E_{xy} denotes the number of agents ranking alternative x over y. For all pairs $(x, y) \in A \times A$ such that $x \neq y$ we must have $E_{xy} + E_{yx} = |N|$ (the diagonal elements of E are defined to equal 0). Hence, matrix E tabulates the results of all head-to-head contests between alternatives under truthful preferences. Now, denote by E' the altered matrix w.r.t. to E, in which agent i misreports her true preferences $R_i = abc$ by submitting $R'_i \neq R_i$. We have the following five possibilities:

(I) $R'_i = bac$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca}, E'_{cb} = E_{cb}$;

(II) $R'_i = bca$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca} + 1, E'_{cb} = E_{cb}$;

(III) $R'_i = acb$, implying $E'_{ab} = E_{ab}$, $E'_{ca} = E_{ca}, E'_{cb} = E_{cb} + 1$.

(IV) $R'_i = cba$, implying $E'_{ab} = E_{ab} - 1$, $E'_{ca} = E_{ca} + 1, E'_{cb} = E_{cb} + 1$.

(V) $R'_i = cab$, implying $E'_{ab} = E_{ab}$, $E'_{ca} = E_{ca} + 1, E'_{cb} = E_{cb} + 1$.

Now, since $g(R_N) = cab$ and $g(R'_i, R_{-i}) = bac$, it must be the case that:

$$E_{ca} + E_{cb} + E_{ab} \geq E_{ac} + E_{bc} + E_{ba} \tag{9}$$
$$E'_{ca} + E'_{cb} + E'_{ab} \leq E'_{ac} + E'_{bc} + E'_{ba}. \tag{10}$$

Given agent i's five possible modifications to matrix E listed above, the only way that Eqs. (9)-(10) do not lead to a contradiction is if either case (I) or (II) applies.[21] If case (II) applies then we must have $E_{ca} + E_{cb} + E_{ab} = E'_{ca} + E'_{cb} + E'_{ab}$ in turn implying that both Eqs. (9)-(10) are equalities. But then we cannot have $g(R_N) = cab$ and $g(R'_i, R_{-i}) = bac$ (this would imply that $cab \succeq bac \succeq cab$, a contradiction).

Thus it must be that case (I) applies. Since $g(R_N) = cab$ we must have $E_{ab} \geq E_{ba}$ (otherwise, $f(R_N) \neq cab$ because ordering cba would have better Kemeny performance for profile R_N). For similar reasons, we must also have $E_{ca} + E_{cb} \geq E_{ac} + E_{bc}$, and $E_{ca} \geq E_{ac}$.

We now distinguish between two cases:

1. $E_{ca} + E_{cb} > E_{ac} + E_{bc}$. In this case we cannot have $bac \neq f(R'_i, R_{-i})$, since ordering cba would have a better Kemeny score for profile (R'_i, R_{-i}).

2. $E_{ca} + E_{cb} = E_{ac} + E_{bc}$. Here, suppose first that $E_{ca} > E_{ac}$. Then we cannot have $bac \in K(R'_i, R_{-i})$ since bca would have better Kemeny performance for profile (R'_i, R_{-i}). Hence, it must be that $E_{ac} = E_{ca}$ implying $E_{cb} = E_{bc}$. Thus, $|N|$ must be even. If $E_{ab} = E_{ba}$, then $\succeq$ must rank cab first, and bac before cba or bca. If $E_{ab} = E_{ba} + 2$, then $\succeq$ must rank bac first, and cab before abc or acb. In either case, the ordering $\succeq$ is not regular.

Now, suppose that $\succeq$ fails regularity. For ease of exposition and without loss of generality, suppose the first-ranked ordering of $\succeq$ is cab and $bac \succeq cba$ and $bac \succeq bca$ and consider the associated $\succeq$-Condorcet-Kemeny rule (call it g). Construct a profile R_N such that $E_{ac} = E_{ca}$, $E_{cb} = E_{bc}$, and $E_{ab} = E_{ba}$ and where there exists an agent i with preferences $R_i = abc$. We will have $g(R_N) = cab$. Suppose this agent misreports by submitting $R'_i = bac$, leading to $g(R'_i, R_{-i}) = bac$ and implying that the rule is not K-SP.

Now we address part (2). Suppose g is a $\succeq$-fixed-benchmark rule. Without loss of generality, suppose that $R_i = abc$ and agent i can profitably Kemeny misreport. Since f satisfies Btw-SP [5], the only way that K-SP can be violated is if $f(R_N) = cab$ and

[21] Recall that pairs of elements symmetric to the main diagonals of E and E' must sum to $|N|$.

$f(R_i', R_{-i}) = bac$ or $f(R_N) = bca$ and $f(R_i', R_{-i}) = acb$. Suppose that the former case holds (the latter is handled with a similar argument).

Since f satisfies Btw-SP, we will have $f(R_i', R_{-i}) = bac \Rightarrow f(bac, R_{-i}) = bac$. Now distinguish between the following two cases:

(i) $bac \succeq cab$. Here, $f(R_N) = cab$ implies that bac violated local unanimity in profile R_N. Since $(b, a) \notin R_i$, this means that either $(c, b) \in \bigcap_{i \in N} R_i$ or $(c, a) \in \bigcap_{i \in N} R_i$ or both. But this contradicts $R_i = abc$.

(ii) $cab \succeq bac$. Here, $f(R_N) = cab$ implies that there exist $j, k \neq i$ such that $(c, a) \in R_j$ and $(c, b) \in R_k$. Since $f(bac, R_{-i}) \neq cab = f(abc, R_{-i})$, this means that $bac \cap \bigcap_{j \neq i} R_j = \{(b, a)\}$ and $abc \cap \bigcap_{j \neq i} R_j = \emptyset$. Hence, ordering cab is ranked first and $f(bac, R_{-i}) = bac$ implies that $bac \succeq bca$ and $bac \succeq cba$. Thus, $\succeq$ is not regular.

Now, suppose we have a $\succeq$-fixed-benchmark rule, call it f, such that $\succeq$ is not regular. For ease of exposition, and without loss of generality, suppose that $\succeq$ ranks cab first and bac before both bca and cba. Consider now the profile R_N, where $R_i = abc$, $(b, a) \in R_j$ for all $j \neq i$ and $\bigcap_{l \in N} R_l = \emptyset$. Then $g(R_N) = cab$. Now, suppose agent i misreports and submits $R_i' = bac$, leading to $bac \cap \bigcap_{j \neq i} R_j = \{(b, a)\}$. The ordering $\succeq$ ensures that $g(R_i', R_{-i}) = bac$ leading to a violation of K-SP. $\square$

Proof of Theorem 6: Consider two agents and three alternatives, say $N = \{1, 2\}$ and $A = \{a, b, c\}$.

Consider the opposite profile $R_N = (R_1, R_2) = (abc, cba)$. By preference selection, $f(R_N) \in \{R_1, R_2\}$, say $f(R_N) = cba$. Now by K-SP and preference selection, $f(abc, bca) = bca$. Again by K-SP and preference selection we have $f(acb, bca) = bca$. Now applying the same arguments repetitively we obtain $f(cba, abc) = abc$ which is now a contradiction to anonymity (as $f(abc, cba) = cba$).

Now for arbitrary number m of alternatives, we may enlarge the above profiles by letting all agents rank the same m-3 alternatives in the same order at the bottom and by preference selection $f(R_N)$ has to rank those alternatives at the bottom with the same ranking. But then we can do the same arguments as above.

For an arbitrary number n of agents, if n is even then half of the agents play the role of agent 1 and half of the agents play the role of agent 2 and this induces a K-SP, preference selection and anonymous rule, a contradiction to the above. $\square$

Before proceeding with the proof of Theorem 7, we consider the special case of three alternatives and three agents and show Proposition 4.

Proof of Proposition 4: For part (i) it is easy to verify that the described rules satisfy all the properties.

For the other direction, we first show $f(abc, cba, cba) = cba$. If $f(abc, cba, cba) \neq cba$, then by preference selection, $f(abc, cba, cba) = abc$. Then by neutrality, $f(cba, abc, abc) = cba$. Now using both preference selection and K-SP it can be shown that $f(bac, cab, cab) = bac$, $f(acb, bca, bca) = acb$ and f is dictatorial with dictator 1, i.e. $f(R_N) = R_1$ for all $R_N \in \mathcal{R}^N$, a contradiction. Hence, $f(abc, cba, cba) = cba$. Similarly, we obtain $f(cba, abc, cba) = cba$ and $f(cba, cba, abc) = cba$. Using preference selection and K-SP we then obtain for any preference $R \in \mathcal{R}$ and any profile $R_N \in \mathcal{R}^N$ such that for $N = \{i, j, k\}$ we have $R_i = R_j = R$ and $R_k = -R$, $f(R_N) = R$. Furthermore, by preference selection $f(abc, cab, bca) \in \{abc, cab, bca\}$, say $f(abc, cab, bca) = abc$, and then by neutrality for all $R_N \in \triangle'$, $f(R_N) = R_1$, and similar for the triangular profiles in $\triangle''$. Now it is easy to see that f is a median rule with agent-based tie-breaking.

For part (ii) it is easy to verify that the described rules satisfy all the properties.

For the other direction, note that by anonymity, for any $R_N, R'_N \in \hat{\triangle}'$, $f(R_N) = f(R'_N) \equiv \hat{R}'_0$, and for any $R_N, R'_N \in \hat{\triangle}''$, $f(R_N) = f(R'_N) \equiv \hat{R}''_0$. For any $R \in \mathcal{R}^N \setminus \triangle$, if the median is not chosen, then by preference selection and anonymity, say $f(R) = R_1 \neq R_2, R_3$ and $R_1 \notin [R_2, R_3]$. Using K-SP and preference selection, then without loss of generality, $f(abc, acb, cab) = abc$. Applying K-SP and preference selection, we obtain $f(bac, acb, cab) = bac$ and $f(bac, acb, cba) = bac$. On the other hand we may use K-SP and preference selection repetitively (by moving agent 1 first and then agents 2 and 3) to obtain $f(cab, cba, bca) = cab$. But now $f(acb, cba, bca) = acb$ and by K-SP, $f(acb, cba, bac) \neq bac$. This is a contradiction to anonymity as $f(bac, acb, cab) = bac$. □

Proof of Theorem 7: Let $N = \{1, 2, 3\}$ and $A = \{a, b, c, d\}$. Suppose that f satisfies K-SP, preference selection and anonymity. Let $\underline{f}$ defined for profiles where all agents rank d at the bottom. By Proposition 4 (ii), $\underline{f}$ must be median rule with preference-based tie-breaking.

Consider the following profile $R_N = (R_1, R_2, R_3) = (abcd, cabd, bcad)$. Without loss of generality (by preference selection), let $\underline{f}(R_N) = R_1 = f(R_N)$. Consider the profile $\hat{R}_N = (\hat{R}_1, \hat{R}_2, \hat{R}_3) = (dabc, abdc, badc)$.

Starting from profile R_N we have for $R'_1 = abdc$ by K-SP and preference selection $f(R'_1, R_2, R_3) = R'_1$ (as $\delta(R'_1, R_1) = 1$, $\delta(R'_1, R_2) > 1$ and $\delta(R'_1, R_3) > 1$). Similarly, for $R''_1 = adbc$ we obtain $f(R''_1, R_2, R_3) = R''_1$ and finally $f(\hat{R}_1, R_2, R_3) = \hat{R}_1$. Then we have

$\delta(R_3, \hat{R}_1) = 5 > \max\{\delta(R_3, R_2), \delta(R_3, \hat{R}_3)\}$, and from K-SP and preference selection we obtain $f(\hat{R}_1, R_2, \hat{R}_3) = \hat{R}_1$. Similarly, we then have $\delta(R_2, \hat{R}_1) = 5 > \max\{\delta(R_2, \hat{R}_3), \delta(R_2, \hat{R}_2)\}$, and from K-SP and preference selection we obtain $f(\hat{R}_1, \hat{R}_2, \hat{R}_3) = \hat{R}_1$. At profile $\hat{R}_N$ all agents rank c at the bottom and $f(\hat{R}_N)$ restricted to $\{a, b, d\}$ shall be the median of $\hat{R}_N$ restricted to $\{a, b, d\}$, but $f_1(\hat{R}_N) = d$ and agents 2 and 3 rank d last among $\{a, b, d\}$, a contradiction (as the median rule would require $f_3(\hat{R}_N) = d$.

Now if $|N| = 3k$ and $m \geq 4$, then k agents play the role of agent 1 (by reporting the same preference), k agents play the role of agent 2 and k agents play the role of agent 3 and we obtain a three-agents rule satisfying K-SP, preference selection and AN. By letting all agents rank the same $m - 4$ alternatives at the bottom, then we obtain a contradiction as above (since by preference selection those alternatives must be ranked at the bottom of the social ranking as well). $\square$

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Online Appendix for "Strategy-proof preference aggregation and the anonymity-neutrality tradeoff"

A Can and Storcken (2018) axiomatization of Kemeny distance

Let function $\alpha : \mathcal{R} \times \mathcal{R} \mapsto \Re$ denote the distance between two orderings. Consider now the following four conditions on α.

Condition 1 *(Metric conditions). For all $R, R', R'' \in \mathcal{R}$, we have:*

(i) Non-negativity: $\alpha(R, R') \geq 0$.

(ii) Identity of indiscernibles: $\alpha(R, R') = 0$ *if and only if* $R = R'$.

(iii) Symmetry: $\alpha(R, R') = \alpha(R', R)$.

(iv) Triangle inequality: $\alpha(R, R'') \leq \alpha(R, R') + \alpha(R', R'')$.

Condition 2 *(Betweeness). For all $R, R', R'' \in \mathcal{R}$ such that $R' \in [R, R'']$ we have* $\alpha(R, R'') = \alpha(R, R') + \alpha(R', R'')$.

Condition 3 *(Neutrality). For all $R, R' \in \mathcal{R}$ and all permutations π on A, we have* $\alpha(R, R') = \alpha(\pi R, \pi R')$.

Condition 4 *(Normalization).* $\min\{\alpha(R, R') : R \neq R'\} = 1$.

We now state the characterization of Kemeny distance by Can and Storcken [14], who in turn build on earlier work by Kemeny and Snell [21].

Theorem 1 *(Can and Storcken [14]) A distance function α satisfies Conditions 1-2-3-4 if and only α is the Kemeny distance.*

B Example of a rule that is Local K-SP but not Min-SP

Suppose $m = 4$ and consider a function $f : \mathcal{R} \mapsto \mathcal{R}$ satisfying:

$$
\begin{aligned}
&f(abcd) = adbc, \quad f(abdc) = badc, \quad f(acbd) = acdb, \quad f(adbc) = adcb \\
&f(bacd) = badc, \quad f(bdca) = bcda, \quad f(cbad) = cabd, \quad f(cdab) = dcab \\
&f(dbca) = dabc, \quad f(dcba) = cdab, \quad f(dacb) = dcab, \quad f(dbac) = dabc \\
&f(R) = R, \text{ for all other } R \in \mathcal{R}
\end{aligned}
$$

Define the rule $g : \mathcal{R}^N \mapsto \mathcal{R}$ such that for all $R_N \in \mathcal{R}^N$, we have $g(R_N) = f(R_1)$. To verify its stated properties we only need to check deviations by agent 1. To check Local K-SP it suffices to examine the profitability of adjacent deviations for agent 1 when R_1 is such that $f(R_1) \neq R_1$. There are 12*3=36 such deviations and none of them imply a violation of Local K-SP. For example, when $R_1 = abcd$, we have $g(R_N) = f(abcd) = adbc$, $\delta(R_1, g(R_N)) = 2$, and three possible local misreports: $R'_1 \in \{bacd, acbd, abdc\}$. If $R'_1 = bacd$ then $g(R'_1, R_{-1}) = badc$; if $R'_1 = acbd$ then $g(R'_1, R_{-1}) = acdb$; and if $R'_1 = abdc$ then $g(R'_1, R_{-1}) = badc$. In all three cases, the Kemeny distance between R_1 and $g(R'_1, R_{-1})$ is exactly equal to 2.

On the other hand, if $R_1 = adbc$, and agent 1 misreports $R'_1 = abcd$, then $g(R_N) \neq R_1$ and $g(R'_1, R_{-1}) = R_1$, in violation of Min-SP. Note that $\delta(R_1, R'_1) = 2$, so that this violation of Min-SP has no bearing on Local K-SP. Note, finally, that the rule g is not onto (e.g., there is no profile R_N such that $g(R_N) = abcd$).

C Integer programming formulation and application to $n = 2, m = 4$ case

A rule f is a function $f : \mathcal{R}^N \mapsto \mathcal{R}$, determining for every profile a social ordering. If a rule f, when applied to profile R_N, ranks a_i before a_j then we write $a_i f(R_N) a_j$. Equivalently, we may introduce the binary variables $d^f_{R_N}(a_i, a_j) \in \{0, 1\}$ and require that :

$$d^f_{R_N}(a_i, a_j) = 1 \Leftrightarrow a_i f(R_N) a_j.$$

In this way we can express f by associating with it a vector $\boldsymbol{d}^f \equiv \left[d^f_{R_N}(x, y)\right]$ where R_N ranges throughout $\mathcal{R}^N$ and $(a_i, a_j) \in A \times A$ with $i \neq j$. As a result $\boldsymbol{d}^f$ is a vector of dimension $(m!)^n \cdot m(m-1)$ that must satisfy the binary integer requirement $d^f_{R_N}(a_i, a_j) \in \{0, 1\}$ for all profiles R_N and pairs of alternatives (a_i, a_j).[22]

Each rule f is uniquely defined by its corresponding $\boldsymbol{d}$ vector. For instance, take a dictatorial rule which, no matter what other agents submit, always selects agent 1's ordering. This rule is described by a $\boldsymbol{d}$ vector such that for all profiles R_N and pairs (a_i, a_j) we have $d_{R_N}(a_i, a_j) = 1$ if and only if $(a_i, a_j) \in R_1$.

We define the rule properties that were discussed in the main text, each time translating them to the integer-vector framework mentioned above.[23]

[22] In what follows for convenience we drop the superscript f from the vector $\boldsymbol{d}^f$.

[23] Completeness and Transitivity are explicitly listed because the integer-vector formulation does not a priori ensure they hold.

(C) **Completeness**. For all profiles $R_N \in \mathcal{R}^N$ and pairs of alternatives $(a_i, a_j) \in A \times A$ and $i \neq j$, either $a_i f(R_N) a_j$ or $a_j f(R_N) a_i$.

Equivalently, for all $R_N \in \mathcal{R}^N$:

$$d_{R_N}(a_i, a_j) + d_{R_N}(a_j, a_i) = 1, \;\; \forall (a_i, a_j) \in A \times A, \; i \neq j. \tag{11}$$

(T) **Transitivity**. For all profiles $R_N \in \mathcal{R}^N$ and triplets of alternatives $(a_i, a_j, a_k) \in A \times A \times A$, if $a_i f(R_N) a_j$ and $a_j f(R_N) a_k$, then $a_i f(R_N) a_k$.

Equivalently for all $R_N \in \mathcal{R}^N$:

$$d_{R_N}(a_i, a_j) + d_{R_N}(a_j, a_k) + d_{R_N}(a_k, a_i) \leq 2, \;\; \forall (a_i, a_j, a_k) \in A \times A \times A, \; i \neq j, k, j \neq k. \tag{12}$$

(A) **Anonymity**. Consider the set of permutations on N, and call it Σ_N. For any element $\sigma \in \Sigma_N$ and profile R_N, let $\sigma(R_N)$ denote the profile in which agent 1 is given the ordering of agent $\sigma(1)$, agent 2 of $\sigma(2)$ and so on. I.e., $\sigma(R_N) = (R_{\sigma(1)}, R_{\sigma(2)}, ..., R_{\sigma(n)})$.

Anonymity requires that for all $R_N \in \mathcal{R}^N$ we have:

$$d_{R_N}(a_i, a_j) = d_{\sigma(R_N)}(a_i, a_j), \;\; \forall \sigma \in \Sigma_N, \; (a_i, a_j) \in A \times A, \; i \neq j. \tag{13}$$

(N) **Neutrality**. Consider now the set of permutations on A, call it Π_A. For any element $\pi \in \Pi_A$ and profile R_N, let $\pi(R_N)$ denote the profile in which agents' rankings are rewritten in a way that alternative a_1 is now named $\pi(a_1)$, alternative a_2 is named $\pi(a_2)$ and so on. That is, $\pi(R_N) = (\pi(R_1), \pi(R_2), ..., \pi(R_n))$, where $\pi(R_k) = \{(\pi(a_i), \pi(a_j)) : \; \forall (a_i, a_j) \in R_k)\}$. Neutrality requires that for all $R_N \in \mathcal{R}^N$ we have:

$$d_{R_N}(a_i, a_j) = d_{\pi(R_N)}(\pi(a_i), \pi(a_j)), \;\; \forall \pi \in \Pi_A, \; (a_i, a_j) \in A \times A, \; i \neq j. \tag{14}$$

(U) **Unanimity**. If all agents submit the same ordering then the rule also assigns this ordering.

Equivalently, for all $R_N \in \mathcal{R}_N$ and $R \in \mathcal{R}$:

$$\{R_k = R, \; \forall k = 1, 2..., n\} \; \Rightarrow \; \{d_{R_N}(a_i, a_j) = 1, \;\; \forall (a_i, a_j) \in R\}. \tag{15}$$

(LU) **Local Unanimity**. If all agents prefer an alternative a_i to a_j then the rule must also respect this binary comparison.

Equivalently, for all $R_N \in \mathcal{R}^N$ and $(a_i, a_j) \in A \times A$:

$$\{a_i R_k a_j, \; \forall k = 1, 2, ..., n\} \; \Rightarrow \; d_{R_N}(a_i, a_j) = 1. \tag{16}$$

Clearly, LU implies the U property above.

(K-SP) **Kemeny strategy-proofness**. Consider a profile $R_N = (R_1, R_2, ..., R_n)$ and within this profile an agent k with preferences R_k. Denote by R_{-k} the preferences of all other agents, i.e., $R_{-k} = (R_1, R_2, .., R_{k-1}, R_{k+1}, ..., R_n)$. Suppose now agent k misreports his preferences by submitting an ordering $R'_k \neq R_k$. This will give rise to a profile $(R'_k, R_{-k}) \equiv (R_1, R_2, .., R_{k-1}, R'_k, R_{k+1}, ..., R_n)$.

K-SP requires that for all $R_N \in \mathcal{R}^N$ and $k \in N$ all such misreports not be profitable so that:

$$\sum_{(a_i,a_j)\in R_k} d_{R_N}(a_i,a_j) \geq \sum_{(a_i,a_j)\in R_k} d_{(R'_k,R_{-k})}(a_i,a_j), \ \ \forall R'_k \neq R_k. \tag{17}$$

The Integer Program. Set $A = \{a,b,c,d\}$ and $N = \{1,2\}$. We wish to explore whether there exists a rule satisfying Anonymity, Local Unanimity and K-SP.

Using an integer-programming framework, this problem can be reformulated in the following way: Does there exist a 0-1 $\boldsymbol{d}$ vector that simultaneously satisfies (C)-(T)-(A)-(LU)-(K-SP)? Since $m = 4, n = 2$, the associated $\boldsymbol{d}$ vector will have dimension $(4!)^2 * 4 * 3 = 6912$. To answer this question, we set up an integer program. One particularly simple way of doing this is to we introduce a scalar variable $y \geq 0$ and define problem $\mathcal{P}$:

$$\begin{aligned}
\mathcal{P} = \min_{\mathbf{d},y} \quad & y \\
\text{s.t.} \quad & \text{(C)-(T)-(A)-(LU)} \\
& \sum_{(a_i,a_j)\in R_k} d_{R_N}(a_i,a_j) - \sum_{(a_i,a_j)\in R_k} d_{(R'_k,R_{-k})}(a_i,a_j) + y \geq 0, \ \ \forall R_N \in \mathcal{R}^N, \ k \in N, \ R'_k \neq R_k. \\
& \mathbf{d} \in \{0,1\}, \ y \geq 0.
\end{aligned}$$

If $\mathcal{P}$ yields an optimal solution $(\boldsymbol{d}^*, y^*)$ with $y^* > 0$, we will know that anonymity, local unanimity and K-SP cannot be simultaneously satisfied. Conversely, if $y^* = 0$ then the vector $\boldsymbol{d}^*$ defines a rule that satisfies the desired properties. It is thus clear that finding an optimal solution for $\mathcal{P}$ is equivalent to resolving the existence of an anonymous, locally unanimous, K-SP rule.

So our goal is to solve mixed-integer program $\mathcal{P}$ or closely-related variants thereof. To this end, we use Matlab and its built-in mixed-integer linear programming solver **intlinprog** to solve a more involved version of $\mathcal{P}$ in which there is a non-negative variable y_h for each K-SP constraint h,[24] and a linear objective function $\boldsymbol{c} \cdot \boldsymbol{y}$, where $\boldsymbol{c} > \boldsymbol{0}$. We do this for computational reasons, as the simpler model seemed to be yielding suboptimal results.

$$\begin{aligned}
\mathcal{P}' = \min_{\mathbf{d},\boldsymbol{y}} \quad & \boldsymbol{c} \cdot \boldsymbol{y} \\
\text{s.t.} \quad & \text{(C)-(T)-(A)-(LU)} \\
& \sum_{(a_i,a_j)\in R_k} d_{R_N}(a_i,a_j) - \sum_{(a_i,a_j)\in R_k} d_{(R'_k,R_{-k})}(a_i,a_j) + y_h \geq 0, \ \ \forall R_N \in \mathcal{R}^N, \ k \in N, \ R'_k \neq R_k. \\
& \mathbf{d} \in \{0,1\}, \ \ \boldsymbol{y} \geq \boldsymbol{0}.
\end{aligned}$$

Problem $\mathcal{P}'$ shares the appealing properties of $\mathcal{P}$: the existence of an anonymous, locally unanimous, K-SP rule is equivalent to an optimal solution $(\boldsymbol{d}^*, \boldsymbol{y}^*)$ of $\mathcal{P}'$ such that $\boldsymbol{y}^* = \boldsymbol{0}$. After a few iterations and various modifications of vector $\boldsymbol{c}$, the solver converges to the rule discussed in the text and listed in the Excel

[24]There are $(4!)^2 * 2 * 23 = 26496$ such constraints.

file, and we have a proof of existence.

Below we give the complete formal definition of the rule found as a solution to the above problem. First, we define fixed-status-quo rules with tie-breaking for three alternatives.

Definition 10 *Let $N = \{1, 2\}$ and $A = \{a_1, a_2, a_3\}$. Fix a status-quo ordering R_0 and denote by $r_k(R)$ the k'th-ranked alternative of $R \in \mathcal{R}$. The R_0***-fixed-status-quo rule with tie-breaking***, SQ^{R_0}, is the anonymous rule such that for all $R_N = (R_1, R_2) \in \mathcal{R}^N$:*

(i) if R_0 is locally unanimous for R_N, then $SQ^{R_0}(R_N) = R_0$;

(ii) if R_0 is not locally unanimous for R_N and $R_1 \in [R_0, R_2]$, then $SQ^{R_0}(R_N) = R_1$;

(iii) if R_0 is not locally unanimous for R_N and $R_2 \in [R_0, R_1]$, then $SQ^{R_0}(R_N) = R_2$; and

(iv) otherwise, we have $R_1 \neq R_2$ and $\delta(R_1, R_0) = \delta(R_2, R_0) = 2$, and we set $SQ^{R_0}(R_N) = R_i$, where $i \in N$ is such that $r_3(R_i) = r_1(R_0)$.[25]

It is obvious that the above rules satisfy local unanimity and anonymity, and one can also check K-SP.

Second, for four alternatives we provide below a formal description of the anonymous, locally unanimous and K-SP rule.

Definition 11 *Let $N = \{1, 2\}$ and $A = \{a, b, c, d\}$. Specify $a \in A$ as the losing alternative. For all non-losing alternatives $x \in \{b, c, d\}$, introduce the orderings $\underline{R}_0^x$ and $\overline{R}_0^x$ as follows:*

$$\begin{aligned} \underline{R}_0^b &= dca, \quad \overline{R}_0^b = cda \\ \underline{R}_0^c &= bda, \quad \overline{R}_0^c = dba \\ \underline{R}_0^d &= cba, \quad \overline{R}_0^d = bca. \end{aligned}$$

Let f be an anonymous rule such that, for any profile $R_N = (R_1, R_2) \in \mathcal{R}^N$,

(I) if $R_1 \cap R_2 = \{(b, a), (c, a), (d, a)\}$, then there are three kinds of possible profiles[26] *and the rule sets:*

$$f(bcda, dbca) = cdba, \quad f(bdca, cdba) = dbca, \ f(cbda, dbca) = bcda;$$

(II) if $R_1 \cap R_2 = \{(a, b), (a, c), (a, d)\}$, then there are three kinds of possible profiles and the rule sets:

$$f(abcd, adcb) = abdc, \quad f(acbd, adbc) = adbc, \ f(acdb, abdc) = acbd;$$

(III) if $R_1 \cap R_2 \cap \{(a, b), (a, c), (a, d)\} = \{(a, x)\}$ (where $x \in \{b, c, d\}$), then

$$f(R_1, R_2) = SQ^{\underline{R}_0^x}(R_{1|A\setminus\{x\}}, R_{2|A\setminus\{x\}})\, x,$$

i.e., the non-losing alternative x is placed at the bottom of the social ordering and is preceded by the outcome of the $\underline{R}_0^x$-fixed-order-status-quo rule applied to profile $(R_{1|A\setminus\{x\}}, R_{2|A\setminus\{x\}})$;

[25] To illustrate part (iv) of the definition, if $R_0 = bca$, then the R_0-fixed-status-quo rule with tie-breaking is given by $SQ^{R_0}(abc, cab) = cab$.

[26] Recall that the rule is anonymous.

(IV) if $R_1 \cap R_2 \cap \{(a,b),(a,c),(a,d)\} = \{(a,y),(a,z)\}$ (where $\{b,c,d\} = \{x,y,z\}$), then

$$f(R_1, R_2) = x\ SQ^{\overline{R}_0^x}(R_{1|A\setminus\{x\}}, R_{2|A\setminus\{x\}}),$$

i.e., the non-losing alternative x is placed at the top of the social ordering and is succeeded by the outcome of the $\overline{R}_0^x$-fixed-status-quo rule applied to profile $(R_{1|A\setminus\{x\}}, R_{2|A\setminus\{x\}})$;

(V) if $R_1 \cap R_2 = \emptyset$, then there are twelve possible profiles and the rule sets:

$$\begin{aligned}
&f(abcd, dcba) = cdba,\ f(abdc, cdba) = dbca,\ f(acbd, dbca) = bcda,\ f(acdb, bdca) = dbca\\
&f(adbc, cbda) = bcda,\ f(adcb, bcda) = cdba,\ f(bacd, dcab) = cdba,\ f(badc, cdab) = dbca\\
&f(bcad, dacb) = cdba,\ f(bdac, cadb) = dbca,\ f(cabd, dbac) = bcda,\ f(cbad, dabc) = bcda.
\end{aligned}$$

The rule's output for any other profile is given in the excel-file but can also be derived from the fact that losing alternative a is always placed as low as possible subject to local unanimity, from points (I)-(II)-(III)-(IV)-(V), and from properties anonymity, local unanimity and K-SP of the rule.