DEMS Department of Economics, Management and Statistics

DEMS WORKING PAPER SERIES

Dynamical analysis of healthcare policy effects in an integrated economic-epidemiological model

Fausto Cavalli, Ahmad Naimzada, and Daniela Visetti

No. 521 – June 2023

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

Dynamical analysis of healthcare policy effects in an integrated economic-epidemiological model

Fausto Cavalli*[1], Ahmad Naimzada†[1], and Daniela Visetti‡[2]

[1]Department of Economics, Management and Statistics, University of Milano Bicocca, Center for European Studies (CefES),University of Milano Bicocca
[2]Department of Economics, Management and Statistics, University of Milano Bicocca

Abstract

We study the static and dynamical properties of a model that describes the interaction between the economic and epidemiological domains. The epidemiological sphere is represented by a susceptible-infected-susceptible model, while the economic domain consists of an overlapping generations model in which the workers correspond to the non infected population of adults. The productivity of the firms and the propensity to save for retirement of the households are negatively affected by the disease spread.

A capital tax is levied and the collected resources are used to curb the spread of the outbreak. We show that multiple endemic steady states can arise from the interaction between the two domains, and different stable endemic attractors can coexist with the stable disease free steady state. We study analytically and numerically the complex dynamics and the evolution of the basins of attraction in the case of multistability. We show that the effect of taxation can be beneficial both from the epidemiological and the economic points of view, as it can give rise to new steady states characterized by reduced shares of infected people and increased capital level, it can simplify the dynamical behaviors and reduce the size of the basins of attraction of those outcomes in which large shares of infected people and low capital levels are observed.

*fausto.cavalli@unimib.it
†ahmad.naimzada@unimib.it
‡daniela.visetti@unimib.it

1 Introduction

Over the last few years, various events are having a significant impact on the trend in economic growth. The outbreak of the recent COVID-19 pandemic has required the management of a global health emergency and the consequent costs to the global economy. The challenges posed by environmental issues, climate change and pollution require tackling the problems of sustainable economic growth and effective ecological transition.

There is growing evidence that the economic, epidemiological and environmental spheres cannot be regarded as independent. The trend of a disease spread and the capability to recover from it essentially depend on the planned investments in healthcare, on their timeliness and effectiveness. Likewise, lockdowns and restrictions imposed by an out-of-control health situation have a highly negative impact on economic performance. Similar considerations may also be made concerning the interaction between the economic and environmental spheres. In addition to this, it should not be overlooked that the epidemiological and environmental aspects also influence each other (just think of how the environmental factors can favor the spread of viral agents).

Moreover, it must be emphasized how these interplays are characterized by an inherently dynamical nature. We can observe cyclical trends, consecutive waves, peaks and falls, and very irregular dynamics. In addition, seemingly similar contexts may evolve in very different ways, making the initial conditions relevant to the study and prediction of phenomena.

The literature has been studying the phenomena of interaction between different spheres for some time. Without claiming to be exhaustive, concerning the interaction between the economic and environmental domains, we can mention the seminal work by John and Pecchenino [19], in which the conflict between the economic growth and the quality of the environment is studied through an overlapping generations model (OLG). This gave rise to an outgrowing research strand, see for example [23, 16, 22, 8]. The researches about health and economic growth mainly ground on two approaches for the economic sphere modelling, namely OLG and Solow growth models. Concerning the OLG modelling, we can mention the contributions by Chakraborty [9], who studied the effect of considering an endogenous probability to survive, and by Gori et al. [18], who considered a Diamond OLG model coupled with a dynamical equation of infection prevalence. Concerning Solow modelling, we can refer to the contribution by Carmona and Leon [7], who studied the effect of a disease on the economic growth by encompassing in a continuous time Solow model either a Susceptible-Infected-Susceptible (SIS) or a Susceptible-Infected-Recovered (SIR) model for the epidemiological side. Conversely, Bischi et al. [3], considered the time discrete growth model and described the health domain through an SIS model. Finally, Goenka and Liu [17] considered the framework in Lucas [21] in which they included the dynamics of disease transmission. For contributions strictly related to the COVID-19 pandemic we also mention [15, 2], while for surveys, we refer the interested reader to [4, 6, 13]. Finally, a model in which the interaction among the economic, epidemiological and environmental sphere has been considered is studied by Davin et al. [11], in which the role of the public debt was taken into account as well.

However, in the aforementioned literature, except in a few cases, the dynamical aspects have been completely neglected or the investigations have gone as far as stability issues, without studying what can characterize out-of-equilibrium dynamics. The purpose of this contribution is to start placing the attention on the dynamical complexity that characterizes the phenomena of interaction between the different spheres. To this end, we focus on a seemingly simple framework, in which the economic domain, described by an OLG model, interacts with the epidemiological one, outlined by an SIS model. The baseline resulting model is essentially that in [11], in which, in this first step of our research path, we neglect the interaction with the environmental domain and we assume no new public debt is created. To the best of our knowledge, this is the first time that the dynamics of coupled OLG-SIS model are investigated. Even in this simplified framework, the model presents a wide range of interesting results, both statically and dynamically, which deserve to be studied autonomously and differ from those in [11]. We stress that, as in [11], we do not have in mind a particular epidemic spread, but we just aim to address the problem from a theoretical point of view, without the goal to provide empirical comparison of the dynamical outcomes we obtain. Along the lines of [11], the epidemiological domain affects the economic one both with regard to the firms (by reducing the size of the labour force and the effectiveness of the production process) and to the households (by affecting the agent preferences through the probability to survive at the old age). The converse interaction is driven by the taxation of the production, which is used to support health policies.

We investigate the resulting model at different levels. Firstly, we study it from a general perspective, to understand what are the possible static and dynamical outcomes. The main results concern the possibility to have multiple coexisting steady states[1] characterized by a positive fraction of infected people (endemic steady states), together with a potentially stable steady state in which no agent is infected (disease free steady state). Moreover, steady states can become unstable by means of a flip bifurcation. We also provide conditions that regulate the behavior of economic and epidemiological observables (capital and share of susceptible agents) on increasing the taxation level. Subsequently, we focus on two cases of study to better illustrate

[1] We note that this is quite different from what happens in [11], as when the environmental domain is considered, just one endemic steady state is possible.

the interpretation of the results. In the former one, we consider exogenous total productivity factor and household preferences. This allows focusing on the effects of the reduction of the disease spread thanks to taxation, and we show that a suitable level of taxation is beneficial from the economic and epidemiological perspectives. Furthermore, rising taxes has a stabilizing effect on potentially chaotic dynamics. Even if it can increase the complexity of the static scenario by introducing multiple endemic steady states, the new steady states introduced by taxation are preferable, in the sense that they are characterized by reduced levels of infection and possibly increased capital. Moreover, those favored steady states have increasingly large basins of attractions as the taxation increases, so they more likely realize. In the latter case of study, we consider endogenous productivity factor and preferences, showing that, even if this can introduce further elements of complexity, the main outcomes are confirmed.

The remainder of the contribution is organized as follows. In Section 2 we introduce the baseline model, which is studied in Section 3. In Section 4 we discuss two cases of study. Finally, in Section 5 we report conclusions and some possible research lines. In Appendix we collect the proofs of the propositions.

2 The model

The model we study encompasses two domains, the epidemiological one, described by a Susceptible-Infected-Susceptible (SIS) model, and the economic one, consisting of an Overlapping Generations (OLG) model. The model we consider is basically a reduced version of the one proposed by Davin et al. in [11], in which the environmental domain is included as well. For the reader's sake, we detail the baseline model, even if it can be obtained along the lines of that in Davin et al. in [11].

For both domains, we consider a constant in time population formed by $2N$ individuals, which are divided into two fixed groups consisting of adults and elderly people. In what follows, we discuss the dynamical equation modelling each domain.

Epidemiological domain

The epidemiological domain is described by an SIS model. At each discrete time t, adults and elderly are divided into susceptible and infected people, whose numbers are identified by S_t and I_t, respectively. We have that $0 \leq S_t, I_t \leq N$ and $S_t + I_t = N$. We assume that susceptibility to infection is independent of age, so that S_t and I_t are the same for both adults and elderly individuals. The classic SIS model can be written in the following way:

$$\begin{cases} S_{t+1} = S_t \left(1 - \theta \frac{I_t}{N}\right) + \gamma I_t, \\ I_{t+1} = (1-\gamma) I_t + \theta \frac{I_t}{N} S_t, \\ S_0, I_0 > 0, \quad S_0 + I_0 = N, \end{cases} \tag{1}$$

where $\theta > 0$ is the contact rate, and $0 < \gamma \leq 1$ is the recovery rate. We recall that condition $\theta < (1+\sqrt{\gamma})^2$ (see [1]) is required to have $0 \leq S_t, I_t \leq N$ for any t.

Since the population is constant, we can rephrase model (1) by setting $s_t = \frac{S_t}{N}$ and $i_t = \frac{I_t}{N}$, so that we obtain:

$$\begin{cases} s_{t+1} = s_t(1 - \theta i_t) + \gamma i_t, \\ i_{t+1} = (1-\gamma) i_t + \theta i_t s_t, \\ s_0, i_0 > 0, \quad s_0 + i_0 = 1. \end{cases} \tag{2}$$

In the classic SIS model the contact rate is exogenous, while in the present model we take into account public health policies to control the epidemic. To this end, we assume that the contact rate $\theta : [0, +\infty) \to (0, \gamma), g_t \mapsto \theta(g_t)$ is a positive[2], strictly decreasing C^1 function of the public expenditures per adult $g_t = \frac{G_t}{N}$, where G_t denotes the health expenditures.

More precisely, we assume that the contact rate decreases with respect to the government expenditure, so that for given fraction of susceptible and infectious people, the rate of new infections $\theta(g_t)s_t i_t$ is increasingly smaller thanks to the government interventions. Model (2) can be described by considering variable s_t alone, so the dynamics of the epidemiological domain correspond to

$$s_{t+1} = s_t(1 - \theta(g_t) i_t) + \gamma i_t. \tag{3}$$

Since θ is decreasing, its maximum value is attained at $g_t = 0$, and value $\theta(0)$ will be used as the benchmark situation without government intervention.

Economic domain

The economy is described by an OLG model with production à la Diamond (see e.g. [12]). Every adult is endowed with a logarithmic utility function

$$u(c_t, d_{t+1}) = \ln c_t + \beta(s_{t+1}) \ln d_{t+1}, \tag{4}$$

where c_t and d_{t+1} are, respectively, the consumption in the present period and the consumption in the next period, as a retired person, while, according to Davin et al. in [11], $\beta : [0, 1] \to (0, 1]$

[2] We require that θ is always strictly positive for any government expenditure, i.e. that it is not possible to have a null contact rate, which would mean that for suitably large interventions, the disease disappears. We stress that such a scenario would be quite extreme, so we avoid discussing it, and, in any case, assuming $\theta(g) > 0$ does not prevent the occurrence of disease-free steady states.

represents the probability of staying healthy and correspondingly the willingness to save money for the old age (see also [9]). Function β is C^1, increasing ($\beta'(x) \geq 0$ for any $x \in [0,1]$) and concave, so that the smaller the share of infectious people during period $t+1$ is, the larger the discount factor is, with decreasing marginal returns. Consumption choices depend on usual budget constraints.

The sum of the savings σ_t and the consumption c_t of each adult must be equal to the labour income Ω_t. The labour income is $\Omega_t = w_t$, the wage paid by the firm, if the person is healthy, while $\Omega_t = 0$, if the person is infected. A second budget constraint links the savings of one period to the consumption in the next one, where r_{t+1} is the marginal productivity of capital. The budget constraints are given by the following equations:

$$\begin{aligned} \sigma_t + c_t &= \Omega_t, \\ d_{t+1} &= \frac{r_{t+1}}{\beta(s_{t+1})}\sigma_t. \end{aligned} \tag{5}$$

Adults maximize the utility function u subject to the budget constraints (5), and this provides

$$\begin{aligned} c_t &= \frac{\Omega_t}{1+\beta(s_{t+1})}, \\ \sigma_t &= \frac{\beta(s_{t+1})}{1+\beta(s_{t+1})}\Omega_t, \\ d_{t+1} &= \frac{r_{t+1}}{1+\beta(s_{t+1})}\Omega_t. \end{aligned} \tag{6}$$

Firms produce an output Y_t according to $Y_t(L_t, K_t) = A(s_t)L_t^{1-a}K_t^a$, where L_t is the labour, K_t is the capital and $a \in (0,1)$. Function $A : [0,1] \to (0,1]$ represents the factor productivity and takes the value $A(1) = 1$, i.e. we normalize the factor productivity when there are no infected people. It is an increasing C^1 function, and it encompasses the productivity fall due to a large share of infected people (for a discussion on this, we refer to [11, 10]).

Since the government taxes production at a constant rate $\tau \in [0,1]$, firms maximize the profit

$$\pi(L_t, K_t) = (1-\tau)Y_t(L_t, K_t) - r_tK_t - w_tL_t,$$

from which we obtain

$$\begin{aligned} r_t &= a(1-\tau)A(s_t)\left(\frac{L_t}{K_t}\right)^{1-a}, \\ w_t &= (1-a)(1-\tau)A(s_t)\left(\frac{K_t}{L_t}\right)^{a}. \end{aligned}$$

If we define the capital per adult $k_t = \frac{K_t}{N}$ and we assume that every adult and healthy agent works, so that at equilibrium one has $L_t = S_t$, the previous equations can be written as

$$r_t = r(s_t, k_t) = a(1-\tau)A(s_t)\left(\frac{s_t}{k_t}\right)^{1-a}, \qquad w_t = w(s_t, k_t) = (1-a)(1-\tau)A(s_t)\left(\frac{k_t}{s_t}\right)^{a}, \tag{7}$$

with $r, w : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$.

Finally, introducing the budget constraint for the government[3] $G_t = \tau Y_t$, we can obtain the expression for the government expenditures

$$g_t = g(s_t, k_t) = \tau A(s_t) s_t^{1-a} k_t^a. \tag{8}$$

The savings of the adults are the capital for period $t+1$, so the average of the savings is $\bar{\sigma}_t = k_{t+1}$ and, using (6), we can write

$$k_{t+1} = \frac{\beta(s_{t+1})}{1+\beta(s_{t+1})} w(s_t, \chi_t) s_t. \tag{9}$$

Collecting equations (3) and (9) we obtain the two-dimensional discrete dynamical system $M : [0,1] \times [0,+\infty) \to [0,1] \times [0,+\infty)$, $(s_t, k_t) \mapsto M(s_t, k_t)$, with map M defined by

$$\begin{cases} s_{t+1} = s_t[1 - \theta(g(s_t, k_t))(1-s_t)] + \gamma(1-s_t), \\ k_{t+1} = \frac{\beta(s_{t+1})}{1+\beta(s_{t+1})}(1-a)(1-\tau)A(s_t)k_t^a s_t^{1-a}, \end{cases} \tag{10}$$

in which $g(s_t, k_t)$ is given in (8).

Before studying model (10) in a completely general setting, we recall some classic results on the steady states of the SIS model (2) and of the OLG Diamond model with no taxation, exogenous factor productivity $A = 1$, exogenous discount factor β and normalized labour $l = 1$.

Proposition 2.1. *Model*

$$s_{t+1} = s_t(1 - \theta(1-s_t)) + \gamma(1-s_t)$$

has the disease free steady state $s^* = 1$ coexisting, for $\theta > \gamma$, with an endemic steady state $s^* = \gamma/\theta$. The disease free steady state is locally asymptotically stable for $\theta > \gamma$, while the endemic steady state is locally asymptotically stable provided that $\theta - \gamma < 2$.

Proposition 2.2. *The steady states of model*

$$k_{t+1} = \frac{\beta}{1+\beta}(1-a)k_t^a$$

are $k^* = 0$, which is repelling, and $k^* = \left[\frac{\beta}{1+\beta}(1-a)\right]^{\frac{1}{1-a}}$, which is globally stable.

[3]We recall that, differently from [11], in the present contribution we do not study the possibility for the government to issue a debt.

3 General static and dynamical analysis

In this section we consider model (10) from a general perspective, without making specific assumptions on functions θ, A and β. Our goal is to highlight the main differences with the results related to the SIS and Diamond model, both concerning the possible sets of steady states and their dynamical properties, mainly focusing on the role of the health policy. As we will see, such a general setting can give rise to very intricate evolution of scenarios, which however can be explained and discussed by taking into account simple situations. In this section we then focus on the properties of System (10) from a mathematical point of view, providing a first, stylized discussion of them. Also with the help of numerical investigations, we will deepen the explanation of the results in Sections 4.1 and 4.2 focusing on some economically relevant cases of study.

3.1 Steady states

Firstly, we provide the conditions that define the possible steady states of model (10).

Proposition 3.1. *Model (10) always has disease free steady states* $\boldsymbol{\xi}^*_{df} = (s^*_{df}, k^*_{df})$ *with*

$$s^*_{df} = 1, \quad k^*_{df} = \left(\frac{\beta(1)}{1+\beta(1)}(1-a)(1-\tau)\right)^{\frac{1}{1-a}} \tag{11}$$

and $\boldsymbol{\xi}^*_{df,0} = (s^*_{df,0}, k^*_{df,0})$ *with*

$$s^*_{df,0} = 1, \quad k^*_{df,0} = 0,$$

while endemic steady states $\boldsymbol{\xi}^* = (s^*, k^*)$ *solve conditions*

$$s^*\theta(g(s^*, k^*)) = \gamma, \; k^* = \left(\frac{\beta(s^*)}{1+\beta(s^*)}A(s^*)(1-a)(1-\tau)\right)^{\frac{1}{1-a}} s^*. \tag{12}$$

where g is defined in (8). In addition, if $\gamma < \theta(0)$, steady state $\boldsymbol{\xi}^*_0 = (s^*_0, k^*_0)$ with $s^*_0 = \gamma/\theta(0), k^*_0 = 0$ exists.

The main difference between the results of Proposition 2.1 and Proposition 3.1 is the possible emergence of multiple endemic steady states, which is fostered by the endogenization of the contact rate θ (depending on the government expenditure), of the factor productivity and of the discount factor β. Let $k^* : [0,1] \to [0,+\infty), s \mapsto k^*(s)$ be the function defined by the second equation in (12), i.e. $k^*(s) = \left(\frac{\beta(s)}{1+\beta(s)}A(s)(1-a)(1-\tau)\right)^{\frac{1}{1-a}} s$.

We note that the condition $s\theta(g(s, k^*(s))) = \gamma$ that must be fulfilled at an endemic steady state is a generalization of condition $s\theta = \gamma$ with exogenous θ, and again represents a balance between the new infections and recoveries. In particular, $g(s,k^*)$ represents the government expenditure corresponding to the steady state capital level k^* if the number of susceptible agents is s. Let us introduce function $g^*:[0,1]^2\to\mathbb{R},(s,\tau)\mapsto g^*(s,\tau)$ defined by

$$g^*(s,\tau)=\tau\left(\frac{\beta(s)}{1+\beta(s)}(1-a)(1-\tau)\right)^{\frac{a}{1-a}}A(s)^{\frac{1}{1-a}}s, \tag{13}$$

in which the right hand side corresponds to $g(s,k^*(s))$. Since we want to study the role of the taxation rate on the emergence/disappearance of endemic steady states and on their comparative statics, in function g^* we make explicit the dependency of $g(s,k^*(s))$ on τ.

Note that $g^*(s,0)\equiv 0$ (as we have null taxation and hence null government expenditures), $g^*(s,1)\equiv 0$ (as in this case the amount of collected resources at each time t would correspond to the whole capital, which quickly becomes null) and that, for any $s\in[0,1]$, there holds $0\le g^*(s)<1$.

For a given $\tau\in[0,1]$, from (12), we have that endemic steady states of model (10) are one-to-one corresponding to the solutions $s\in(0,1)$ to equation $\gamma-s\theta(g(s,k^*(s)))=0$. This suggests introducing function $\varphi:[0,1]^2\to\mathbb{R},(s,\tau)\mapsto\varphi(s,\tau)$ defined by

$$\varphi(s,\tau)=\gamma-s\theta(g^*(s,\tau)), \tag{14}$$

where function g^* is defined by the right hand side in (13). Moreover, for each $\tau\in[0,1]$, we denote by $\boldsymbol{\xi}_i^*(\tau)=(s_i^*(\tau),k_i^*(\tau)),i=1,\dots\nu(\tau)\in\mathbb{N}$ the endemic steady states of model (10), indexed in ascending order with respect to the fraction of susceptible agents, and we count the endemic steady states corresponding to the extremum points for function φ twice[4]. Accordingly, in what follows we write $\boldsymbol{\xi}_i^*(\tau)<\boldsymbol{\xi}_j^*(\tau)$ when $s_i^*(\tau)<s_j^*(\tau)$. In this case we want to stress the dependence of k^* on τ, instead of the usual dependence on s: this is the motivation for the abuse of notation $k^*(\tau)$.

We note that from the sign of function φ we can obtain information about the balance between the fraction $\gamma(1-s_t)$ of recovered people and that of new contagions $s_t\theta(g(s_t,k_t))(1-s_t)$, as the sign of function φ is the same of $\gamma(1-s_t)-s_t\theta(g(s_t,k_t))(1-s_t)$. This means that φ is positive (respectively, negative) if and only if the fraction of susceptible people increases (respectively, decreases).

We start showing a property of endemic steady states.

[4]Conversely, when a zero of φ is an inflection point with horizontal tangent line, we consider this solution as a unique one. As it will become evident from the subsequent analysis, at those points no bifurcation can occur, while at maximum or minimum points of φ fold bifurcation can occur for the endemic steady states of (10).

Proposition 3.2. *Vector* $\boldsymbol{\xi}^*(\tau) = (s^*(\tau), k^*(\tau))$ *is an endemic steady state of* (10) *for* $\tau \in [0, 1-a)$ *if and only if there exists an endemic steady state* $\boldsymbol{\xi}^*(\bar{\tau}) = (s^*(\bar{\tau}), k^*(\bar{\tau}))$ *of* (10) *for* $\bar{\tau} \in (1-a, 1]$ *with* $s^*(\tau) = s^*(\bar{\tau})$.

Moreover, let $\boldsymbol{\xi}^*(\tau_1)$ *and* $\boldsymbol{\xi}^*(\tau_2)$ *be two endemic steady states, obtained for* $\tau_1, \tau_2 \in [0, 1-a)$, *and let* $\boldsymbol{\xi}^*(\bar{\tau}_1)$ *and* $\boldsymbol{\xi}^*(\bar{\tau}_2)$ *be the corresponding endemic steady states (i.e. having the same fraction of susceptible agents) for* $\bar{\tau}_1, \bar{\tau}_2 \in (1-a, 1]$. *If* $\tau_1 < \tau_2$, *there holds* $\bar{\tau}_1 > \bar{\tau}_2$.

The property described in the previous proposition follows from the fact that fixed $s \in (0, 1)$ the function $g^*(s, \tau)$ is strictly increasing for $\tau \in [0, 1-a]$ and strictly decreasing for $\tau \in [1-a, 1]$ and the same holds true for function $\varphi(s, \tau)$. As a consequence of Proposition 3.2, it is sufficient to study what happens for $\tau \in [0, 1-a]$, as the scenarios occurring for $\tau \in [1-a, 1]$ are the same as those found for $\tau \in [0, 1-a]$, just occurring in reverse order[5].

Let us introduce function $E_\theta : [0, +\infty) \to \mathbb{R}, g \mapsto E_\theta(g)$ representing the elasticity of θ at g, defined by

$$E_\theta(g) = \frac{g\theta'(g)}{\theta(g)} \tag{15}$$

and, given $\tau \in (0, 1)$, function $E_{g^*} : (0, 1] \to \mathbb{R}, s \mapsto E_{g^*}(s)$ representing the elasticity of g^* at s, defined by

$$E_{g^*}(s) = \frac{s\frac{\partial g^*}{\partial s}(s, \tau)}{g^*(s, \tau)} = \frac{a}{1-a}\frac{\beta'(s)}{\beta(s)(1+\beta(s))}s + \frac{1}{1-a}\frac{A'(s)}{A(s)}s + 1. \tag{16}$$

We remark that, since if k is a constant, it holds true that $E_{kf}(s) = E_f(s)$ for any function $f(s)$ and from (13), we have that $E_{g^*}(s)$ actually does not depend on τ. Consequently, it is possible to consider a continuous extension of function E_{g^*} to $\tau = 0$, which will be still identified by E_{g^*} in what follows.

We note that $E_\theta(g) \leq 0$, with $E_\theta(0) = 0$, so when we discuss the role of the elasticity of θ, we refer to $|E_\theta(g)|$, i.e. with "inelastic" and "elastic" contact rate we respectively mean $|E_\theta(g)| < 1$ and $|E_\theta(g)| > 1$. Conversely, we note that $E_{g^*}(s) \geq 1$, with $E_{g^*}(s) = 1$ if A and β are constant functions.

In the next proposition we focus on some characteristics of the endemic steady states[6]

Proposition 3.3. *(i) For each* $\tau \in [0, 1-a]$, *a necessary condition for the existence of more than one endemic steady state is that* $E_\theta(g^*(s))E_{g^*}(s) = -1$ *for some* $s \in (0, 1)$.

[5] We stress that each couple corresponding steady states for $\tau < 1-a$ and $\tau > 1-a$ has in common the share of susceptible agents, but in general, they have different capital levels. However, the comparative statics will provide additional reasons for which considering taxation rates larger than $1-a$ is always detrimental.

[6] We remark that in what follows we focus on situations in which there is a finite number of endemic steady states, since it is the most relevant from the economic point of view.

(ii) *There are no endemic steady states for any* τ *if and only if* $\gamma \geq \theta(0)$.

(iii) *If* $E_\theta(g^*(s,\tau))E_{g^*}(s,\tau) = -1$ *exactly at* $N(\tau)$ *values of* s, *there exist endemic steady states* $\boldsymbol{\xi}_1^*(\tau) \leq \boldsymbol{\xi}_2^*(\tau) \leq \ldots \leq \boldsymbol{\xi}_{\nu(\tau)}^*(\tau)$ *with* $\nu(\tau) \leq N(\tau) + 1$.

(iv) *Let* $\boldsymbol{\xi}_1^*(\tau) \leq \boldsymbol{\xi}_2^*(\tau) \leq \ldots \leq \boldsymbol{\xi}_{\nu(\tau)}^*(\tau)$ *be the endemic steady states of* (10) *for some* $\tau \in [0, 1-a]$. *If all the inequalities are strict, we have that for* $1 \leq 2i \leq \nu(\tau) - 1$ $s \mapsto \varphi(s,\tau)$ *is decreasing at* s_{2i+1}^* *and* $|E_\theta(g^*(s_{2i+1}^*,\tau))E_{g^*}(s_{2i+1}^*)| \leq 1$ *(respectively, for* $1 \leq 2i \leq \nu(\tau)$ *is increasing at* s_{2i}^* *and* $|E_\theta(g^*(s_{2i}^*,\tau))E_{g^*}(s_{2i}^*)| \geq 1$*). If* $\boldsymbol{\xi}_{2i-1}^*(\tau) = \boldsymbol{\xi}_{2i}^*(\tau)$ *for* $1 < 2i \leq \nu(\tau)$, *then* $s_{2i-1}^* = s_{2i}^*$ *is a minimum point. If* $\boldsymbol{\xi}_{2i}^*(\tau) = \boldsymbol{\xi}_{2i+1}^*(\tau)$ *for* $1 \leq 2i < \nu(\tau)$, *then* $s_{2i}^* = s_{2i+1}^*$ *is a maximum point.*

Proposition 3.3 highlights that a crucial element for the occurrence of multiple endemic steady states is related to the joint effect of the elasticity of the contact rate with respect to the government expenditure and of the elasticity of g^* with respect to the number of susceptible agents. If we neglect the effect of the epidemic on A and β, multiple endemic steady states can occur only if at some $s \in (0,1)$ we have $E_\theta(g^*(s)) = -1$. The intuition of this is that if the responsiveness of the contact rate with respect to the government expenditure is small (i.e. $|E_\theta(g^*(s))| < 1$), this slightly affects the scenario with an exogenous θ, and hence there is a unique endemic steady state. Conversely, an increased effectiveness of the government intervention opens the possibility to have additional endemic steady states that, as we will see, are characterized by larger fraction of susceptible agents, and hence are more desirable from the healthcare point of view. Moreover, since $E_{g^*}(s) \geq 1$, the endogenization of A and β can strengthen the possibility for the occurrence of multiple endemic steady states. We note that in [11], when the interaction with the environmental side is also considered, such a multiplicity of endemic steady states is not possible[7].

We draw the attention on point (iii), when multiple, non coincident endemic steady states exist, they are characterized by alternating large and small values of $|E_\theta(g^*(s))E_{g^*}(s)|$, respectively corresponding to decreasing and increasing parts of the graph of function $s \mapsto \varphi(s,\tau)$. This characterization will be crucial for discussing stability of endemic steady states[8].

Finally, if $\gamma \geq \theta(0)$, just the disease free steady state is possible, independently of the taxation rate, which suggests that $\tau = 0$ is the best policy in this case. In addition to this, as we will see

[7]The level of complexity described by Proposition 3.3 is not observed in the existing literature. In [18] a unique endemic steady state is observed, while at most two endemic steady states occur in [20]. At most three steady states characterized by positive capital level are possible in [3], but just one of them can be stable.

[8]It will become evident that such an alternating behavior, with minor adjustments, also generalizes to the situations in which some endemic steady states correspond to the extremum points for function φ.

in Proposition 3.8, if $\gamma > \theta(0)$, $\boldsymbol{\xi}^*_{df}$ is always stable, so, from now on, we will just focus on the case of $\gamma < \theta(0)$.

In the next propositions we investigate how endemic steady states can evolve on increasing τ. In doing this, we recall that, for each τ, there is a one-to-one correspondence between the solutions $s \in (0,1)$ to $\varphi(s,\tau) = 0$ and the endemic steady states of (10). There is then a close link between the emergence of new endemic steady states and the bifurcations of the one dimensional recurrence equation

$$s_{t+1} = s_t + (1 - s_t)\varphi(s_t, \tau). \tag{17}$$

In order to avoid situations in which more than a couple of endemic steady states emerge/vanish at a given steady state $\boldsymbol{\xi}^*(\tau)$, it is necessary to assume that if a steady state $s^*(\tau)$ for (17) is an extremum point for $s \mapsto \varphi(s,\tau)$, this function is either strictly convex or concave on a neighborhood of s^* for taxation rates suitably close to τ. This is guaranteed by imposing that $\frac{\partial^2\varphi(s,\tau)}{\partial s^2} \neq 0$, so we assume

$$\begin{aligned}&\textit{if } \theta(g^*(s^*,\tau_0)) + s^*\theta'(g^*(s^*,\tau_0))\frac{\partial g^*}{\partial s}(s^*,\tau_0) = 0, \textit{ there holds}\\ &\quad\theta''\left(g^*(s^*,\tau_0)\right)\left(\frac{\partial g^*}{\partial s}(s^*,\tau_0)\right)^2 s^* + \theta'\left(g^*(s^*,\tau_0)\right)\left[\frac{\partial^2 g^*}{\partial s^2}(s^*,\tau_0)s^* + 2\frac{\partial g^*}{\partial s}(s^*,\tau_0)\right] \neq 0.\end{aligned} \tag{18}$$

Firstly, we consider what can happen on a neighborhood of an endemic steady state.

Proposition 3.4. *Let θ, A and β be C^2 functions, $\tau_0 \in (0, 1-a)$ and $\boldsymbol{\xi}^*(\tau_0) = (s^*, k^*)$ be an endemic steady state at which (18) holds true.*

Then there exists a neighborhood $\Omega = (s^* - \delta, s^* + \delta) \times (\tau_0 - \epsilon, \tau_0 + \epsilon) \subset (0,1) \times [0, 1-a)$ of (s^*, τ_0), $\epsilon, \delta > 0$, such that exactly one of the following scenarios occur:

- *(a1) there is a unique endemic steady state $\boldsymbol{\xi}^*(\tau) = (s(\tau), k(\tau))$, for every $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$, with $(s(\tau), \tau)$ in Ω;*
- *(a2) there are no endemic steady states $\boldsymbol{\xi}^*(\tau) = (s, k)$ for $\tau \in (\tau_0 - \epsilon, \tau_0)$, with (s, τ) in Ω, and there are two endemic steady states $\boldsymbol{\xi}^*_i(\tau) = (s_i(\tau), k_i(\tau))$, $i = 1, 2$, for $\tau \in (\tau_0, \tau_0 + \epsilon)$, with $(s_i(\tau), \tau)$ in Ω, $i = 1, 2$ and $s_1(\tau) \leq s^* \leq s_2(\tau)$;*
- *(a3) there are two endemic steady states $\boldsymbol{\xi}^*_i(\tau) = (s_i(\tau), k_i(\tau))$, $i = 1, 2$, for $\tau \in (\tau_0 - \epsilon, \tau_0)$, with $(s_i(\tau), \tau)$ in Ω, $i = 1, 2$, $s_1(\tau) \leq s^* \leq s_2(\tau)$ and there are no endemic steady states $\boldsymbol{\xi}^*(\tau) = (s, k)$ for $\tau \in (\tau_0, \tau_0 + \epsilon)$, with (s, τ) in Ω.*

Proposition 3.4 establishes a first possible way through which endemic steady states create/vanish.

From the proof of Proposition 3.4, we have that cases (b) and (c) are linked to the fold bifurcations of steady states $s^* \in (0,1)$ for (17). We stress that it is possible to show that even if condition (18) is not fulfilled, no pitchfork bifurcation of $s^* \in (0,1)$ for (17) can occur.

Proposition 3.5. *Let θ, A and β be C^2 functions, $\tau_0 \in (0, 1-a)$ and $\boldsymbol{\xi}^*_{df}(\tau_0) = (1, k^*_{df})$ be a disease free steady state at which (18) holds true for $s^* = 1$.*

Then there exists $\Omega = (\tilde{s}, 1) \times (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \subset (0,1) \times [0, 1-a)$, $\epsilon > 0$, such that exactly one of the following scenarios occur:

(b1) there are no endemic steady states $\boldsymbol{\xi}^*(\tau) = (s, k)$ for $\tau \in (\tau_0 - \epsilon, \tau_0 + \epsilon)$, with (s, τ) in Ω;

(b2) there are no endemic steady states $\boldsymbol{\xi}^*(\tau) = (s, k)$ for $\tau < \bar{\tau}$, with (s, τ) in Ω and there is a unique endemic steady state $\boldsymbol{\xi}^*(\tau) = (s(\tau), k(\tau))$ for $\tau > \bar{\tau}$, with $(s(\tau), \tau)$ in Ω;

(b3) there is a unique endemic steady state $\boldsymbol{\xi}^*(\tau) = (s(\tau), k(\tau))$ for $\tau < \bar{\tau}$, with $(s(\tau), \tau)$ in Ω, and there are no endemic steady states $\boldsymbol{\xi}^*(\tau) = (s, k)$ for $\tau > \bar{\tau}$, with (s, τ) in Ω.

In cases (b2) and (b3) an endemic steady state exists for, respectively, $\tau > \bar{\tau}$ and $\tau < \bar{\tau}$. These two cases are related to transcritical bifurcations of $s^* = 1$ for dynamical equation (17). Even if the fold and the transcritical bifurcations we mentioned in cases (a2), (a3), (b2) and (b3) of both Propositions 3.4 and 3.5 are related to the recurrence equation (17), they actually provide an insight on possible bifurcations through which the number of endemic steady states $\boldsymbol{\xi}^*$ of model (10) changes[9]. This will become more evident from the dynamical analysis and from the numerical investigations carried on in Sections 4.1 and 4.2.

Concerning the comparative statics of endemic steady states, we have the following result.

As we will see in Proposition 3.9, we just focus on steady states at which $|E_\theta(g^*(s^*, \tau))| < 1$, since those at which $|E_\theta(g^*(s^*, \tau))| > 1$ are repelling and do not play an active role. We avoid to explicitly discuss what happens when $E_\theta(g^*(s^*, \tau)) = -1$, since it corresponds either to repelling steady states or to situations comparable to the case of $|E_\theta(g^*(s^*, \tau))| < 1$.

Proposition 3.6. *Let $I \subset [0, 1-a]$ be a range of values for which an endemic steady state $\boldsymbol{\xi}^*(\tau) = (s^*(\tau), k^*(\tau))$ characterized by $E_\theta(g^*(s^*(\tau), \tau))E_{g^*}(s^*(\tau)) > -1$ exists for any $\tau \in I$. We have that*

[9] We avoid to provide the precise proof of the occurrence of a fold and transcritical bifurcations, because this would require to deal with high order non degeneracy and transversality conditions, which are not analytically possible for the present model. In Propositions 3.8 and 3.9 we limit ourselves to showing the occurrence of an eigenvalue $\lambda = 1$ for the Jacobian matrix of (10) evaluated at the endemic steady states corresponding to solutions to $\varphi(s, \bar{\tau}) = 0$. This, together with the numerical investigations, provides plausible insights on the possible bifurcations.

- *$s^*(\tau)$ increases;*
- *$k^*(\tau)$ increases if*

$$E_\theta(g^*(s^*(\tau),\tau)) < -\frac{\tau}{1-a+E_A(s^*(\tau))+(1-\tau)E_{\beta/(1+\beta)}(s^*(\tau))} \tag{19}$$

for $\tau \neq 0$ and if

$$\theta'(0) < -\frac{\theta(0)}{\left(\frac{\beta(s(0))}{1+\beta(s(0))}(1-a)\right)^{\frac{a}{1-a}} A(s(0))^{\frac{1}{1-a}} s(0)\left[E_{\beta/(1+\beta)}(s(0))+E_A(s(0))+1-a\right]} \tag{20}$$

for $\tau = 0$, while it decreases when (19) (respectively (20)) is fulfilled with the opposite inequality $>$.

Conversely, if we consider $\boldsymbol{\xi}^*(\tau)$ for $\tau > 1-a$ both $s^*(\tau)$ and $k^*(\tau)$ decrease.

We firstly note that Proposition 3.6 shows that any increase of the taxation rate above $\tau = 1-a$ is detrimental, as it results in a deterioration of both the epidemiological sphere (as the number of infected people increases) and the economic one (as the steady state capital level decreases). The explanation is that above a certain threshold, the marginal increase in collected resources due to the larger taxation rate becomes smaller than the marginal decrease of the capital level, and hence the government expenditure decreases. We stress that this is the unique relevant "asymmetry" in the behavior of $\boldsymbol{\xi}^*(\tau)$ for $\tau \in [1-a,1]$ with respect to $\tau \in [0,1-a]$ and, as remarked after Proposition 3.2, is related just to component k^*. As long as a given endemic steady state exists, the related fraction of susceptible people increases. This is predictable, as for $\tau \in [0,1-a]$ the government investment on healthcare increases, hence the epidemiological situation improves.

The behavior of the capital level is conversely ambiguous, as it depends on condition (19), which is fulfilled if the elasticity of θ is suitably large. For now, we do not add further details on this, we will deepen the discussion in Sections 4.1 and 4.2.

3.2 Stability

In this section we provide analytical conditions for stability of both endemic and disease free steady states.

Proposition 3.7. *Both the disease free $\boldsymbol{\xi}^*_{df,0}$ and endemic $\boldsymbol{\xi}^*_0$ steady states defined in Proposition 3.1 are unstable.*

The previous proposition confirms for the two dimensional model (10) the instability of the null capital level steady state of the uncoupled OLG model studied in Proposition 2.2. Concerning $\boldsymbol{\xi}^*_{df}$, we have the next result.

Proposition 3.8. *The disease free steady state $\boldsymbol{\xi}^*_{df}$ defined in (11) is locally asymptotically stable provided that*

$$\gamma > \theta\left(\tau\left[\frac{\beta(1)}{1+\beta(1)}(1-a)(1-\tau)\right]^{\frac{a}{1-a}}\right) \tag{21}$$

Condition (21) is fulfilled only if there is an even number of endemic steady states.

Before discussing condition (21) we study stability for the endemic steady state.

Proposition 3.9. *An endemic steady state $\boldsymbol{\xi}^* = (s^*(\tau), k^*(\tau))$ with positive capital is locally asymptotically stable provided that*

$$\begin{cases} E_\theta(g^*(s^*(\tau),\tau))E_{g^*}(s^*(\tau)) > -1 \\ [\theta(g^*(s^*(\tau),\tau)) - \gamma]\left[1 + \frac{1-a}{1+a}E_\theta(g^*(s^*(\tau),\tau))E_{g^*}(s^*(\tau))\right] < 2 \end{cases} \tag{22}$$

We start assuming that $\tau \in [0, 1-a]$. The former condition in (22) implies that $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau) < 0$, so any endemic steady state at which $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau) \geq 0$ is unstable. Recalling Proposition 3.3 (iv), any endemic steady state $\boldsymbol{\xi}^*_{2i}, i > 0$ is unstable, in particular repelling.

Noting that the first condition in (22) implies that $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau) < 0$, it can be explained as follows. We discussed (just before Proposition 3.2) how the positive/negative sign of φ determines the increase/decrease in the number of susceptible agents. This means that, if (22) holds true, on a left neighborhood of s^*, we have that φ is positive, and hence the susceptible population increases. A symmetric situation takes place for $s > s^*$, in which case the fraction of infected people grows. So, if $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau)$ is not too large, the trajectories converge toward s^*. Conversely, if $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau) < 0$, the two previous behaviors for $s < s^*$ and $s > s^*$ swap, and s^* cannot be stable.

Similarly, close to $s^* = 1$, under condition (21) we have that the fraction of susceptible population increases, as the new recoveries are more than the new infections, and so the disease free steady state is stable. Conversely, if condition (21) is fulfilled with the opposite inequality $<$, we have the converse situation and the disease free steady state is repelling.

We remark that in the exogenous case just the case of $\frac{\partial\varphi}{\partial s}(s^*(\tau),\tau) < 0$ occurs. Recalling Propositions 3.4 and 3.5 and as we will see in the numerical simulations in Sections 4.1 and 4.2, we have that when the former condition in (22) becomes an equality, either a fold or a transcritical bifurcation occurs, with the emergence or disappearance of, respectively, a new couple of endemic steady states or a single one. We stress that fold bifurcations do not affect the stability of the disease free steady state, while with the occurrence of a transcritical bifurcation a previously unstable disease free steady state becomes stable.

The latter condition in (22) is related to the emergence of a flip bifurcation, and is a generalization of the classic SIS model one, corresponding to $\theta - \gamma < 2$. We stress that since whenever an endemic steady state exists the third inequality in (34) holds true, no Neimark-Sacker bifurcation can occur and stability can be recovered/lost just through a flip bifurcation.

Concerning the last condition in (22), we note that $[\theta(g^*(s^*(\tau), \tau)) - \gamma]$ decreases as τ increases, since the government expenditure increases and θ is strictly decreasing. Moreover, factor $1 + \frac{1-a}{1+a} E_\theta(g^*(s^*(\tau), \tau)) E_{g^*}(s^*(\tau))$ is equal to 1 if $\tau = 1$ and is always less than 1 for $\tau > 0$. This means that the last condition in (22) is more restrictive for $\tau = 0$ when compared to any $\tau > 0$. This allows concluding that taxation does not introduce instability with respect to the scenario of no taxation, and hence the government intervention has a potentially stabilizing effect. This means that if $\boldsymbol{\xi}_1^*(\tau)$ is stable for $\tau = 0$, it will be stable for any $\tau > 0$, as long it exists on some interval $[0, \bar{\tau})$. Note that this does not mean that if an endemic steady state is stable for some $\bar{\tau} > 0$ (i.e. for some given healthcare policy) it will remain stable for any $\tau > \bar{\tau}$. However, we have been able to show through simulations that an increase of τ is destabilizing only by adopting quite extreme choices of functions A and β, with lack of economic relevance. This suggests that such a possibility is not interesting from an economic point of view, so we avoid discussing it further.

Note that, as a consequence of Proposition 3.2, if we increase τ on $[0, 1-a]$ we have that the sequence of bifurcations occurring on a given endemic steady state $\boldsymbol{\xi}^*(\tau)$ existing on some interval $I \subset [0, 1-a]$, takes place in *reverse order* on the corresponding interval $\bar{I} \subset [1-a, 1]$ where $\boldsymbol{\xi}^*(\bar{\tau})$ is defined.

4 Two cases of study

In what follows we consider two economically relevant cases of study, obtained by considering more specific assumptions on the contact rate, preferences and factor productivity functions. We progressively introduce the elements of complexity in order to better understand the outcomes.

4.1 The role of endogenous contact rate

In this section we study the effect of the contact rate endogenization, with the aim of emphasizing the direct effect of the economic domain on the epidemiological one. To illustrate the essential elements characterizing this, we consider a simplified setting for (10), obtained by introducing the following two assumptions:

Assumption 1. *Function $E_\theta(g) : [0, +\infty) \to \mathbb{R}$ defined by (15) is strictly decreasing.*

Assumption 2. *Factor productivity and probability to survive are constant functions, respectively $A(s) \equiv 1$ and $\beta(s) \equiv \bar{\beta}$.*

Under Assumption 1, we have a greater responsiveness $|E_\theta(g)|$ of the contact rate with respect to a marginal increase of the government expenditure as the current expenditure grows[10]. This means that a unitary increase of the current expenditure g_2 has the effect to reduce θ proportionally more than a unitary increase of any expenditure $g_1 < g_2$. This depicts a reasonable situation in which it is easier to deal with an epidemic if prevention, containment and resilience measures are already in place than if they are not.

Assumption 2 allows to reduce the impact of the epidemiological domain onto the economic one to the fact that $l_t = s_t$, i.e. labour corresponds to the fraction of susceptible agents.

In the general case, we provided only a local analysis of how the set of steady states can change as τ increases. Thanks to Assumptions 1 and 2, it is possible to have a global description for $\tau \in [0, 1-a]$.

In what follows, when we write "Scenario n_1-n_2-..." with $n_i \in \mathbb{N}$ we mean that, on increasing τ, we initially have n_1 endemic steady states, we then have n_2 endemic steady states and so on. As an example, Scenario "1-2-0" means that there exist $\tau_1, \tau_2 \in (0, 1-a)$ with $\tau_1 < \tau_2$ such that a unique endemic steady state exists for $\tau \in [0, \tau_1]$, two endemic steady states exist for $\tau \in (\tau_1, \tau_2]$ and no endemic steady state exists for $\tau \in (\tau_2, 1-a]$.

Proposition 4.1. *Let $\gamma < \theta(0)$. Under Assumptions 1 and 2, on increasing $\tau \in [0, 1-a]$, only the following scenarios are possible:*

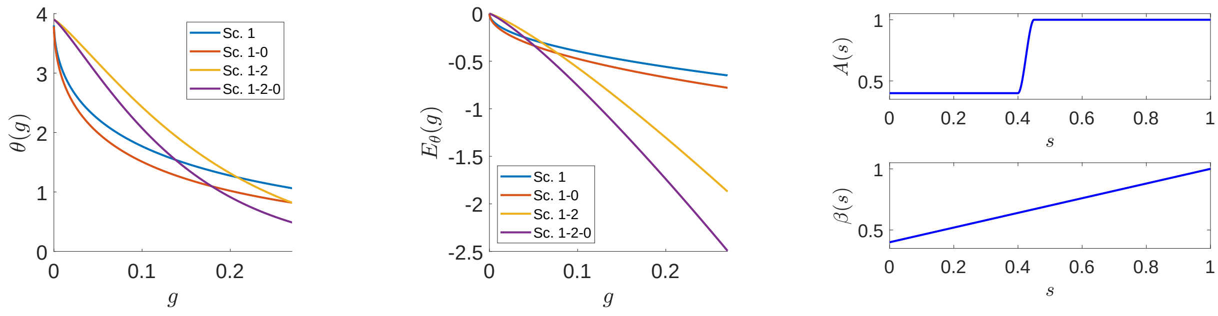
Scenario 1: *There exists a unique endemic steady state for any τ.*

Scenario 1-0: *There exists $\tau_1 \in (0, 1-a)$ such that a unique endemic steady state exists for $\tau \in [0, \tau_1)$ and no endemic steady state exists for $\tau \in [\tau_1, 1-a]$.*

Scenario 1-2: *There exists $\tau_1 \in (0, 1-a)$ such that a unique endemic steady state exists for $\tau \in [0, \tau_1]$ and two endemic steady states exist for $\tau \in (\tau_1, 1-a]$.*

Scenario 1-2-0: *There exist $\tau_1, \tau_2 \in (0, 1-a)$ with $\tau_1 < \tau_2$ such that a unique endemic steady state exists for $\tau \in [0, \tau_1]$, two endemic steady states exist for $\tau \in (\tau_1, \tau_2]$ and no endemic steady state exists for $\tau \in (\tau_2, 1-a]$.*

[10]We stress that Assumption 1 is equivalent to requiring that θ is not "too convex". If θ is twice differentiable, we have that $E'_\theta(g) < 0$ is equivalent to $\theta''(g) < \frac{g(\theta'(g))^2 - \theta(g)\theta'(g)}{g\theta(g)}$, where the right hand side is positive.



(a) (b) (c)

Figure 1: Panels (a) and (b): graphs of functions θ and E_θ. Panel (c): graphs of endogenous functions A and β.

We stress that each scenario describes the sequence of endemic steady states up to $\tau = 1 - a$, while for $\tau > 1 - a$ the sequence is repeated in reverse order. Proposition 4.1 shows that multiple endemic steady states can occur just due to the endogenization of the contact rate, even in the simple case of a monotonic θ. Recalling the comments in Section 3 relative to Proposition 3.4 and Proposition 3.5, we note that Proposition 4.1 actually clarifies the sequence of transcritical/fold bifurcations occurring to $\boldsymbol{\xi}_1^*$ and $\boldsymbol{\xi}_{df}^*$. We remark that when $\boldsymbol{\xi}_2^*$ exists, from point (iv) of Proposition 3.3 and Proposition 3.9, it is repelling while $\boldsymbol{\xi}_{df}^*$ becomes stable (see also Proposition 3.8). So, differently from the classic SIS model, we have that a non repelling endemic steady state can coexist with a stable disease free steady state.

We cast an insight on the occurrence of each scenario with the help of numerical simulations. As an example of function θ that fulfills Assumption 1 we consider

$$\theta(g) = \theta_0 e^{-\theta_1(\theta_2+g)^\alpha}, \tag{23}$$

with $\theta_i > 0, i = 0, 1, 2$ and $\alpha > 0$, for which we have $E_\theta(g) = -\alpha\theta_1 g(\theta_2 + g)^{\alpha-1}$. A direct check shows that $E_\theta(g)$ is strictly decreasing and $|E_\theta(g)|$ is strictly concave (respectively, convex) for $\alpha < 1$ (respectively, $\alpha > 1$).

For all the simulations, we set $a = 0.3$ (belonging to the empirically relevant range of output elasticity of capital, see [5]) and $\gamma = 1$. In each scenario, parameters used for function θ are listed in Table 1, while under Assumption 2 we set $A = 1$ and $\bar{\beta} = 1$. The corresponding graphs of functions θ and E_θ are reported in Figures 1 (a) and (b).

In the first column of Figure 2 we report simulations related to each scenario provided by Proposition 4.1. Figures 2 (a,d,g,j) report function φ for different values of τ using different colors. In particular, the blue and the red colors are respectively used for the extreme taxation rates, namely $\tau = 0$ and $\tau = 1 - a = 0.7$, while the yellow and purple colors are used to represent φ for those taxation rates for which the number of solutions to $\varphi(s, \tau) = 0$ changes. Recalling

	Function θ			
Scenario	α	θ_0	θ_1	θ_2
"1"	0.5	3.9	2.5	10^{-4}
"1-0"	0.5	3.9	3	10^{-4}
"1-2"	1.2	3.9	7	10^{-4}
"1-2-0"	1.2	3.9	10	10^{-4}

Function β	$n = 1$	$\beta_0 = 1$	$\beta_1 = 10^{-4}$
Function A	$\underline{A} = 0.4$	$s_{A,1} = 0.4$	$s_{A,2} = 0.5$

Table 1: Left table: parameters related to function θ defined in (23). Right table: parameter related to functions A and β defined in (27) and (26).

the comments to Propositions 3.4 and 3.5, in what follows, we describe these changes in the numerousness of steady states as a fold or a transcritical bifurcation, since they occur for map (17) and the subsequent bifurcation diagrams will confirm this for model (10).

In the simulation related to Scenario "1" reported Figure 2 (a), the graph of function φ is always decreasing, and lies between the blue and the red graphs, as a consequence of which there is always a unique endemic steady state and neither fold nor transcritical bifurcations occur.

In the simulation related to Scenario "1-0" reported in Figure 2 (d), we have that, as τ increases, function $s \mapsto \varphi(s, \tau)$ is decreasing for each $\tau \lesssim 0.4$, so there is a unique endemic steady state. For $\tau \approx 0.4$ a transcritical bifurcation occurs, after which no endemic steady state exists for $0.4 \lesssim \tau \leq 0.7$, as φ lies above the yellow curve and it can no more intersect the horizontal axis.

In Scenario "1-2" reported in Figure 2 (g) we have that φ becomes U-shaped for a value of τ smaller than that for which φ passes through point $(1, 0)$ (in this case, $\tau \approx 0.52$). This means that for $\tau \lesssim 0.52$ function φ intersects the horizontal axis at a unique point and a unique endemic steady state exists, while for $\tau \gtrsim 0.52$ the two intersections provide two endemic steady states as a consequence of the transcritical bifurcation occurred at $\tau \approx 0.52$. Since the minimum of function $\varphi(s, 1-a)$ lies below the horizontal axis for $0.52 \lesssim \tau \leq 0.7$, we still have two endemic steady states. This is the main difference with Scenario "1-2-0". If we look at Figure 2 (j), we can see that for $\tau \lesssim 0.36$ and $0.36 \lesssim \tau \lesssim 0.42$ the behavior of φ is the same as what happens in Scenario "1-2", with one zero when the graph of φ lies between the blue and the yellow graphs and two zeros when it lies between the yellow and purple graphs, being the latter tangent to the horizontal axis. For $0.42 \lesssim \tau \leq 0.7$, we have $\varphi(s, \tau) > 0$ for any $s \in (0, 1)$ and hence no endemic steady state is possible. In this scenario, a transcritical bifurcation ($\tau \approx 0.36$) followed by a fold one ($\tau \approx 0.42$) occurred.

Concerning comparative statics, with exogenous functions A and β condition (19) (under

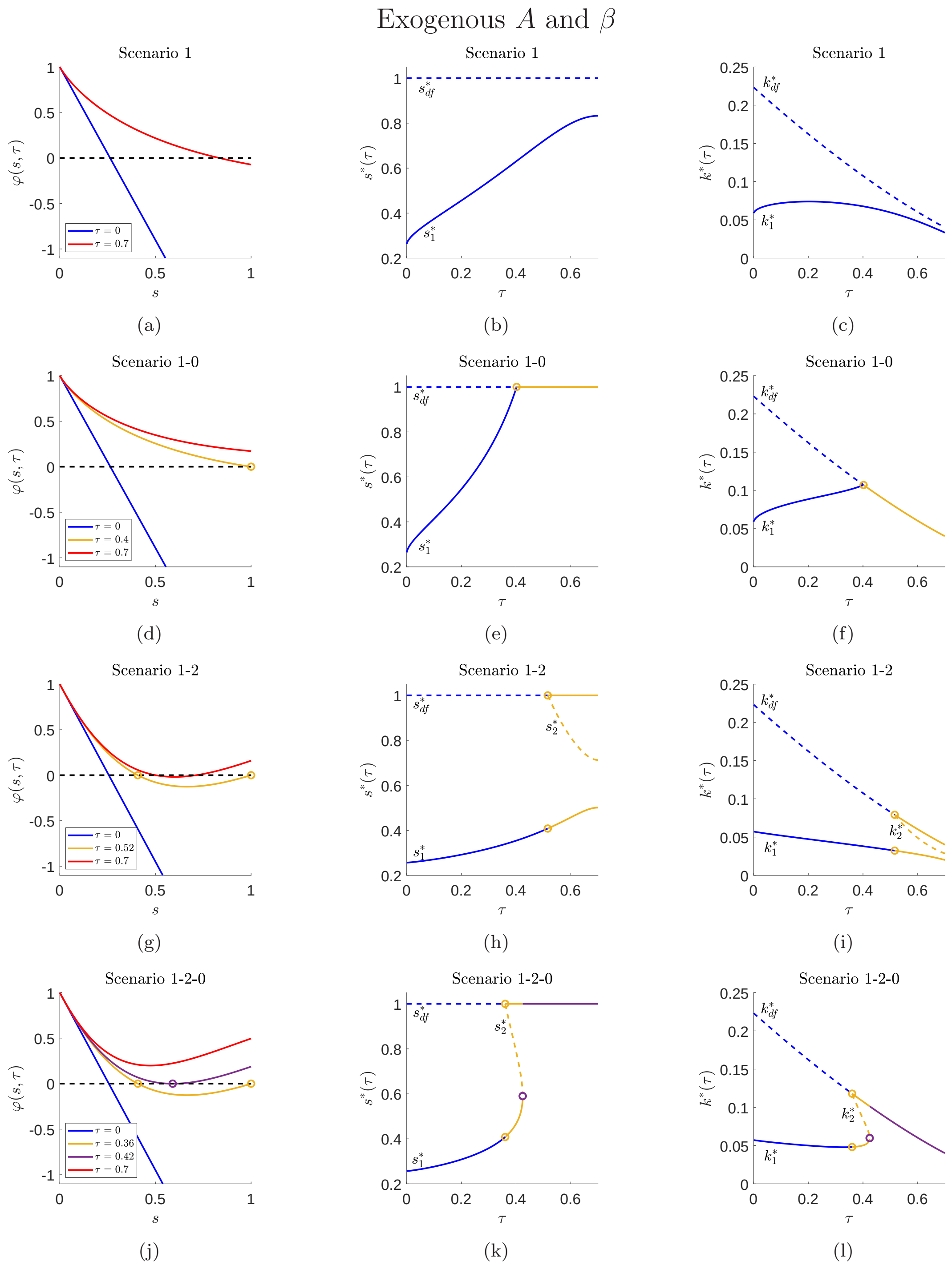


Figure 2: Simulations related to the Scenarios of Proposition 4.1. Left column: graphs of function φ for $\tau = 0$ (blue color) and $\tau = 1 - a$ (red color), and the intermediate values of τ for which we have a change in the number of endemic steady states (possible colors yellow and purple). Middle and right columns: evolution of fraction of susceptible agents (middle) and of capital level (right) for the endemic and the disease free steady states.

which $k^*(\tau)$ increases) simplifies as

$$E_\theta(g^*(s^*(\tau),\tau)) < -\frac{\tau}{1-a}. \tag{24}$$

We report the evolution of $s^*(\tau)$ and $k^*(\tau)$ respectively in the middle and right columns of Figure 2, related to the same settings (in particular, the same function θ) used for the simulations in the first column. Components of $\boldsymbol{\xi}_i^*$ are numerically computed on varying τ. On each row, the colors that are used in the Figures in the middle and right columns refer to those of the first column. We stress that colors are used to denote the number of steady states, and not a particular steady state, so a curve related to a given steady state can be differently colored as τ increases. Moreover, in Figures 2 (b,e,h,k) we also represent the horizontal line $s^*_{df} = 1$, corresponding to the disease free steady state, while in Figures 2 (c,f,i,l) we report the decreasing line $k^*_{df} = \left(\frac{\bar{\beta}(1-a)(1-\tau)}{\bar{\beta}+1}\right)^{\frac{1}{1-a}}$ (it is always the topmost line). Finally, recalling Propositions 3.8 and 3.9, we use a dashed line to represent steady states for those values of τ for which they are repelling, and we do not discuss them.

In agreement with Proposition 3.6, we can see that each curve $s_1^*(\tau)$ is always strictly increasing for $\tau \in [0, a-1]$.

Conversely, we already noted that the behavior of the capital level is ambiguous, as it depends on condition (24). As long as $\boldsymbol{\xi}^*$ exists, condition (24) may be always fulfilled, it may never hold true, or it can become valid or not as τ changes. We recall that both $E_\theta(g)$ and the right hand side of (24) are negative, so condition (24) requires the elasticity of θ to be suitably elastic. To understand this, we note that k^* has a positive direct dependence on s^* and a negative one on τ. However, as τ increases, the number of healthy people (and hence the number of workers) increases, and this can counterbalance the direct negative effect on capital of increasing taxation. This is possible only if the increase of s^* with respect to τ is quick enough, which, in turns, is possible if θ is sufficiently elastic to an increase of the government expenditure. We note that the threshold given by the right hand side in (24) is increasingly more restrictive as τ increases, so (24) requires that the responsiveness of θ increases at least proportionally to the growth of the taxation rate.

Condition (19) cannot be made more explicit from the analytical point of view. However, it is possible to say what is the effect on k_1^* (related to the unique steady state existing for $\tau = 0$) when a health policy is introduced.

Let $\boldsymbol{\xi}^*(\tau) = (s^*(\tau), k^*(\tau))$ be a continuous curve of endemic steady states for τ in a right neighborhood of 0 and let $s^*(0) \neq 0$. Under the hypotheses of Proposition 3.6 we have that

$k_1^*(\tau)$ increases at $\tau = 0$ provided that

$$\theta'(0) < -\frac{\theta(0)}{(1-a)^{\frac{1}{1-a}}\left(\frac{\overline{\beta}}{1+\overline{\beta}}\right)^{\frac{a}{1-a}}(1-a)^{\frac{1}{1-a}}s^*(0)+1-a}, \tag{25}$$

while it decreases if (25) holds true with $>$ inequality.

Previous equation provides a bound on the derivative of θ to obtain an increase of the capital level when we pass from a situation without health policy ($\tau = 0$) to that in which taxation is used for healthcare. If (25) holds true, we have that the steady state capital level is better off with a policy intervention. On the other hand, if (25) is valid with the converse inequality $>$, we have that $k_1^*(\tau)$ initially decreases when taxation is introduced. This is due to a small initial effect on the epidemiological sphere of the government intervention. However, this does not mean that $k_1^*(\tau)$ will go on decreasing for any τ, as well as (25) does not guarantee that $k_1^*(\tau)$ is increasing for any τ. In addition to this, new stable endemic steady states may emerge and the disease free steady state can recover stability, giving rise to non repelling steady states characterized by higher capital levels.

Recalling (19), the global behavior of k_i^* is strongly related to the convexity/concavity[11] of the elasticity of θ. Having in mind Proposition 3.6, we look again at Figure 2.

Concerning Scenario "1", in Figure 2 (b) we have that, except for extreme values of τ, s_1^* increases at an approximately constant rate. Conversely, the graph of k_1^* is hump-shaped. A direct check would show that condition (25) initially holds true, so k_1^* initially increases. However, since in Scenario "1" function $E_\theta(g)$ is concave, we have decreasing marginal responsiveness as taxation increases, and hence condition (19) is no more valid for suitably large values of τ and k_1^* decreases. For this simulation, the optimal capital level is actually attained for approximately $\tau = 0.2$.

Let us move to Scenario "1-0", in which we used a concave function θ as in Scenario "1" (α is the same), but in Scenario "1-0" it has larger elasticity (see also blue and red graphs in Figure 1 (a,b)). This leads s_1^* to increase more quickly than in the former case (Figure 2 (e) compared to Figure 2 (b)), and hence in Scenario "1-0" $\boldsymbol{\xi}_1^*$ collides with $\boldsymbol{\xi}_{df}^*$, and does not exist any more. The endemic steady state then becomes the unique (non repelling) steady state (yellow curve). In this case, the quick increase of disposable workforce thanks to the decrease of infected agents

[11] We stress that it does not only depend on the concavity/convexity of θ, as

$$\frac{d^2E_\theta(g^*(s^*(\tau),\tau))}{d\tau^2} = E'_\theta(g^*)\left(\frac{\partial^2 g^*}{\partial\tau^2}+2\frac{\partial^2 g^*}{\partial s^*\partial\tau}(s^*)'+\frac{\partial^2 g^*}{\partial s^2}((s^*)')^2+\frac{\partial g^*}{\partial s}(s^*)''\right)+E''_\theta(g^*)\left(\frac{\partial g^*}{\partial s}(s^*)'+\frac{\partial g^*}{\partial\tau}\right)^2$$

so it is also affected by the shape of $g^*(s^*(\tau),\tau)$. However, ceteris paribus, a suitably large positive or negative $E''_\theta(g^*)$ is able to guarantee the concavity or the convexity of $E_\theta(g^*(s^*(\tau),\tau))$.

is able to compensate the growing taxation rate, and hence k_1^* increases (blue solid curve, Figure 2 (f)) until there is only the disease free steady state (yellow solid curve). In this simulation, the optimal taxation rate is at $\tau \approx 0.4$ and its increase above this threshold is neither beneficial to the epidemiological sphere nor to the economic one.

Concerning Scenario "1-2", in Figure 2 (h), regarding the endemic steady state $\boldsymbol{\xi}_1^*$, we have that s_1^* (lower solid blue-yellow line) increases more slowly than in Scenario "1". For the simulation related to Scenario "1-2" we considered a function θ for which condition (25) is fulfilled with the opposite inequality and whose elasticity is convex (see Figure 1 (b)). As a consequence of this, k_1^* initially decreases and continues to decrease (solid blue-yellow line Figure 2 (i)). However, at $\tau \approx 0.52$, the transcritical bifurcation leads to the emergence of the new, repelling endemic steady state $\boldsymbol{\xi}_2^*$ (dashed yellow line). This allows $\boldsymbol{\xi}_{df}^*$ to recover stability and to become dynamically relevant (topmost solid yellow lines in Figure 2 (h,i)). The outcome is that at $\tau \approx 0.52$ steady state $\boldsymbol{\xi}_{df}^*$ is characterized by the highest capital level, and, from the static point of view, $\tau \approx 0.52$ turns out to be the best taxation rate both from the epidemiological and the economic points of view.

Finally, regarding Scenario "1-2-0", in Figure 2 (k), we have that s_1^* (lower solid blue-yellow line) initially increases quite slowly, but its growth accelerates more significantly than in Scenario "1-2". The reason for this is that in both simulations θ has convex elasticity (same parameter α), but in that related to Scenario "1-2-0" elasticity is larger (larger parameter θ_1, see Figure 1 (b)). So, even if condition (25) is fulfilled with the opposite inequality and k_1^* initially decreases, the number of healthy agents that can work increases more and more and counterbalance the effect of taxation. This results in a U-shaped graph for k_1^* (blue-yellow bottom solid line in Figure 2 (l)), which starting from $\tau \approx 0.35$ increases and at $\tau \approx 0.42$ we have that k_1^* is greater than in absence of health policy. In any case, also in this Scenario, at $\tau \approx 0.36$ the disease free steady state becomes stable and this corresponds to the "best" taxation rate both from the epidemiological and the economic points of view.

To conclude, the static numerical investigation shows that a suitable health policy has a beneficial effect on the capital level as well. We stress that it is possible to obtain simulations in which the best capital level is obtained for $\tau = 0$, both with or without coexistence between different non repelling steady states. This occurs when the elasticity of the contact rate is significantly small, and hence the beneficial effect of health policy on the epidemiological domain is not significant.

Concerning the dynamical behaviors of (10) under Assumptions 1 and 2, we start noting that stability condition (22) simplifies to

$$\begin{cases} E_\theta(g^*(s^*(\tau),\tau)) > -1 \\ (\theta(g^*(s^*(\tau),\tau)-\gamma)\left[1+\frac{1-a}{1+a}E_\theta(g^*(s^*(\tau),\tau))\right] < 2 \end{cases}$$

As long as the first condition holds, since E_θ is strictly decreasing, $\tau \in [0, 1-a]$ has a potentially stabilizing effect. The unique possible scenarios, in addition to those in which $\boldsymbol{\xi}^*$ is stable or unstable for any τ, are those in which $\boldsymbol{\xi}^*$ is unstable for $\tau < \bar{\tau} \in (0, 1-a)$ and recovers stability through a period halving bifurcation[12].

Now we reconsider from the dynamical point of view the settings we used for the simulations reported in Figure 2 to describe the Scenarios of Proposition 4.1. In Figure 3 we study the dynamical behavior[13] through bifurcations diagrams related to endemic steady states $\boldsymbol{\xi}_1^*$ (black color) and, when it is stable $\boldsymbol{\xi}_{df}^*$ (red color). Each diagram is obtained on varying $\tau \in I \subset [0, 1-a]$ (respectively, $\tau \in I_{df} \subset [0, 1-a]$), where I is the range of values of τ for which the endemic steady state $\boldsymbol{\xi}_1^*$ exists (respectively, $\boldsymbol{\xi}_{df}^*$ is stable). The initial conditions are chosen suitably close to $\boldsymbol{\xi}_1^*$ and $\boldsymbol{\xi}_{df}^*$, respectively. When multiple attractors coexist, we report some diagrams for the basins of attractions of $\boldsymbol{\xi}_1^*$ (green color) and $\boldsymbol{\xi}_{df}^*$ (yellow color).

We remark that since all the simulations are obtained with $\gamma = 1$ and $\theta(0) \in [3.9, 4]$, without healthcare policy ($\tau = 0$), steady state $\boldsymbol{\xi}_1^*$ inherits the instability of the corresponding endemic steady state in the isolated SIS equation, and endogenous oscillations are transmitted to the economic domain.

This is what can be observed in all the black bifurcation diagrams reported in Figures 3 (a,c,d,g). In these cases, for $\tau = 0$ we observe chaotic trajectories, as evident from the time series in Figure 3 (b). In line with the theoretical insights, τ has a stabilizing effect, and complex dynamics simplify into periodic ones and finally converge to the endemic steady state. We remark that from Figure 3 (c) we have a further confirmation of the transcritical bifurcation occurring at $\boldsymbol{\xi}_{df}^*$, while from Figure 3 (g), we can realize that $\boldsymbol{\xi}_1^*$ disappears at $\tau \approx 0.42$ as a consequence of a fold bifurcation, when it merges with the repelling $\boldsymbol{\xi}_2^*$, which entered the set of feasible values through a transcritical bifurcation at $\tau \approx 0.36$.

When $\boldsymbol{\xi}_1^*$ coexists with stable $\boldsymbol{\xi}_{df}^*$, if we look at the basins of attraction reported in Figures 3 (e,f,h,i), we can see that, as τ increases, the basin of $\boldsymbol{\xi}_1^*$ shrinks and that of $\boldsymbol{\xi}_{df}^*$ grows. Even if not evident from Figures (e,h), if s_0 is close to 0, convergence is toward the disease free steady

[12] We stress that, recalling Proposition 3.2, for $\tau \in [1-a, 1]$ we have a symmetric behavior with respect to that on $[0, 1-a]$. This means that if $\boldsymbol{\xi}^*$ is unstable for $\tau \in [0, \tau_1)$ and stable for $\tau \in (\tau_1, 1-a]$, there will be a $\bar{\tau}_1 \in (1-a, 1)$ such that $\boldsymbol{\xi}^*$ is stable for $\tau \in (1-a, \bar{\tau}_1)$ and unstable for $\tau \in (\bar{\tau}_1, 1]$.

[13] As for the static case, the goal of the present Section is not to provide a systematic description of all the dynamical situations that may occur for each endemic steady state configuration. The simulations we present aim to show the common effects of τ on dynamics, which essentially hold true for any numerical test we performed.

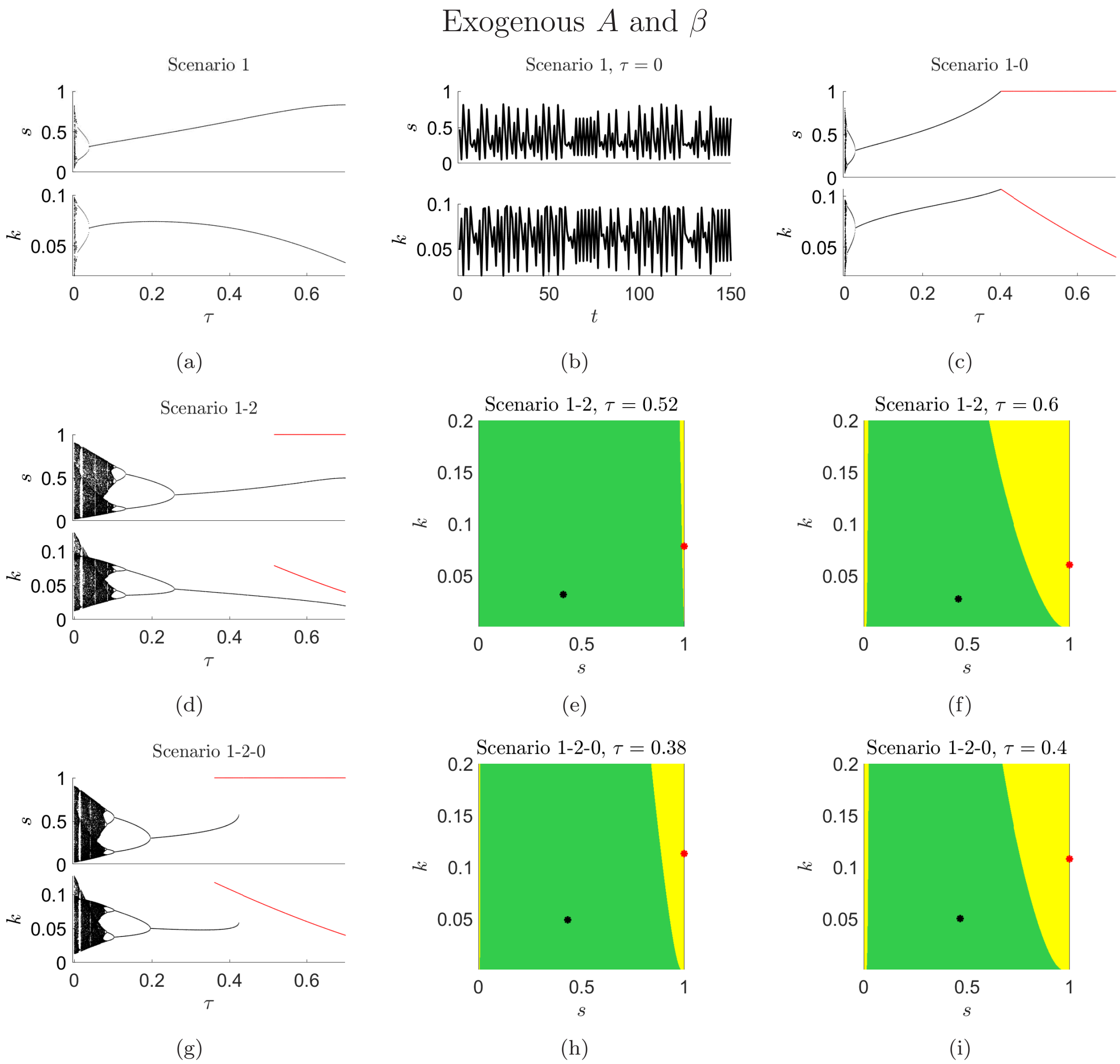
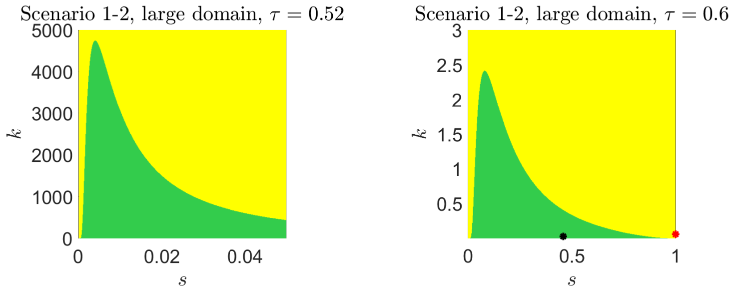


Figure 3: Panels (a,b): Scenario “1”, bifurcation diagrams (a) and time series (b). Panel (b): Scenario “1-0”, bifurcation diagrams. Panels (d,e,f): Scenario “1-2”, bifurcation diagrams (d) and basins of attraction (e,f). Panel (g,h): Scenario “1-2-0”, bifurcation diagrams (g) and basins of attraction (h,i). In the bifurcation diagrams, black color refers to attractors related to $\boldsymbol{\xi}_1^*$, red color to $\boldsymbol{\xi}_{df}^*$. In the basins of attraction, green color denotes the basins for the attractors related to $\boldsymbol{\xi}_1^*$, yellow color to $\boldsymbol{\xi}_{df}^*$.

state $\boldsymbol{\xi}_{df}^*$. This latter case can be explained as follows: if the fraction of susceptible people is initially small, a few new infections are possible, which means that the fraction of population that recovers is larger than the one that will be infected, especially if the recovery rate is large, as in the proposed simulations. This leads to a significant decrease of the infected people and this, together with a prompt health policy, can drive the epidemiological trajectories toward the steady state characterized by a larger fraction of susceptibles. We stress that we always observed numerically that the basins are connected sets and convergence toward $\boldsymbol{\xi}_1^*$ occurs if the initial

Exogenous A and β



(a) (b)

Figure 4: Basins of attraction for a large range of initial capital values, related to the simulation reported in Figures 3 (e,f).

epidemiological situation is close to that at $\boldsymbol{\xi}_1^*$.

As an example, we report in Figure 4 the basins related to Figures 3 (e,f) obtained for a large range of initial values of k. It's worth noting that to appreciate connectedness, very large initial capital levels may have to be considered.

This first set of simulations, even if characterized by a relatively small complexity degree in terms of dynamics and coexistence, points out some interesting features about the role of taxation rate.

a) τ exhibits a local stabilizing effect. As τ increases, complex dynamics arising around unstable endemic steady states gradually simplifies and local stability is recovered.

b) τ can increase the global complexity of the scenario, but to the detriment of less desirable steady states. We already observed in the static analysis that increasing τ gives rise to endemic steady states characterized by more favorable capital levels and fractions of susceptibles. The dynamical analysis shows that trajectories more likely converge toward them as τ increases, as their basins of attraction grow.

c) If the epidemiological scenario is characterized by a significant spread of the disease and/or the health policy is not suitably effective, trajectories could be locked in an "endemic trap", i.e. the long run evolution may not be the best possible, both from the epidemiological and economic point of view.

4.2 The role of endogenous factor productivity and probability to survive

In this section we lay emphasis on the effect of the endogenization of functions A and β, in order to explain the influence of the epidemiological domain on the economic one. Concerning θ, we still consider Assumption 1.

In [11], function A was chosen as a piecewise constant function, to describe the fact that above a certain threshold of infected agents, there is an abrupt fall in the productivity. In line with this, but to provide a more realistic gradual transition between high and low levels of productivity, we assume that A has a sigmoidal shape, in which the inflection point represents the fraction of susceptible agents corresponding to the maximum fall in the factor productivity. Having in mind this and recalling that β is increasing and concave, to focus on the simplest setting arising we make in addition the following assumption.

Assumption 3. *Functions A and β are such that $E_{g^*}(s)$ has at most two monotonicity changes.*

We stress that due to the possible change of concavity for A, it would be too restrictive to assume that $E_{g^*}(s)$ changes its monotonicity at most once. Moreover, the case of more than two monotonicity changes simply consists of a replication of the phenomena we are going to show.

In what follows we consider

$$\beta(s) = \beta_0 + \frac{1-\beta_0}{(1+\beta_1)^n}(s+\beta_1)^n, \tag{26}$$

with $n \in (0,1]$, $\beta_0 \in (0,1)$ and $\beta_1 > 0$, so that $\beta(s)$ is differentiable and $\beta(1) = 1$ and

$$A(s) = \begin{cases} \underline{A} & 0 < s < s_{A,1}, \\ \frac{\chi_3 s^3 + \chi_2 s^2 + \chi_1 s + \chi_0}{(s_{A,2} - s_{A,1})^3} & s_{A,1} \le s \le s_{A,2}, \\ 1 & s_{A,2} < s < 1, \end{cases} \tag{27}$$

with $\chi_3 = -2\left(1-\underline{A}\right)$, $\chi_2 = 3(s_{A,1} + s_{A,2})\left(1-\underline{A}\right)s_{A,1}^3$, $\chi_1 = -6 s_{A,1} s_{A,2}\left(1-\underline{A}\right)$ and $\chi_0 = -s_{A,1}^3 + 3 s_{A,1}^2 s_{A,2} - 3\underline{A} s_{A,1} s_{A,2}^2 + \underline{A} s_{A,2}^3$.

Note that the seemingly complicated expression for $A(s)$ simply consists of a cubic polynomial connecting with regularity two constant pieces. The parameters we use in all the simulations are reported in Table 1, and the resulting graphs of $A(s)$ and $\beta(s)$ are depicted in Figure 1 (c).

In the simulations we report in this section we can have at most 3 coexisting endemic steady states. In general, with endogenous A and β it would be intricate to arrange a Proposition like 4.1, and remaining analytical results do not simplify with respect to those reported in Section 3. Moreover, the role of endogenization of factor productivity and probability to survive can be better understood in terms of a perturbation of the case with constant A and β, so we just discuss the effect on the numerical outcomes reported in Section 4.1.

We compare each panel of Figure 5 with the corresponding one of Figure 2 as we take into account functions (26) and (27). Firstly, we focus on the changes occurred at function φ by comparing Figures 2 (a,d,g,j) and Figures 5 (a,d,g,j). If we look at the rightmost parts of the red graphs in corresponding Scenarios, we can see that they are very similar. This resemblance occurs also for graphs corresponding to intermediate values of τ (e.g. for $\tau = 0.4$ in Figures 2 (d) - 5 (d), for $\tau = 0.52$ in Figures 2 (g) - 5 (g) and for $\tau = 0.36$ in Figures 2 (j) - 5 (j)). The reason of this is that s_i^* is large enough to avoid the fall in the productivity (the factor is the same, $A(s) = 1$), and the probability to survive is close to 1. We stress that choosing $n = 1$ in (26) we have that the effect of endogenizing β is the maximum possible when we are close to $s = 1$. This suggests that the effect of an endogenous probability to survive is quite mild and does not significantly affect the endemic steady state configuration characterized by sufficiently large values of s^*. This is also confirmed by comparing the regions with $s^* > 0.6$ of Figures 2-5(b,e,h,k) and with $k^* > 0.08$ Figures 2-5 (c,f,i,l), as they are essentially the same.

Conversely, if we compare each panel in Figure 5 with the corresponding one in Figure 2 for small values of s, we can see that the situation is quite different. The double change of monotonicity occurring for φ in all the graphs reported in Figures 5 (a,d,g,j) for $s < 0.5$ is a consequence of the fall in the factor productivity in the presence of a few susceptible agents. As a consequence of this, k is small and the effect is that the government intervention is quite ineffective. This results in endemic steady states characterized by small shares of susceptible agents, as we can note by comparing the curves for s_1^* in Figures 2-5(b,e,h,k). In addition, now s_1^* increases much more slowly, and this results in corresponding levels of k_1^* that are essentially flat or decreasing. In most Scenarios, $\boldsymbol{\xi}_1^*$ is still present also for large values of τ, but coexists with $\boldsymbol{\xi}_3^*$. Note that if we compare the graphs related to $\boldsymbol{\xi}_3^*$ in Figure 5 with those related to $\boldsymbol{\xi}_1^*$ in Figure 2 (when they both exist), we can conclude that they resemble very much. This is because introducing an endogenous factor productivity actually gives rise for small values of τ to a new endemic steady state ($\boldsymbol{\xi}_1^*$ in Figure 5), with strongly reduced share of susceptibles and capital levels with respect to those of $\boldsymbol{\xi}_1^*$ in Figure 2. Such a new steady state coexists with $\boldsymbol{\xi}_3^*$ for larger values of τ, and hence the complexity of the endemic steady state scenarios increases, mainly through fold bifurcations. Differently from the exogenous SIS and from the framework studied in Section 4.1, multiple non repelling endemic steady states can occur.

We remark that from Figures 5 (c,f,i,l) the taxation rates that provide the highest non repelling steady state capital level are essentially the same we highlighted for the corresponding Figures 2 (c,f,i,l). What is different, is that too mild healthcare policies have much more harmful effects.

Now we study the effects of endogenizing A and β on dynamics.

Comparing the black bifurcation diagrams in Figures 6 (a,d,g,i) with those for the corresponding scenarios in Figure 3 (a,c,d,g), we can conclude that endogenous factor productivity and survive probability have a destabilizing effect on endemic steady state $\boldsymbol{\xi}_1^*$, as the ranges of values for which it is unstable is larger in Figures 6 (a,d,g,i) than in Figures 3 (a,c,d,g). In some cases, we still have a period halving bifurcation, i.e. increasing the role of the health policy allows recovering the local stability of $\boldsymbol{\xi}_1^*$, but this does no more hold true for all the scenarios.

From the theoretical point of view, under Assumptions 1 and 3, $E_\theta(g^*(s,\tau))(E_{g^*}(s))$ can have multiple monotonicity changes. This means that an endemic steady state may be stable for some $\tau_1 \in (0, 1-a)$ and unstable for some $\tau_2 \in (\tau_1, 1-a)$, with a possible destabilizing role of τ that could not occur with constant A and β. However, even if this is possible according to (22), numerical simulations show that it just occurs for quite extreme function choices, otherwise, we typically find that increasing τ stabilizes an unstable endemic steady state or it remains unstable.

If we look at the basins of attraction reported in Figures 6 (b,c), we can see that even with nonlinear functions A and β, as τ increases the basins of "least desirable" endemic steady states ($\boldsymbol{\xi}_1^*$) shrink and those of "more desirable" ones ($\boldsymbol{\xi}_3^*$) grow. We checked that the same occurs also for Scenarios "1-0", "1-2" and "1-2-1". In addition to this, in the present cases we can have coexistence between complex attractors and (possibly multiple) stable steady states (Figures 6 (h,j,k,l))

When pertinent, the three outcomes a, b, c highlighted at the end of Section 4.1 are still valid.

5 Conclusions and future perspectives

By investigating a simple model for the interaction between the economic and epidemiological domains, we highlighted several hints that should be kept in mind. First, when considered as isolated, the OLG model has a unique, globally stable equilibrium, while in the SIS model a unique endemic steady state coexists with a disease free one, but one of them is always repelling, so no path dependency in convergence can be observed. Conversely, the coupling of the two domains can give rise to multiple steady states, and to the possible coexistence of several non repelling endemic steady states with the stable disease free one. The second evidence is that the static analysis alone can be misleading. On varying the taxation level the number of possible steady states can change and τ can affect their stability, their basins of attraction and the possible complexity of the arising trajectories. This also means that, if the initial situation varies, the final outcomes may change significantly. The very same starting point may evolve toward a disease free desirable outcome as well as remain locked in an endemic trap. In these cases, promptly triggering healthcare intervention may foster the evolution toward a disease free desirable outcome.

Effective policy-making cannot ignore all the foregoing aspects. Indeed, this contribution is just a first step and can be improved in many directions. For example, limiting to the epidemiological and economic spheres, both the baseline models describing each domain can be improved, to allow for studying more empirically relevant results. Moreover, the interaction with additional domains, like that environmental, can be included. With regard to this, it is interesting to understand how the static and dynamical outcomes of the present contribution are affected by the new element of complexity. Finally, the effectiveness of endogenous policies that can properly react to different scenarios may be taken into account, to see if they are able to select the most desirable scenarios.

Appendix

Proof of Prop. 3.1. Setting $s_{t+1} = s_t = s$ and $k_{t+1} = k_t = k$ in (10) we find that the steady states are solution of

$$\begin{cases} (1-s)[s\theta(g(s,k)) - \gamma] = 0, \\ k = (1-a)(1-\tau)\dfrac{\beta(s)}{1+\beta(s)}A(s)k^a s^{1-a} \end{cases} \tag{28}$$

from which we have that the latter equation is solved by $k = 0$ and $k = k^*$ defined in (12). The former equation is solved by $s = 1$ and $s\theta(g(s,k)) = \gamma$. Replacing $k = 0$ and $k = k^*$ in $g(s,k)$ and combining the possible components of the solutions allows concluding. □

Proof of Prop. 3.2. Since

$$\frac{\partial g^*(s,\tau)}{\partial \tau} = (1-a-\tau)\left(\frac{\beta(s)}{1+\beta(s)}\right)^{\frac{a}{1-a}} [(1-a)(1-\tau)]^{\frac{2a-1}{1-a}} (A(s))^{\frac{1}{1-a}} s \tag{29}$$

we have that, for $s = 0$ function g^* is constant, while for any given $s \in (0,1]$, function $\tau \mapsto g^*(s,\tau)$ is strictly increasing for $\tau \in [0, 1-a]$ and strictly decreasing for $\tau \in [1-a, 1]$, with $g^*(s,0) = g^*(s,1) = 0$. For any $\tau \in [0, 1-a)$ there is a unique $\bar{\tau} \in (1-a, 1]$ for which $g^*(s,\tau) = g^*(s,\bar{\tau})$, and vice-versa. The choice of $\bar{\tau}$ is independent of s, so that $g^*(s,\tau) = g^*(s,\bar{\tau})$ for any $s \in [0,1]$. This shows that, for each τ, there is a one-to-one correspondence between the solutions $s \in (0,1)$ to $\varphi(s,\tau) = 0$ and to $\varphi(s,\bar{\tau}) = 0$, which proves the first part of the proposition.

Let $\bar{\tau}_1, \bar{\tau}_2 \in (1-a, 1]$ be the unique values for which $\varphi(s,\tau_1) = \varphi(s,\bar{\tau}_1)$ and $\varphi(s,\tau_2) = \varphi(s,\bar{\tau}_2)$ for any s. Indeed, $\varphi(s,\tau_1) = 0$ and $\varphi(s,\bar{\tau}_1) = 0$ have the same solutions, as well as $\varphi(s,\tau_2) = 0$ and $\varphi(s,\bar{\tau}_2) = 0$. If $\boldsymbol{\xi}^*(\tau_1) = (s_1^*, k_1^*)$, we have $\varphi(s_1^*, \tau_1) = 0$. From the first part of the proof, there exists a unique endemic steady state $\boldsymbol{\xi}^*(\bar{\tau}_1) = (s^*(\bar{\tau}_1), k^*(\bar{\tau}_1))$ occurring for $\bar{\tau}_1 \in [1-a, 1]$.

The same for $\boldsymbol{\xi}^*(\tau_2)$ and $\boldsymbol{\xi}^*(\bar{\tau}_2)$. Since $\varphi(s,\tau)$ (for $s \neq 0$) is strictly increasing on $[0, 1-a]$ and strictly decreasing on $[1-a, 1]$, we have that from $\tau_1 < \tau_2$ we have $\bar{\tau}_1 > \bar{\tau}_2$. □

Proof of Prop. 3.3. Recalling the link between endemic steady states and solutions $s \in (0,1)$ to $\varphi(s,\tau) = 0$, we focus on this latter problem.

$$\frac{\partial \varphi}{\partial s} = -\theta(g^*(s,\tau)) - s\frac{\partial g^*(s,\tau)}{\partial s}\theta'(g^*(s,\tau)) = -\theta(g^*(s,\tau))(1 + E_\theta(g^*(s,\tau))E_{g^*}(s)). \tag{30}$$

For a given τ, to have multiple solutions to $\varphi(s,\tau) = 0$, function $s \mapsto \varphi(s,\tau)$ must not be strictly monotonic. From (30), φ can have a monotonicity change only if $E_\theta(g^*(s,\tau))E_{g^*}(s) = -1$ for some s and this proves (i).

Recalling Proposition 3.2 we have that $\varphi(s,0) \leq \varphi(s,\tau)$ for any $\tau \in [0, 1-a]$. Note that $\varphi(s,0) = \gamma - s\theta(0) > 0$ for any $s \in (0,1)$ if and only if $\gamma - \theta(0) \geq 0$. This allows concluding that for any $\tau \in [0, 1-a]$ we have $\varphi(s,\tau) \geq \varphi(s,0) > 0$ for any $s \in (0,1)$, so $\varphi(s,\tau) = 0$ has no solutions $s \in (0,1)$ for any τ and (ii) is proved.

Note that $\varphi(0,\tau) > 0$ and $\frac{\partial \varphi}{\partial s}(0,\tau) < 0$ for any $\tau \in [0, 1-a]$. If all the $N(\tau)$ (isolated) stationary points $s_i(\tau), i = 1, \ldots, N(\tau)$ correspond to a change of monotonicity, so that the function is decreasing, then increasing and so on, if moreover $\varphi(s_1(\tau),\tau) < 0$, $\varphi(s_2(\tau),\tau) > 0$ and so on, then $\nu(\tau) = N(\tau)$ or $\nu(\tau) = N(\tau) + 1$, depending weather there is a last zero in $(s_{N(\tau)}, 1)$ or not. If a change of monotonicity or a change of sign of $\varphi(s_i(\tau),\tau)$ do not happen, then the number of zeros $\nu(\tau)$ will be less than $N(\tau) + 1$. This provides (iii). Finally, recalling that we count twice the solutions to $\varphi(s,\tau) = 0$ if and only if they are extremum points, assuming $\boldsymbol{\xi}_i^* < \boldsymbol{\xi}_j^*$ for $i \neq j$ guarantees that no solution to $\varphi(s,\tau) = 0$ is an extremum point. So (iv) immediately follows from (30) and the previous considerations. □

Proof of Prop. 3.4 and 3.5. We recall that $\tau \mapsto \varphi(s,\tau)$ is strictly increasing with respect to τ on $[0, 1-a]$ for any $s \in (0,1]$.

Let s^* be such that $\varphi(s^*,\bar{\tau}) = 0$ and $\frac{\partial \varphi}{\partial s}(s^*,\bar{\tau}) \neq 0$. If $s^* < 1$, the implicit function theorem applies and gives (a1). If $\bar{s} = 1$, the implicit function theorem gives either (b2) or (b3), because for some values of τ there holds $s(\tau) > \bar{s} = 1$.

Let now $\frac{\partial \varphi}{\partial s}(s^*,\bar{\tau}) = 0$ and the second partial derivative be

$$\frac{\partial^2 \varphi}{\partial s^2}(s^*,\bar{\tau}) = -\theta''\left(g^*(s^*,\bar{\tau})\right)\left(\frac{\partial g^*}{\partial s}(s^*,\bar{\tau})\right)^2 s^* - \theta'\left(g^*(s^*,\bar{\tau})\right)\left[\frac{\partial^2 g^*}{\partial s^2}(s^*,\bar{\tau})s^* + 2\frac{\partial g^*}{\partial s}(s^*,\bar{\tau})\right] < 0.$$

Then s^* is a maximum point for $\varphi(s,\bar{\tau})$. It is immediate that, for any $\tau < \bar{\tau}$, $\varphi(s,\tau)$ has no solutions in a neighborhood of s^*. We choose $\epsilon > 0$ and $\delta > 0$ such that on $(s^*-\delta, s^*+\delta) \times [\bar{\tau}, \bar{\tau}+\epsilon)$ the second partial derivative $\frac{\partial^2 \varphi}{\partial s^2}(s,\tau)$ is negative. If necessary, we choose a smaller ϵ, so that

$\varphi(s^* - \delta/2, \tau)$ and $\varphi(s^* + \delta/2, \tau)$ are negative for $\tau \in (\bar{\tau}, \bar{\tau} + \epsilon)$. Now, for any $\tau \in (\bar{\tau}, \bar{\tau} + \epsilon)$, we have that $\varphi(s^* \pm \delta/2, \tau) < 0$ and $\varphi(s^*, \tau) > 0$. By the continuity of φ there exist $s^* - \delta/2 < s_1(\tau) < s^* < s_2(\tau) < s^* + \delta/2$ such that $\varphi(s_i(\tau), \tau) = 0$, $i = 1, 2$. The zeros must be unique because the second derivative is negative. This gives either (a2) or (b2).

The case in which s^* is minimum point is analogous and gives either (a3) or (b3).

Finally, case (b1) happens when the first equation in (28) is fulfilled because $1 - s = 0$, but $\varphi(1, \tau_0)$ is not zero. □

Proof of Prop. 3.6. From now on, for readability, in the next computations we drop the superscript *, but we assume to be at a steady state.

We have

$$\frac{\partial \varphi}{\partial \tau}(s, \tau) = -s\theta'(g^*(s, \tau))\frac{\partial g^*}{\partial \tau}(s, \tau) = -s\theta'(g^*(s, \tau))\frac{g^*(s, \tau)(1 - a - \tau)}{\tau(1 - \tau)(1 - a)} \tag{31}$$

in which we rearranged the expression in (29). Using (30) and (31) we obtain

$$\frac{ds}{d\tau}(\tau) = -\frac{\frac{\partial \varphi}{\partial \tau}(s, \tau)}{\frac{\partial \varphi}{\partial s}(s, \tau)} = -\frac{sE_\theta(g^*(s, \tau))(1 - a - \tau)}{\tau(1 - \tau)(1 - a)(1 + E_\theta(g^*(s, \tau))E_{g^*}(s))} \tag{32}$$

which allows concluding about the comparative statics of s^* for $\tau \neq 0$. In general, it is possible to write

$$\frac{E_\theta(g^*(s, \tau))}{\tau} = \frac{\left(\frac{\beta(s)}{1+\beta(s)}(1 - a)(1 - \tau)\right)^{\frac{a}{1-a}} A(s)^{\frac{1}{1-a}} s\theta'(g^*(s, \tau))}{\theta(g^*(s, \tau)} \tag{33}$$

which means that the comparative statics of s^* is valid also for $\tau = 0$.

Let

$$k(s(\tau), \tau) = \left(\frac{\beta(s(\tau))}{1 + \beta(s(\tau))} A(s(\tau))(1 - a)(1 - \tau)\right)^{\frac{1}{1-a}} s(\tau).$$

Direct computation provides

$$\frac{\partial k}{\partial \tau}(s, \tau) = -\frac{k(s, \tau)}{(1 - a)(1 - \tau)}, \quad \frac{\partial k}{\partial s}(s, \tau) = \frac{k(s, \tau)}{s}\left[\frac{1}{1 - a}E_{\beta/(1+\beta)}(s) + \frac{1}{1 - a}E_A(s) + 1\right].$$

Since $\frac{dk}{d\tau}(s(\tau), \tau) = \frac{\partial k}{\partial \tau}(s(\tau), \tau) + \frac{\partial k}{\partial s}(s(\tau), \tau)\frac{ds}{d\tau}(\tau)$, noting that $\frac{\partial k}{\partial \tau}(s, \tau) \leq 0$, $\frac{\partial k}{\partial s}(s, \tau) \geq 0$ and if $\tau > 1 - a$ we have $\frac{ds}{d\tau}(\tau) < 0$, we conclude that $\frac{dk}{d\tau}(\tau) < 0$ if $\tau > 1 - a$.

Moreover, recalling (32), (16) and from

$$E_{g^*}(s) = \frac{a}{1 - a}E_{\beta/(1+\beta)}(s) + \frac{1}{1 - a}E_A(s) + 1$$

we have

$$\frac{dk}{d\tau}(s(\tau), \tau) = -k(s(\tau), \tau)\frac{\tau + E_\theta(g^*(s(\tau), \tau))\left[(1 - \tau)E_{\beta/(1+\beta)}(s(\tau)) + E_A(s(\tau)) + 1 - a\right]}{\tau(1 - \tau)(1 - a)\left[1 + E_\theta(g^*(s(\tau), \tau))E_{g^*}(s(\tau))\right]}$$

which provides comparative statics for k^*, with $\tau \neq 0$. The sign of $\frac{dk}{d\tau}(s(0),0)$ is the same of

$$\frac{dk}{d\tau}(s(0),0)\frac{1}{k(s(0),0)}.$$

Using (33), it is also possible to write

$$\frac{dk}{d\tau}(s(\tau),\tau) = -k(s(\tau),\tau)\frac{B(s(\tau),\tau)}{(1-\tau)(1-a)\left[1+E_\theta(g^*(s(\tau),\tau))E_{g^*}(s(\tau))\right]},$$

where

$$\begin{aligned}B(s(\tau),\tau) =& 1+\frac{\theta'(g^*)}{\theta(g^*)}\left(\frac{\beta(s(\tau))}{1+\beta(s(\tau))}(1-a)(1-\tau)\right)^{\frac{a}{1-a}} A(s(\tau))^{\frac{1}{1-a}}s(\tau)\left[(1-\tau)E_{\beta/(1+\beta)}(s(\tau))\right.\\ &\left.+E_A(s(\tau))+1-a\right]\end{aligned}$$

and we have written g^* instead of $g^*(s(\tau),\tau)$. This provides comparative statics for k^*, with $\tau = 0$. We remark that (20) is well defined since $s^*(\tau) \geq \gamma/\theta(0) > 0$ for any τ. □

We denote the map defined by the right-hand side of equation (10) in the following way

$$\begin{aligned}F_1(s,k) &= s[1-\theta(g(s,k))(1-s)]+\gamma(1-s),\\ F_2(s,k) &= \frac{\beta(F_1(s,k))}{1+\beta(F_1(s,k))}(1-a)(1-\tau)A(s)k^a s^{1-a}.\end{aligned}$$

The entries of its Jacobian matrix $J(s,k)$ will be denoted $J_{ij}(s,k)$, with $i,j=1,2$ and are

$$\begin{aligned}J_{11}(s,k) =& 1-\gamma+(2s-1)\theta(g(s,k))\\ &-s(1-s)\theta'(g(s,k))\left[\tau A'(s)s^{1-a}k^a+(1-a)\tau A(s)\left(\frac{k}{s}\right)^a\right]\\ J_{12}(s,k) =& -a\tau s(1-s)\theta'(g(s,k))A(s)\left(\frac{s}{k}\right)^{1-a}\\ J_{21}(s,k) =& (1-a)(1-\tau)\frac{\beta(F_1(s,k))}{1+\beta(F_1(s,k))}\left[A'(s)k^a s^{1-a}+(1-a)A(s)\left(\frac{k}{s}\right)^a\right]\\ &+(1-a)(1-\tau)\frac{\beta'(F_1(s,k))}{[1+\beta(F_1(s,k))]^2}A(s)k^a s^{1-a}J_{11}\\ J_{22}(s,k) =& (1-a)(1-\tau)\frac{\beta(F_1(s,k))}{1+\beta(F_1(s,k))}aA(s)\left(\frac{s}{k}\right)^{1-a}\\ &+(1-a)(1-\tau)\frac{\beta'(F_1(s,k))}{[1+\beta(F_1(s,k))]^2}A(s)k^a s^{1-a}J_{12}\end{aligned}$$

and in what follows we identify by J^* and $J^*_{ij}, i,j=1,2$ the Jacobian matrix and its entries, respectively, evaluated at a steady state.

Proof. of Prop. 3.7. We start recalling (see e.g. [14]) that stability requires $1-\mathrm{tr}(J^*)+\det(J^*) > 0$. We have

$$1 - \mathrm{tr}(J(1,k)) + \det(J(1,k)) = \frac{1}{k^{1-a}}(\gamma - \theta(\tau k^a)) \left[k^{1-a} - (1-a)(1-\tau)a\frac{\beta(1)}{1+\beta(1)}\right]$$

and hence $\lim_{k\to 0^+} 1 - \mathrm{tr}(J(1,k)) + \det(J(1,k)) = -\infty$, so $\boldsymbol{\xi}^*_{df,0}$ is unstable.

Similarly, we have

$$1 - \mathrm{tr}\left(J\left(\frac{\gamma}{\theta(0)}, k\right)\right) + \det\left(J\left(\frac{\gamma}{\theta(0)}, k\right)\right) = -z_1 k^{a-1} + z_2 k^{2a-1} + z_3$$

where we set

$$z_1 = \frac{a\gamma\beta\left(\frac{\gamma(\theta(0)+\gamma\sigma-\gamma\theta(0)-\sigma\theta(0)+\theta(0)^2)}{\theta(0)^2}\right)A\left(\frac{\gamma}{\theta(0)}\right)(1-a)(1-\tau)(\gamma\theta(0)-2\gamma\sigma+\sigma\theta(0))}{\theta(0)^2\left(\frac{\gamma}{\theta(0)}\right)^a\left(\beta\left(\frac{\gamma(\theta(0)+\gamma\sigma-\gamma\theta(0)-\sigma\theta(0)+\theta(0)^2)}{\theta(0)^2}\right)+1\right)}$$

$$z_2 = \frac{a\gamma^3\tau\beta'\left(\frac{\gamma(\theta(0)+\gamma\sigma-\gamma\theta(0)-\sigma\theta(0)+\theta(0)^2)}{\theta(0)^2}\right)\theta'(\sigma)A\left(\frac{\gamma}{\theta(0)}\right)^2(\gamma-\theta(0))(1-a)(1-\tau)}{\theta(0)^4\left(\beta\left(\frac{\gamma(\theta(0)+\gamma\sigma-\gamma\theta(0)-\sigma\theta(0)+\theta(0)^2)}{\theta(0)^2}\right)+1\right)^2\left(\frac{\gamma}{\theta(0)}\right)^{2a}}$$

$$z_3 = \gamma - \theta(\sigma)\left(\frac{\gamma}{\theta(0)} - 1\right) - \frac{\gamma\tau k^a\left(\theta(\sigma)+\theta'(\sigma)\left(\frac{\gamma^{1-a}A'\left(\frac{\gamma}{\theta(0)}\right)}{\theta(0)^{1-a}} + \frac{A\left(\frac{\gamma}{\theta(0)}\right)(1-a)}{\left(\frac{\gamma}{\theta(0)}\right)^a}\right)(\gamma-\theta(0))\right)}{\theta^2(0)}$$

with

$$\sigma = \theta\left(\frac{\gamma^{1-a}k^a\tau A\left(\frac{\gamma}{\theta(0)}\right)}{\theta(0)^{1-a}}\right)$$

We note that for suitably small values of k, we must study the sign of z_1. Recalling that by Proposition 3.1 the steady state $\boldsymbol{\xi}^*_0$ exists only if $\gamma < \theta(0)$, z_1 is positive and we can conclude that $\lim_{k\to 0^+} 1 - \mathrm{tr}(J(\gamma/\theta(0),k)) + \det(J(\gamma/\theta(0),k)) = -\infty$ and hence $\boldsymbol{\xi}^*_0$ is unstable. □

Proof of Prop. 3.8. We evaluate the Jacobian matrix at the disease free stationary state (s_{df}, k_{df}):

$$J_{11} = 1 - \gamma + \theta\left(\tau\left[\frac{\beta(1)}{1+\beta(1)}(1-a)(1-\tau)\right]^{\frac{a}{1-a}}\right)$$

$$J_{12} = 0$$

$$J_{21} = (1-a)(1-\tau)\frac{\beta(1)}{1+\beta(1)}(A'(1)+1-a)\left[\frac{\beta(1)}{1+\beta(1)}(1-a)(1-\tau)\right]^{\frac{a}{1-a}} + (1-a)(1-\tau)\frac{\beta'(1)}{[1+\beta(1)]^2}\left[\frac{\beta(1)}{1+\beta(1)}(1-a)(1-\tau)\right]^{\frac{a}{1-a}} J_{11}$$

$$J_{22} = a$$

Since $\gamma \le 1$ and θ is strictly decreasing, the two multipliers are real and positive.

Condition (21) corresponds to $J_{11} < 1$.

In what follows, we assume that τ is set in $[0, 1-a]$. From the expression of $\varphi(s,\tau)$, condition (21) is equivalent to $\varphi(1,\tau) > 0$, which means that $(1-s)\varphi(s,\tau)$ is strictly positive on a left neighborhood of $s = 1$. We recall that we count twice the solutions corresponding to extremum points of $\varphi(s,\tau)$.

Simple straightforward geometrical considerations provide conclusions about the connection between (21) and the number of endemic steady states. □

Proof of Prop. 3.9. We evaluate the Jacobian matrix at the endemic steady state (s^*, k^*), $J^* = J(s^*, k^*)$, where from now on we will write (s, k) instead of (s^*, k^*). Moreover, we avoid to write the explicit dependence of s^* on τ. We obtain

$$\begin{aligned}
J_{11} =& 1 + \gamma - \theta(g^*(s,\tau)) \\
&- s(1-s)\theta'(g^*(s,\tau))\tau \left[\frac{\beta(s)}{1+\beta(s)} A(s)(1-a)(1-\tau)\right]^{\frac{a}{1-a}} [A'(s)s + (1-a)A(s)] \\
J_{12} =& -\frac{a\tau(1+\beta(s))}{(1-a)(1-\tau)\beta(s)} s(1-s)\theta'(g^*(s,\tau)) \\
J_{21} =& (1-a)(1-\tau)\frac{\beta(s)}{1+\beta(s)} \left[\frac{\beta(s)}{1+\beta(s)} A(s)(1-a)(1-\tau)\right]^{\frac{a}{1-a}} [A'(s)s + (1-a)A(s)] \\
&+ (1-a)(1-\tau)\frac{\beta'(s)}{[1+\beta(s)]^2} A(s) \left[\frac{\beta(s)}{1+\beta(s)} A(s)(1-a)(1-\tau)\right]^{\frac{a}{1-a}} sJ_{11} \\
J_{22} =& a + (1-a)(1-\tau)\frac{\beta'(s)}{[1+\beta(s)]^2} A(s) \left[\frac{\beta(s)}{1+\beta(s)} A(s)(1-a)(1-\tau)\right]^{\frac{a}{1-a}} sJ_{12}
\end{aligned}$$

Since A is increasing, all the entries are positive and there holds

$$\begin{aligned}
\det(J^*) =& a[1 + \gamma - \theta(g^*(s,\tau))] \\
\mathrm{tr}(J^*) =& 1 + \gamma - \theta(g^*(s,\tau)) + a - \frac{g^*(s,\tau)(1-s)}{A(s)} \theta'(g^*(s,\tau))[A'(s)s + (1-a)A(s)] \\
&- \frac{a\beta'(s)}{\beta(s)(1+\beta(s))} g^*(s,\tau)s(1-s)\theta'(g^*(s,\tau))
\end{aligned}$$

which is well-defined as A is strictly positive, and in which we used the expression for $g^*(s,\tau)$. The conditions for stability (see e.g. [14]) are

$$\begin{cases} 1 - \mathrm{tr}(J^*) + \det(J^*) > 0 \\ 1 + \mathrm{tr}(J^*) + \det(J^*) > 0 \\ \det(J^*) < 1 \end{cases} \tag{34}$$

The last inequality is always fulfilled, since at an endemic steady state we have $s = \frac{\gamma}{\theta(g^*(s,\tau))} < 1$.

The first inequality gives

$$\begin{aligned}1-&\mathrm{tr}(J^*)+\det(J^*)=(\theta(g^*(s,\tau))-\gamma)(1-a)\\&+\frac{g^*(s,\tau)(1-s)}{A(s)}\theta'(g^*(s,\tau))[A'(s)s+(1-a)A(s)]\\&+\frac{a\beta'(s)}{\beta(s)(1+\beta(s))}sg^*(s,\tau)(1-s)\theta'(g^*(s,\tau))\\=&(\theta(g^*(s,\tau))-\gamma)(1-a)[1+E_\theta(g^*(s,\tau))(E_{g^*}(s))]>0\end{aligned}$$

and this gives the first inequality in (22).

The second inequality in (34) gives

$$\begin{aligned}1+&\mathrm{tr}(J^*)+\det(J^*)=(1+a)(2+\gamma-\theta(g^*(s,\tau)))\\&-\frac{g^*(s,\tau)(1-s)}{A(s)}\theta'(g^*(s,\tau))[A'(s)s+(1-a)A(s)]\\&-\frac{a\beta'(s)}{\beta(s)(1+\beta(s))}g^*(s,\tau)s(1-s)\theta'(g^*(s,\tau))\\=&(1+a)(2+\gamma-\theta(g^*(s,\tau)))\\&-(\theta(g^*(s,\tau))-\gamma)E_\theta(g^*(s,\tau))(1-a)(E_{g^*}(s))>0\end{aligned}$$

and this can be written as the second inequality in (22). □

Proof of Prop. 4.1. Since $E_\theta(g)$ is strictly decreasing, by (30) and the fact that $E_{g^*}(s)=1$ we have that $s\mapsto\varphi(s,\tau)$ can change its monotonicity at most once, being, for $s\in[0,1]$, either strictly decreasing or decreasing-increasing. Recalling that there is a one-to-one correspondence between the endemic steady states and the solutions to $\varphi(s,\tau)=0$, this means that we can have at most 2 endemic steady states. From geometrical considerations, as τ increases on $[0,1-a]$, solutions to $\varphi(s,\tau)=0$ can emerge/vanish at those $\bar\tau$ at which

a) $\varphi(s,\bar\tau)$ becomes tangent to the horizontal axis at some $s\in[0,1)$, in which case a couple of solutions disappears since $\varphi(s,\bar\tau)$ the tangency point is a minimum point

or

b1) $\varphi(1,\bar\tau)=0$ and $\varphi(s,\bar\tau)$ is decreasing on a left neighborhood of $s=1$, in which case a solution vanishes

b2) $\varphi(1,\bar\tau)=0$ and $\varphi(s,\bar\tau)$ is increasing on a left neighborhood of $s=1$, in which case a new solution emerges.

Moreover, since $\tau\mapsto\varphi(s,\tau)$ is strictly increasing, each case can occur at most once.

If for some taxation rate we have a unique endemic steady state, as τ increases the number of endemic steady states can just change due to cases b) (in fact case a) can happen only starting from two endemic steady states). In this situation, after case b1) we would have no endemic steady states (and their number can not change as τ increases, since case a) cannot occur), while after case b2) we would have two endemic steady states. So the possibilities are: 1, 1-0 or 1-2. In the last situation we may have the occurrence of case a), with consequently no endemic steady states (scenario 1-2-0).

We notice that for $\tau = 0$ function $\varphi(s, 0)$ is linear and has exactly one zero. This proves that all the scenarios start with one endemic steady state. No other evolution is possible and we obtain the four possible scenarios.

□

References

[1] L. J. S. Allen. Some discrete-time SI, SIR, and SIS epidemic models. *Mathematical Biosciences*, 124(1):83–105, 1994.

[2] E. Angelini, M. Damjanović, M. D. Pariès, and S. Zimic. Modelling pandemic risks for policy analysis and forecasting. *Economic Modelling*, 120:106162, 2023.

[3] G. I. Bischi, F. Grassetti, and E. J. Sanchez Carrera. On the economic growth equilibria during the Covid-19 pandemic. *Communications in Nonlinear Science and Numerical Simulation*, 112:106573, 2022.

[4] D. E. Bloom, M. Kuhn, and K. Prettner. *Health and Economic Growth*. Oxford University Press, feb 2019.

[5] B. Bridgman. Is labor's loss capital's gain? Gross versus net labor shares. *Macroecon Dyn*, 22(8):2070–2087, 2018.

[6] B. Callegari and C. Feder. A literature review of pandemics and development: the long-term perspective. *Economics of Disasters and Climate Change*, 6(1):183–212, 2022.

[7] J. Carmona and A. León. Pandemic effects in the Solow growth model. *Bulletin of Economic Research*, 2022.

[8] F. Cavalli, E. Moretto, and A. Naimzada. Dynamical analysis of evolutionary transition toward sustainable technologies. Technical report, DEMS Working paper series N 510, 2023.

[9] S. Chakraborty. Endogenous lifetime and economic growth. *Journal of Economic Theory*, 116(1):119–137, 2004.

[10] M. A. Cole and E. Neumayer. The impact of poor health on total factor productivity. *The Journal of Development Studies*, 42(6):918–938, 2006.

[11] M. Davin, M. Fodha, and T. Seegmuller. Environment, public debt, and epidemics. *Journal of Public Economic Theory*, 2022.

[12] D. De La Croix and P. Michel. *A theory of economic growth*. 2002.

[13] P. Dykas, T. Tokarski, and R. Wisła. *The Solow Model of Economic Growth*. Routledge, 2022.

[14] S. Elaydi. *Discrete chaos*. CRC Press, Taylor & Francis Group, 2007.

[15] S. Federico and G. Ferrari. Taming the spread of an epidemic by lockdown policies. *Journal of Mathematical Economics*, 93:102453, 2021.

[16] M. Fodha and T. Seegmuller. Environmental quality, public debt and economic development. *Environmental and Resource Economics*, 57:487–504, 2013.

[17] A. Goenka and L. Liu. Infectious diseases, human capital and economic growth. *Economic Theory*, 70(1):1–47, 2019.

[18] L. Gori, C. Mammana, P. Manfredi, and E. Michetti. Economic development with deadly communicable diseases and public prevention. *Journal of Public Economic Theory*, 24(5):912–943, 2021.

[19] A. John and R. Pecchenino. An overlapping generations model of growth and the environment. *The Economic Journal*, 104:1393–1410, 1994.

[20] J. Li, Y. Zhao, and H. Zhu. Bifurcation of an SIS model with nonlinear contact rate. *Journal of Mathematical Analysis and Applications*, 432(2):1119–1138, 2015.

[21] R. E. Lucas. On the mechanics of economic development. *Journal of Monetary Economics*, 22(1):3–42, 1988.

[22] P. Zeppini. A discrete choice model of transitions to sustainable technologies. *Journal of Economic Behavior & Organization*, 112:187–203, 2015.

[23] J. Zhang. Environmental sustainability, nonlinear dynamics and chaos. *Economic Theory*, 14:489–500, 1999.

Endogenous A and β

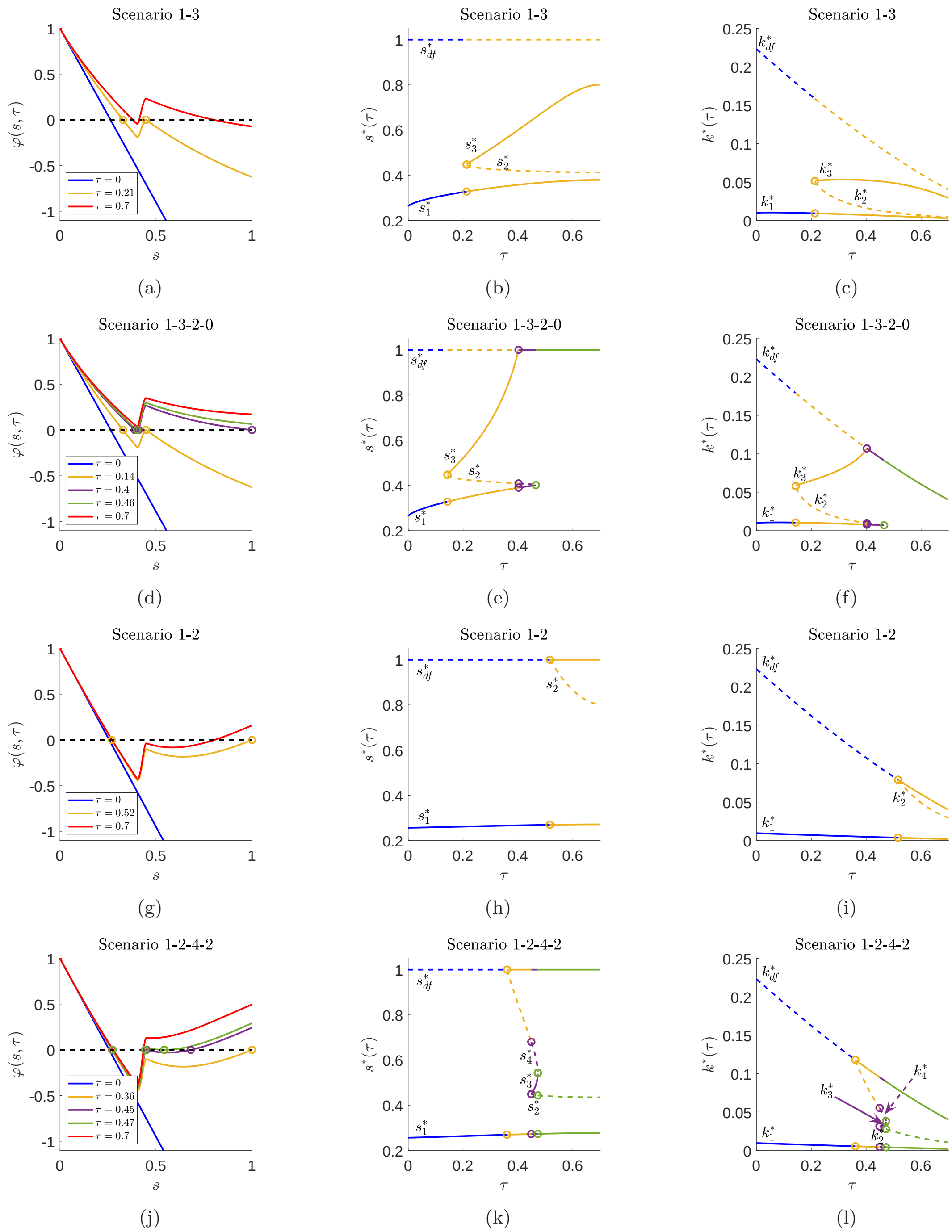


Figure 5: Simulations related to the Scenarios reported in Figure 2: each panel of the present figure is obtained by introducing nonlinear functions A and β for the simulation related to the corresponding panel of Figure 2. Left column: graphs of function φ for $\tau = 0$ (blue color) and $\tau = 1 - a$ (red color), and the intermediate values of τ for which we have a change in the number of endemic steady states (possible colors yellow, purple, green and light blue). Middle and right columns: evolution of the fraction of susceptible agents (middle) and of the capital level (right) for the endemic and the disease free steady states.

Endogenous A and β

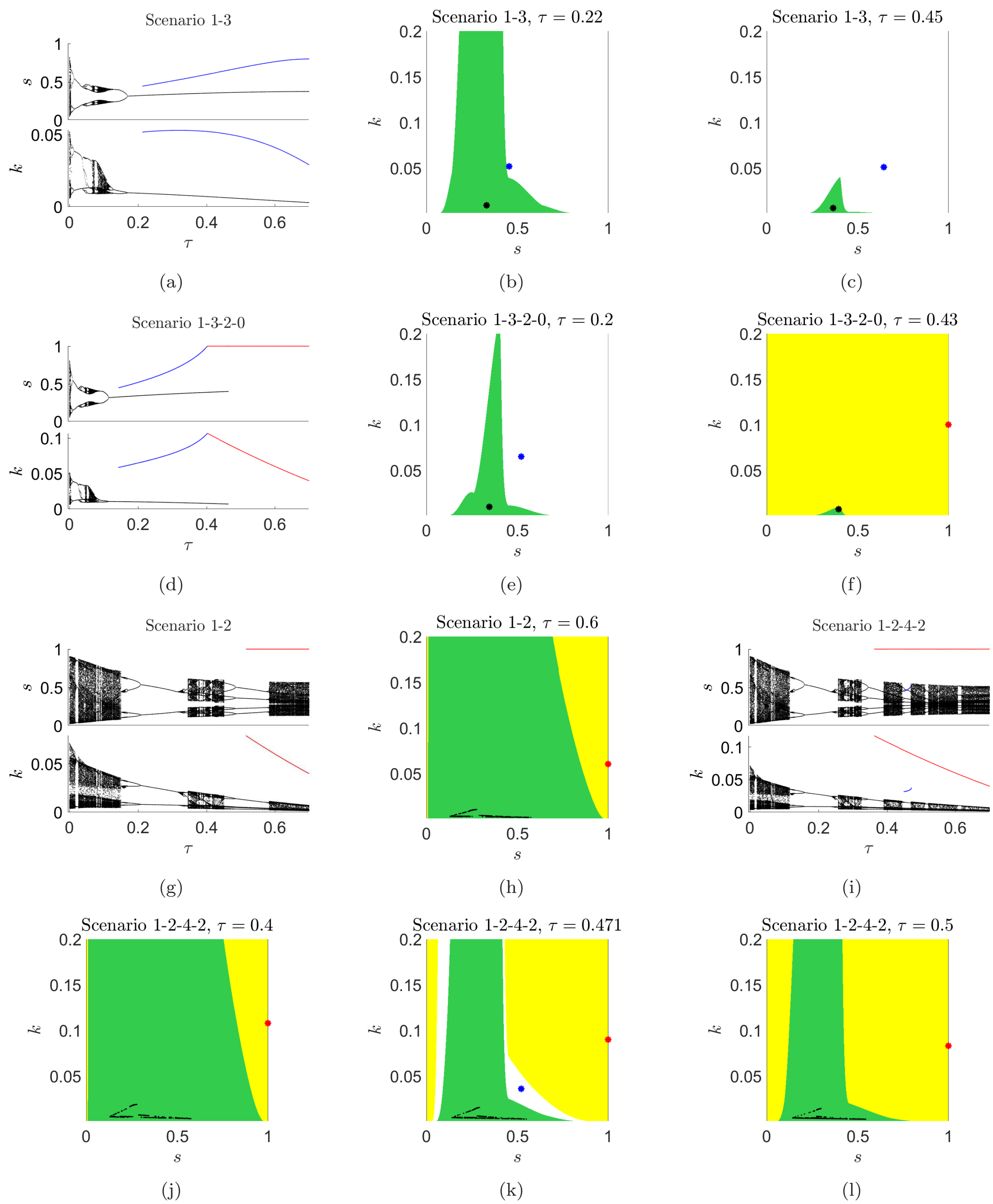


Figure 6: Panels (a,b,c): Scenario "1-3", bifurcation diagrams (a) and basins of attraction (b,c). Panels (d,e,f): Scenario "1-3-2-0", bifurcation diagrams (d) and basins of attraction (e,f). Panels (g,h): Scenario "1-2", bifurcation diagrams (g) and basins of attraction (h). Panels (i,j,k,l): Scenario "1-2-4-2", bifurcation diagrams (i) and basins of attraction (j,k,l). In the bifurcation diagrams, black color refers to attractors related to $\boldsymbol{\xi}_1^*$, blue color to $\boldsymbol{\xi}_3^*$, red color to $\boldsymbol{\xi}_{df}^*$. In the basins of attraction, green color denotes the basins for the attractors related to $\boldsymbol{\xi}_1^*$, white color to $\boldsymbol{\xi}_3^*$, yellow color to $\boldsymbol{\xi}_{df}^*$.