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Subjective expected utility and psychological gambles

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SUBJECTIVE EXPECTED UTILITY AND PSYCHOLOGICAL GAMBLES

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Abstract. We obtain an elementary characterization of expected utility based on a representation of choice in terms of psychological gambles, which requires no assumption other than coherence between ex-ante and ex-post preferences. Weaker version of coherence are associated with various attitudes towards complexity and lead to a characterization of minimax or Choquet expected utility.
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JEL: D81, G12

1. Introduction

An agent who, by effect of his own choice, a gift or some other selection mechanism, is about to receive an item x out of a set X of possible alternatives may view the situation from an *ex-ante* perspective and anticipate the level of satisfaction or disappointment. This psychological process originates from the awareness that, independently from one's own preferences, the actual process of choice is often bound to details which are difficult to fully anticipate. The outcome of an evening out with friends, e.g., is likely to depend upon the preferences of each of the participants, on the personal psychological or health status of the decision maker on that particular evening, on the actual availability of some of the options (like a particular restaurant being open or not) and so on. In these situations agents often rely on their own anticipations and speculate on whether the actual outcome will be preferred to what anticipated[1]. We refer to the mind process of evaluating the potential disappointment implicit in one's anticipations as a psychological gamble.

The starting point of this paper is the elementary fact that any choice process may be equivalently written as a psychological gamble and that assigning a utility to any item x is no different than assessing the probability that the anticipation of an outcome preferred to x will be disappointed. Although this interpretation requires no assumption, it connects the process of choice, which we normally envision as a behavioural process, with the mind experiment of anticipation and disappointment. On the other hand, writing choice problems in terms of gambles naturally opens the door to the possibility of considering more elaborate strategies involving, e.g., multiple events. In other words this reformulation embeds the

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[1]The impact of anticipations on choice has been investigated in both theoretical and experimental papers. Kőszegi and Rabin [19], in accordance with a basic principle of prospect theory, propose a model of choice based on reference dependence. The key point they make is that the reference level depends, rather than on past consumption or endowments, on agents anticipations. Concerning the experimental literature, a recent paper by Sabater-Grande et al. [22] studies the way students form their anticipations of future grades and document the (surprisingly often) rate of disappointment.

problem of choosing from a set into the more complex and rich setting of gambles. The discussion of the axioms to impose in this extended setting are the core of the present paper.

We follow Savage [23] in considering the problem of choice among acts, i.e. mappings from states to consequences. In this setting, the least controversial axiom is monotonicity: if every possible consequence of one of two acts dominates the corresponding consequence determined by its alternative, then such act should be preferred. We interpret this property as a criterion of simple coherence between the utility associated ex-ante with an act and that originating ex-post from its consequences. This basic principle has a natural extension from acts to gambles to which we refer as full coherence. We prove in Theorem 1 that full coherence is necessary and sufficient for the subjective expected utility representation (SEU) to hold. Our proof follows a different path than, e.g., the one proposed by Savage in which, as in most other proofs, a rich structure of the set of acts is required while we just need minimal properties.

Full coherence may be criticized on the basis of the remark that gambles involving different levels of complexity may not be directly comparable. We discuss different attitudes towards complexity, each translating into a corresponding weaker form of coherence. We show that every one of these criteria characterizes a different representation of preferences, including the minimax representation (MEU) of Gilboa and Schmeidler [16], Theorem 3, as well the Choquet expected utility (CEU) of Schmeidler [25], Theorem 4. Our analysis permits thus to unify these important models and to compare them in terms of the different attitudes to complexity involved. The role of complexity complements the traditional interpretation of these models in terms of ambiguity.

Many of our results exploit tools which are widely used in financial theory, particularly asset pricing. The connection with finance, which was first recognized by Gilboa and Samuelson [15], becomes fully clear in the context of gambling which provides a natural bridge between these apparently distinct areas of economic theory. Among other things, the tools we use permit to establish that every capacity is equivalent to a subadditive one, in some appropriate sense, Corollary 3.

In section 2 we introduce and discuss gambles, in section 3 we define full coherence, prove that this is necessary and sufficient for (SEU) and briefly discuss its limits. In section 4 we clarify the terms of the equivalence between full coherence and absence of arbitrage opportunities[2]. Starting from section 5, we discuss deviations from coherence based on aversion to complexity. We prove, in Theorem 2, that a minimal coherence criterion is already enough to recover multiple priors and a slightly stronger property, Θ_0 coherence, is equivalent to (MEU). In section 6 we characterize (CEU) in terms of Θ_1 coherence and implicitly show that (MEU) and (CEU) reflect different attitudes towards complexity. In section 7 we obtain some results on subjective capacities which do not require any form of coherence. Eventually we close in section 8 with a short discussion of the relationship with the existing literature.

1.1. **Notation.** We write $\mathfrak{F}(X,Y)$ to indicate the family of all functions $f : X \to Y$ and $\mathfrak{F}(X)$ for $\mathfrak{F}(X,\mathbb{R})$. The symbol $\mathfrak{F}_0(X)$ designates the collection of functions vanishing outside some finite set which we endow with the norm $\|f\| = \sum_{x\in X}|f(x)|$. The symbol $ba(\Omega)$ (resp. $\mathbb{P}(\Omega)$) denotes the family of

[2] And this in turn is a concept strictly related to the concept of coherent betting scheme due to de Finetti [7], from which we borrow our terminology.

bounded, finitely additive set functions (resp. probabilities) on the power set of Ω. If $\mathcal{H} \subset \mathfrak{F}(\Omega)$ we write

$$ba(\Omega, \mathcal{H}) = \left\{\lambda \in ba(\Omega) : \mathcal{H} \subset L^1(\lambda)\right\} \tag{1}$$

Following de Finetti [7] a set is identified with its characteristic function, so that $A(x)$ is either 1, if $x \in A$, or else 0. If $A \subset X$ and $f, g \in \mathfrak{F}(X)$ we write $f_A g = fA + gA^c$.

2. Psychological Gambles or Anscombe and Aumann need not play roulette.

Throughout the paper we fix the set Ω of *states of nature*, X of *outcomes* (or prizes) and $\mathbf{A} \subset \mathfrak{F}(\Omega, X)$ of *acts*. The main assumption we make is the following

(A1). *Preferences on $\mathbf{A}$ and X are represented by a corresponding utility function, V and u respectively.*

In order to simplify notation, we define order intervals:

$$I(f) = \{h \in \mathbf{A} : V(h) \leq V(f)\} \quad \text{and} \quad I_u(y) = \{x \in X : u(x) \leq u(y)\} \tag{2}$$

and let $I(g, f) = I(f) \setminus I(g)$ and $I_u(x, y) = I_u(y) \setminus I_u(x)$.

The main reason for assuming **(A1)** is the following, obvious implication that we shall use repeatedly: there exists a countable collection $\mathcal{I}$ of non empty order intervals such that any other non empty order interval includes an element of $\mathcal{I}$ as a subset.

Definition 1. *A psychological gamble on $\mathbf{A}$ (or simply a gamble) is just an element of the set*

$$\Theta = \Big\{\theta \in \mathfrak{F}_0(\mathbf{A}, \mathbb{R}_+) : \sum_{f \in \mathbf{A}} \theta(f) \leq 1\Big\}. \tag{3}$$

In the light of the preceding discussion, the quantity $\theta(f)$ is interpreted as the odds posted on the event of receiving an act which is not preferred to our anticipation, f. Thus, θ produces the overall payoff

$$I(\theta) = \sum_{f \in \mathbf{A}} \theta(f) I(f). \tag{4}$$

Of course, the larger its supporting set the more complex is the gamble.

The Dirac function δ_f sitting at f embeds $\mathbf{A}$ into Θ by associating each act with the event $I(f)$. Abusing notation, we shall often identify f with δ_f and view acts as gambles of a particularly simple form. Likewise, the utility function V extends naturally to a linear functional on Θ defined as

$$V(\theta) = \sum_{f \in \mathbf{A}} \theta(f) V(f), \qquad \theta \in \Theta \tag{5}$$

which we interpret as the value of the gamble θ. The next Lemma proves that this interpretation is consistent with the idea that the economic value of a gamble should reflect its perspective yields.

Lemma 1. *The value of a gamble is a linear function of its payoff.*

Proof. It is easily seen that both maps, V and I, assign the same value to gambles θ and θ' which are equivalent according to the following criterion:

$$\sum_{f\sim g}\theta(f)=\sum_{f\sim g}\theta'(f),\qquad g\in\mathbf{A}. \tag{6}$$

Thus, if $I(\theta)=I(\eta)$ we can assume, up to equivalence, assume that the support of θ and η are of the form $\{f_1,\ldots,f_n\}$ and $\{g_1,\ldots,g_k\}$ respectively with $V(f_i)<V(f_{i+1})$ and $V(g_j)<V(g_{j+1})$. If $V(f_1)>V(g_1)$ then letting h be the least valuable act between g_2 and f_1 we obtain $I(\theta)(f_1)=I(\theta)(h)$ while $I(\eta)(f_1)\neq I(\eta)(h)$, a contradiction. Thus necessarily $V(f_1)=V(g_1)$ and $\sum_{i\geq 1}\theta(f_i)=\sum_{j\geq 1}\eta(g_j)$. Applying the same argument recursively we conclude that $n=k$ and that $V(f_\lambda)=V(g_\lambda)$ and $\sum_{i\geq p}\theta(f_i)=\sum_{j\geq p}\eta(g_j)$ for each $p=1,\ldots,n$. Thus, if $I(\theta)=I(\eta)$ then θ and η must be equivalent in the sense of (6) so that $V(\theta)=V(\eta)$ which proves the claim. □

Indirectly Lemma 1 also associates V with a unique, additive (but not necessarily bounded) set function on the ring $\mathscr{R}$ generated by the order intervals in $\mathbf{A}$.

The ease of the extension from acts to gambles and the linearity of the value function $V(\theta)$ suggest that assumption **(A1)** may be avoided by defining preferences directly on Θ and imposing Herstein and Milnor [17] kind of axioms thereon. This would however contribute very little to the behavioural foundations of our approach and would make the importance of the coherence property to be introduced next less transparent.

3. Coherence and its limits.

A gamble may be defined similarly on any partially ordered set. The special feature of Θ is that the payoff of each gamble on $\mathbf{A}$ may be considered from a *conditional* perspective, i.e. by treating the state ω as fixed, and identifying an act f with its consequence $f(\omega)$. The *conditional value* of gamble θ is defined as in (5)[3]

$$u(\theta|\omega)=\sum_{f\in\mathbf{A}}\theta(f)u\big(f(\omega)\big),\qquad \theta\in\Theta. \tag{7}$$

Ranking gambles by their conditional value represents a highly incomplete partial order that we denote by writing

$$\theta\geq_u\eta\quad\text{if and only if}\quad u(\theta|\omega)\geq u(\eta|\omega)\quad\text{for all}\quad \omega\in\Omega. \tag{8}$$

Concerning the structure of the set of acts we assume the following:

(A2). *There exist two constant acts in* $\mathbf{A}$ *with values* $x,y\in X$ *such that (a)* $V(y)>V(x)$, *(b)* ${y_{A}}f, {x_{A}}f\in\mathbf{A}$ *and (c)* $V({y_A}f)\geq V({x_A}f)$ *for each* $A\subset\Omega$ *and* $f\in\mathbf{A}$.

[3] We prefer writing $u(f(\omega))$ rather than $u(f|\omega)$.

Property (*a*) is similar to (**P5**) of Savage while (*b*) is a very weak version of the sure thing principle. By effect of (**A2**) we associate with V the capacity

$$\gamma_V = \frac{V(y_A x) - V(x)}{V(y) - V(x)}, \qquad A \subset \Omega. \tag{9}$$

Differently from the rest of the literature, $\mathbf{A}$ is a quite arbitrary set and, in particular, it need not contain *all* constant or simple acts. Borrowing terminology from Kopylov [20], this qualifies $\mathbf{A}$ as a "small" domain. Although the point may seem a minor one, it is not so, we believe, for three main reasons. First, if Ω is infinite and $\mathbf{A}$ coincides with the whole space $\mathfrak{F}(\Omega, X)$, as in Savage, then the assumption that preferences are complete is extremely restrictive and hardly plausible from a behavioural point of view[4]. Second, if $\mathbf{A}$ does not contain all simple acts, the strategy of proof inaugurated by Savage (and all subsequent authors) – namely to deduce subjective probability from preferences, prove the expected utility representation for simple acts (*gambles*, in his terminology [23, p. 71]) and then extending to general acts – is simply not feasible[5]. Eventually, one cannot identify u with the restriction of V to constant acts. In fact there is no self evident criterion to test whether u and V are compatible with one another. The lack of a direct link between the two utility functions may be appropriate to describe principal/agent situations in which, as suggested in [15], an agent chooses among acts in accordance with the utility function V while the principal evaluates the agent's choice by considering the consequences it produces.

We formulate some general criteria for assessing whether V and u are compatible.

Definition 2. *The utility functions V and u are said to be:*

(a). simply coherent with one another if

$$f \geq_u g \quad \textit{implies} \quad V(f) \geq V(g), \qquad f, g \in \mathbf{A}; \tag{10a}$$

(b). fully coherent if

$$\theta \geq_u \eta \quad \textit{implies} \quad V(\theta) \geq V(\eta), \qquad \theta, \eta \in \Theta. \tag{10b}$$

Simple coherence is a basic condition roughly corresponding to (**P7**) of Savage. In plain words it suggests to consider acts by the consequences they produce and is often interpreted as a monotonicity assumption, as in Anscombe and Aumann [1]. Full coherence is the natural extension of this principle from acts to gambles. Several alternative properties, intermediate between the two, will be considered in the next sections.

Given (**A2**), it is clear that coherence is unaffected if V and u are normalized so that

$$u(x) = V(x) = 0 \quad \text{and} \quad u(y) = V(y) = 1 \tag{11}$$

(and thus $\gamma_V(A) = V(y_A x)$). We thus feel free to assume (11) henceforth without explicitly mentioning.

If f is any act, we shall often use Assumption (**A2**) to obtain the truncations:

$$f^k = f_{\{u(f) \leq k\}} x, \quad f_k = f_{\{u(f) \geq -k\}} x \quad \text{and} \quad f(k) = f_{\{|u(f)| \leq k\}} x, \qquad k \geq 0. \tag{12}$$

[4] A critical discussion of the completeness of preferences axiom, as is well known, is offered by Aumann [2] who refers to completeness as "perhaps the most questionable" axiom.

[5] Even in the subjective approach adopted by Ghirardato et al [14] all simple acts must be available.

Theorem 1. *Assume* **(A1)** *and* **(A2)***. The utility functions V and u are fully coherent if and only if*

$$V(f)=\Phi\big(u(f)\big)+\int_\Omega u(f)dm, \qquad f\in\mathbf{A} \tag{13}$$

in which (a) Φ is a positive linear functional on the linear space $\mathscr{L}_u=\{u(\theta):\theta\in\Theta\}$ and vanishes on bounded functions and (b) $m\in\mathbb{P}(\Omega,\mathscr{L}_u)$. The representation (13) is unique.

Proof. If V and u satisfy (13) they are clearly fully coherent. Conversely, assume that u and V are fully coherent, normalized utility functions on X and $\mathbf{A}$ respectively. By (5) the extension of V and u to gambles may be stretched still by replacing Θ with $\mathfrak{F}_0(\mathbf{A})$. It is an obvious consequence of linearity that V and u are fully coherent relatively to Θ if and only if they are so relatively to $\mathfrak{F}_0(\mathbf{A})$ or, in other words, that if $\zeta\in\mathfrak{F}_0(\mathbf{A})$ and $\zeta\geq_u 0$ then necessarily $V(\zeta)\geq 0$. We show that $\mathfrak{F}_0(\mathbf{A})$ is a vector lattice relatively to $\geq_u$. Fix $\zeta\in\mathfrak{F}_0(\mathbf{A})$, define $A=\{u(\zeta)\geq 0\}$ and define ζ^* implicitly by letting $\zeta^*(f_Ax)=1$ and $\zeta^*(x_Af)=-1$ if $\zeta(f)\neq 0$ and $\zeta^*=0$ elsewhere. It is then clear that $u(\zeta^*)=|u(\zeta)|$.

Regarding the random quantity $u(\zeta)$ as the value of a map $T:\mathfrak{F}_0(\mathbf{A})\to\mathfrak{F}(\Omega)$ we conclude that T is a positive linear operator acting on a vector lattice and, from (10b), that it such that

$$V(\zeta)>0 \quad\text{implies}\quad \sup_\omega u(\zeta|\omega)>0, \qquad \zeta\in\mathfrak{F}_0(\mathbf{A}). \tag{14}$$

In other words the linear functional induced by V on $\mathfrak{F}_0(\mathbf{A})$ is conglomerative with respect to T so that it follows from [3, Theorem 3.3] that there exists a positive linear functional Φ on $\mathscr{L}_u$ which vanishes on bounded functions and a positive, finitely additive set function m on the power set of Ω such that

$$u(\theta|\cdot)\in L(m) \quad\text{and}\quad V(\theta)=\Phi(u(\theta))+\int_\Omega u(\theta)dm, \qquad \theta\in\Theta \tag{15}$$

from which (13) readily follows. Moreover, recalling (9) we clearly see that (15) implies $m=\gamma_V$ so that m is unique and is a probability. For fixed $f\in\mathbf{A}$, it follows from the inclusion $u(f)\in L^1(m)$ that

$$\int u(f)dm=\lim_k\int_{\{|u(f)|\leq k\}}u(f)dm=\lim_k\int u\big(f(k)\big)dm=\lim_k V(f(k)) \tag{16}$$

and therefore

$$\Phi(u(f))=\lim_k V(f_{\{|u(f)|>k\}}x). \tag{17}$$

This proves uniqueness. □

We notice three main facts. First, that the integrability of all elements of $\mathscr{L}_u$ is not an assumption but an endogenous property. Second that, borrowing from the theory of finance, the component $\Phi(u(f))$ may be interpreted as a *bubble in preferences*, i.e. as a criterion for evaluating the unbounded part of the conditional utility of the act. Eventually, even if **(A2)** is a rather mild condition, it could be made more general since, as is clear from the proof, the only mixture operations needed in the proof involve the sets $\{u(f)>0\}$ or their complement so that one may indeed considerably restrict the family of sets for which mixing is allowed (see the discussion in [20]).

Theorem 1 also suggests a close and unexplored connection between expected utility and subjective conditional expectation. A version of (10b) was introduced by de Finetti in [6] under the name of *conglomerability* and its relation with disintegration properties has been studied thoroughly by Dubins [9].

We can dispose of preference bubbles with the aid of the following continuity axiom:

(A3). *For every $f \in \mathbf{A}$ and every order interval I containing f, there exists an order interval J_f that contains x and such that $y_A x \in J_f$ implies $f_{A^c} x \in I$.*

This property, although not strictly necessary for our results, makes it easier the comparison with the literature. We shall occasionally briefly comment on its role. It is easily seen that, under **(A1)** and **(A2)**, the assumption **(A3)** is equivalent to

$$\lim_{\gamma_V(A)\to 0} V(f_{A^c}x) = V(f), \qquad f \in \mathbf{A}. \tag{18}$$

Thus, if we add **(A3)** to (13), the inclusion $\mathscr{L}_u \subset L^1(m)$ implies $\gamma_V(|u(f)| > k) \to 0$ so

$$V(f) = \lim_k V(f(k)) = \lim_k \int_{\{|u(f)|\le k\}} u(f)dm = \int u(f)dm, \qquad f \in \mathbf{A}$$

i.e. $\Phi = 0$. Conversely, if (13) holds with $\Phi = 0$ and if $m(A_n) \to 0$ then

$$V(f) = \int u(f)dm = \lim_n \int_{A_n^c} u(f)dm = \lim_n \int u(f_{A_n^c}x)dm = \lim_n V(f_{A_n^c}).$$

Corollary 1. *Assume* **(A1)** *and* **(A2)** *and let the utility functions V and u be fully coherent. Then* **(A3)** *is satisfied if and only if the representation* (13) *holds with $\Phi = 0$.*

Despite the ease of extending the setting from acts to gambles, from a behavioural point of view these are much more complex objects than acts mainly because they require taking simultaneous positions on several different bets. One may thus conjecture that the ranking of two gambles may not be dictated by the monotonicity criterion $\geq_u$ whenever the gamble producing the higher conditional value is way more complex than its alternative. In other words, deviations from coherence may be justified on the basis of the difference in the cost of complexity inherent in the two gambles to be compared. In the following sections we shall thus formulate weaker notions of coherence, intermediate between simple and full coherence in order to capture the attitude towards complexity.

A first, easy step in this direction consists in decomposing Θ into the union of a family of convex subsets, Θ_λ, each of which closed with respect to mixing and containing the constant acts x and y. The DM may, e.g., interpret gambles belonging to the same subset Θ_λ as being "similar" or satisfying some symmetric (but not necessarily transitive) property, such as comonotonicity. On each Θ_λ a corresponding version of the coherence property may be assumed, namely

Definition 3. *The utility functions V and u are locally coherent given a cover $\{\Theta_\lambda : \lambda \in \Lambda\}$ of Θ if they are coherent relatively to Θ_λ for each $\lambda \in \Lambda$.*

By an easy corollary of Theorem 1, local coherence is equivalent to the existence, for each $\lambda \in \Lambda$, of a pair $(\Phi_\lambda, m_\lambda)$ with the same properties seen above. This is in line with the locally linear representation obtained by Castagnoli and Maccheroni [5].

4. Coherence and arbitrage.

Full coherence has a surprisingly natural translation into the language of financial economics and, in particular, in the notion of an arbitrage opportunity (or lack of). To clarify this point we define the sets

$$\mathcal{Z}_u^1 = \Big\{\zeta \in \mathfrak{F}_0(\mathbf{A}) : \zeta \geq_u 0 \quad \text{and} \quad \|\zeta\| \leq 1\Big\} \quad \text{and} \quad \mathcal{Z}_u = \bigcup_{\lambda>0} \lambda \mathcal{Z}_u^1. \tag{19}$$

and let $\mathcal{P}$ (resp. $\mathcal{P}_+$) be the convex cone spanned by the functions of the form $\delta_b - \delta_a$ where $a, b \in \mathbf{A}$ and $b \succeq a$ (resp. $b \succ a$).

Write $\mathcal{K}_u = -\mathcal{Z}_u - \mathcal{P}$. It is easily seen that V and u are simply coherent if and only if[6]

$$\mathcal{K}_u \cap \mathcal{P}_+ = \varnothing. \tag{20}$$

Full coherence requires a more stringent condition than (20), involving the topology on $\mathfrak{F}_0(\mathbf{A})$ generated by sets of the form

$$U_{H,a} = \Big\{\eta \in \mathfrak{F}_0(\mathbf{A}) : \sum_{f\in\mathbf{A}} \eta(f)H(f) < a\Big\}, \qquad a \in \mathbb{R},\ H \in \mathfrak{F}(\mathbf{A}). \tag{21}$$

Lemma 2. *The utility functions V and u are fully coherent if and only if*

$$\overline{\mathcal{K}}_u \cap \mathcal{P}_+ = \varnothing \tag{22}$$

Proof. If the utility function V on $\mathbf{A}$ is fully coherent with u, then $V(\kappa) \leq 0$ for every $\kappa \in \mathcal{K}_u$ and, by (21), the same inequality extends to the closure $\overline{\mathcal{K}}_u$ so that (22) holds. Conversely, fix $f_0, g_0 \in \mathbf{A}$ such that $V(f_0) > V(g_0)$,let ϕ_0 be a continuous linear functional on Θ such that

$$\phi_0(\delta_{f_0} - \delta_{g_0}) > 0 \geq \sup_{\zeta\in\overline{\mathcal{K}}_u} \phi_0(\zeta) \tag{23}$$

and define

$$V_0(h) = \phi_0(h), \qquad h \in \mathbf{A}. \tag{24}$$

Then, $V_0(f) > V_0(g)$ while, if $V(b) \geq V(a)$, the inclusion $\delta_a - \delta_b \in \mathcal{K}_u$ implies $\phi_0(\delta_a - \delta_b) \leq 0$ and so $V_0(a) \leq V_0(b)$. We can then normalize ϕ_0 so that $|V_0| \leq 1$ on some arbitrary, non empty order interval I_0. By **(A1)** there exist two countable collections of order intervals, $I(g_n, f_n)$ and I_n, such that (*i*) any other non empty order interval $I(g, f)$ satisfies $I(g_n, f_n) \subset I(g, f)$ for some $n \in \mathbb{N}$ and (*ii*) $I_n \subset I_{n+1}$ with $\mathbf{A} \subset \bigcup_n I_n$. Repeating the same step for each n, we obtain a linear functional ϕ_n separating θ_{g_n,f_n} from $\overline{\mathcal{K}}_u$ and such that the function V_n defined as in (24) satisfies $V_n(a) \leq V_n(b)$ for each $V(b) \geq V(a)$ and $|V_n| \leq 1$ on I_n. Let $\phi = \sum_n 2^{-n}\phi_n$ and define V_ϕ as in (24). For each $f \in \mathbf{A}$ there exists N such that $f \in \bigcap_{n\geq N} I_n$. Thus,

$$|V_\phi(f)| \leq \sum_n 2^{-n}|V_n(f)| \leq 1 + \sum_{n<N} 2^{-n}|V_n(f)| < +\infty.$$

[6] Condition (20) is formally identical to the mathematical definition of absence of arbitrage opportunities in mathematical finance. Compare with [21].

Moreover, if $f \succ g$ then

$$V_\phi(f) - V_\phi(g) = \phi(\delta_f - \delta_g) \geq \sup_{I(g_n, f_n) \subset I(g,f)} 2^{-n}\phi_n(\delta_{f_n} - \delta_{g_n}) > 0.$$

In addition, if $b \succeq a$ then, as seen above, $V_n(b) \geq V_n(a)$ for all $n \in \mathbb{N}$. Then, V_ϕ is a utility function. Moreover if $\theta, \eta \in \Theta$ and $\theta \geq_u \eta$ then $\eta - \theta \in \mathcal{K}_u$ so that $V_\phi(\theta) \geq V_\phi(\eta)$ which proves the claim. □

The technique used to prove Lemma 2 is a common tool in mathematical finance known as Yan Theorem, see [28] and it will be used again in what follows.

5. Multiple priors and maxmin preferences.

We shall now start discussing some popular decision models in terms of deviations from coherence. In so doing it is useful to formulate some minimal restrictions in order to rule out pathological situations.

Definition 4. *The utility functions V and u are minimally coherent if*

$$(\forall \theta \in \Theta)(\exists c_\theta \geq 0): \quad \theta \geq_u \eta \quad \textit{implies} \quad V(\theta) + c_\theta \geq V(\eta), \qquad \eta \in \Theta. \tag{25}$$

In the light of the preceding discussion the parameter c_θ describes the cost of complexity inherent in the gamble θ. Definition 4 suggests that deviations from coherence are due to the need to adjust the value of a gamble by its cost, which is assumed to be finite. All definitions of coherence we shall consider henceforth will satisfy this minimal criterion which, despite its simplicity, has interesting implications.

Theorem 2. *Assume* **(A1)**, **(A2)** *and* **(A3)**. *The utility functions V and u are minimally coherent if and only if there exists a convex set $\Lambda \subset ba(\Omega)_+$ such that*

$$\mathscr{L}_u \subset \bigcup_{\lambda \in \Lambda} L^1(\lambda) \quad \textit{and} \quad V(f) \leq \inf_{\lambda \in \Lambda} \int u(f) d\lambda, \qquad f \in \mathbf{A}. \tag{26}$$

Proof. By virtue of (25), the quantity

$$\kappa_0(\theta) = \sup\{V(\eta) : \eta \in \Theta,\ \theta \geq_u \eta\} \tag{27}$$

is well defined and finite for each $\theta \in \Theta$. In order to extend the domain of this functional from Θ to

$$\Xi_1 = \{\xi \in \Theta + \mathscr{Z}_u : \|\xi\| \leq 1\} \quad \text{and} \quad \Xi = \bigcup_{\lambda > 0} \lambda \Xi_1, \tag{28}$$

consider, as in the proof of Theorem 1, V and $\geq_u$ as defined over the whole of $\mathfrak{F}_0(\mathbf{A})$. By definition, each $\xi \in \Xi_1$ admits some $\eta \in \Theta$ such that $\xi \geq_u \eta$, or, equivalently, such that $\xi^+/2 \geq_u (\eta + \xi^-)/2$. Given that both sides of this inequality are elements of Θ, we have $V(\eta) \leq 2\kappa_0(\xi^+/2) - V(\xi^-)$. This guarantees that the quantity $\kappa_1(\xi)$, defined as in (27), is finite for each $\xi \in \Xi_1$. The corresponding functional κ_1 is concave and positively homogeneous and may thus be further extended, by scaling, to a functional κ defined on Ξ. It is easily seen that κ inherits from κ_0 the properties of being convex and

$$\kappa(\zeta) \geq 0 \quad \text{and} \quad \kappa(f) = V(f), \quad \zeta \in \mathscr{Z}_u,\ f \in \mathbf{A}. \tag{29}$$

Fix $\xi \in \Xi$ and define

$$\Xi_\xi = \{\beta \in \mathfrak{F}_0(\mathbf{A}) : \xi + \beta \in \Xi\} \quad \text{and} \quad \kappa_\xi(\beta) = \kappa(\beta + \xi) - \kappa(\xi) \qquad \beta \in \Xi_\xi. \tag{30}$$

Ξ_ξ is convex, closed with respect to truncations and such that $-\zeta \in \Xi_\xi$ whenever $\xi \geq_u \zeta$. Moreover, κ_ξ is concave (and thus such that $\kappa_\xi \geq \kappa$ on Ξ) and it satisfies

$$\kappa_\xi(\zeta) \geq 0 \quad \zeta \in \mathscr{Z}_u^1 \quad \text{and} \quad -\kappa_\xi(-\xi) = \kappa(\xi) = \kappa_\xi(\xi). \tag{31}$$

Let $\beta_1, \dots, \beta_n$ be a collection of elements of Ξ_ξ satisfying $\sup_\omega u(\beta_i|\omega) < +\infty$ and let $w_1, \dots, w_n$ be convex weights. Then, $\beta = \sum_{i=1}^n w_i \beta_i \in \Xi_\xi$ and $\sup_\omega u(\beta|\omega)\delta_y \geq_u \beta$. From all this we deduce

$$\sum_{i=1}^n w_i \kappa_\xi(\beta_i) \leq \kappa_\xi(\beta) \leq \sup_\omega u(\beta|\omega)\kappa_\xi(y).$$

As a consequence of [4, Theorem 4.1] there exists $\lambda_\xi \in ba(\Omega)_+$ such that $\|\lambda_\xi\| = \kappa_\xi(y)$ and that

$$\sup_\omega u(\beta|\omega) < +\infty \quad \text{implies} \quad u(\beta) \in L^1(\lambda_\xi) \quad \text{and} \quad \kappa_\xi(\beta) \leq \int u(\beta) d\lambda_\xi, \qquad \beta \in \Xi_\xi. \tag{32}$$

Using truncations we generalize this conclusion to

$$u(\zeta) \in L^1(\xi) \quad \text{and} \quad \kappa_\xi(\zeta^k) \leq \int u(\zeta) d\lambda_\xi \leq -\kappa_\xi(-\zeta_k), \qquad \xi \geq_u \zeta. \tag{33}$$

Let

$$\Lambda = \mathrm{co}\{\lambda_f : f \in \mathbf{A}\}. \tag{34}$$

It follows that[7]

$$\mathscr{L}_u \subset \bigcup_{\lambda \in \Lambda} L^1(\lambda) \quad \text{and} \quad V(f^k) \leq \kappa(f^k) \leq \inf_{h \in \mathbf{A}} \kappa_h(f^k) = \inf_{\lambda \in \Lambda} \int u(f) d\lambda, \qquad f \in \mathbf{A}. \tag{35}$$

But then (39) follows from **(A3)**.

Conversely, let Λ satisfy (39) and choose $\theta, \eta \in \Theta$ such that $\theta \geq_u \eta$. Then,

$$V(\eta) \leq \sum_{h \in \mathbf{A}} \eta(h) \inf_{\lambda \in \Lambda} \int u(h) d\lambda \leq \inf_{\lambda \in \Lambda} \int u(\eta) d\lambda \leq \inf_{\lambda \in \Lambda} \int u(\theta) d\lambda = V(\theta) + c_\theta$$

where we have implicitly set $c_\theta = \inf_{\lambda \in \Lambda} \int u(\theta) d\lambda - V(\theta) \geq 0$. □

The last line of the proof implicitly provides a precise estimate of the cost of complexity implicit in the gamble θ. Notice that each $\lambda \in \Lambda$ satisfies

$$\gamma_V(A) \leq \lambda(A), \qquad A \subset \Omega \tag{36}$$

but, that notwithstanding, need not a probability and thus an element of $\mathrm{core}(\gamma_V)$.

If gambles bear a complexity cost, c_θ, there may be a class of gambles whom the decision maker considers particularly simple, i.e. for which $c_\theta = 0$. Different agents may of course have different views on what is it

[7] For the sake of mathematical precision, given that $u(f)$ need not be integrable with respect to all $\lambda \in \Lambda$ but $u(f^k)$ is so for every k, the integral on the right hand side of (33) should be written as $\lim_k \int u(f^k) d\lambda$ and may well be $+\infty$ for some λ. This detail is irrelevant since $f \in L^1(\lambda_f)$.

that makes a gamble simple but they may perhaps agree to include among simple gambles those involving, further to y, only one non constant act. Denote this class of gambles by

$$\Theta_0 = \{t\delta_f + (1-t)\delta_y : 0 \le t \le 1,\ f \in \mathbf{A}\}. \tag{37}$$

Definition 5. *The utility functions V and u are Θ_0 coherent if they are minimally coherent and satisfy:*

$$\theta \ge_u \eta \quad \textit{implies} \quad V(\theta) \ge V(\eta), \qquad \theta \in \Theta_0,\ \eta \in \Theta. \tag{38}$$

This property is clearly intermediate between full coherence and simple coherence. It suggests that a gamble which dominates another one should definitely be preferred to the latter only if simple. Notice that this concept is asymmetric and is thus particularly weak.

Theorem 3. *Assume* **(A1)**, **(A2)** *and* **(A3)**. *Then, V and u are Θ_0 coherent if and only if there exists a set $\Lambda \subset \mathbb{P}(\Omega)$ such that*

$$\mathscr{L}_u \subset \bigcup_{\lambda\in\Lambda} L^1(\lambda) \quad \textit{and} \quad V(f) = \inf_{\lambda\in\Lambda}\int_\Omega u(f)d\lambda \qquad f \in \mathbf{A}. \tag{39}$$

Proof. We take advantage of the findings and notation of the proof of Theorem 2. In particular, under (38) $\kappa(\delta_f + \delta_y) = V(f) + 1$ so that $\kappa_f(y) = 1$ and the collection Λ defined in (34) consists of probabilities. From (33) we conclude that

$$\inf_{\lambda\in\Lambda}\int u(f_k)d\lambda \le \int u(f_k)d\lambda_{f_k} \le -\kappa_{f_k}(-f_k) = \kappa(f_k) \le \inf_{\lambda\in\Lambda}\int u(f_k)d\lambda. \tag{40}$$

Exploiting **(A3)** and (38) we conclude

$$V(f) = \lim_k V(f_k) = \lim_k \kappa(f_k) = \lim_k \inf_{\lambda\in\Lambda}\int u(f_k)d\lambda = \inf_{\lambda\in\Lambda}\int u(f)d\lambda.$$

The representation (39) implies minimal coherence, by Theorem 2. In addition, if $\theta \ge_u \eta$ and $\theta \in \Theta_0$ it also implies

$$V(\theta) = \inf_{\lambda\in\Lambda}\int u(\theta)d\lambda \ge \inf_{\lambda\in\Lambda}\int u(\eta)d\lambda \ge \sum_{h\in\mathbf{A}}\eta(h)\inf_{\lambda\in\Lambda}\int u(h)d\lambda = V(\eta).$$

□

A version of the preceding results may be established without invoking **(A3)**.

Corollary 2. *Assume* **(A1)** *and* **(A2)**. *Then, V and u are Θ_0 coherent if and only if there exist (a) a concave functional $\Phi \in \mathfrak{F}(\mathscr{L}_u)$ which vanishes on bounded functions and (b) a convex set $\Lambda \subset \mathbb{P}(\Omega)$ such that*

$$\mathscr{L}_u \subset \bigcup_{\lambda\in\Lambda} L^1(\lambda) \quad \textit{and} \quad V(f) = \Phi(u(f)) + \inf_{\lambda\in\Lambda}\int_\Omega u(f)d\lambda \qquad f \in \mathbf{A}. \tag{41}$$

Proof. The direct implication holds unchanged up to (40). From this we obtain

$$\inf_{\lambda\in\Lambda}\int u(f)d\lambda = \lim_k V(f_k) = V(f) - \Phi(u(f)),$$

the last equality amounting to the definition of $\Phi(u(f))$. In so writing we use the fact that V and u are simply coherent so that $V(f)$ only depends on $u(f)$. The extension to gambles is obtained by letting $\Phi(u(\theta)) = V(\theta) - \liminf_k V(\theta_k)$ which is clearly concave. □

6. Choquet expected utility.

Upon replacing Θ_0 with some another set of gambles in Definition 5 we obtain a different notion of simplicity and of coherence. In a gamble $\theta \in \Theta_0$ there is essentially only one source of randomness. One may extend this principle and consider positions in several different acts which are "similar" to one another in that they are affected by randomness in quite the same way. A popular criterion is comonotonicity and it is obvious that all gambles in Θ_0 are formed by monotonic acts. By Θ_1 we thus designate the set of gambles θ such that any two acts in the support of θ are comonotonic.

Theorem 4. *Assume* **(A1)**, **(A2)** *and* **(A3)**. *Then V and u are Θ_1 coherent if and only if γ_V is a convex capacity and satisfies the Choquet expected utility representation*

$$V(f) = \int u(f) d\gamma_V, \qquad f \in \mathbf{A}. \tag{42}$$

Proof. The proof is quite elementary and is based on a stepwise approximation of $u(f)$. We easily establish

$$\frac{2^{-n}\delta_y + \eta_n^k}{1+k} \geq_u \frac{f(k) + k\delta_y}{1+k} \geq_u \frac{\eta_n^k}{1+k} \tag{43}$$

in which the gamble η_n^k is defined to be identically equal to 2^{-n} at each act

$$f_i = y_{\{u(f) \geq i2^{-n} - k\}} x, \qquad i = 0, \ldots, k2^n - 1 \tag{44}$$

and null elsewhere. Given that all terms intervening in (43) are gambles included in Θ_1 we can invoke Θ_1 coherence so that $V(f(k)) \geq V(\eta_k^n) - k \geq V(f(k)) - 2^{-n}$. In addition it is easy to see that

$$\begin{aligned}
\lim_n V(\eta_n^k) - k &= \lim_n \sum_{i=1}^{k2^n - 1} 2^{-n} \gamma_V \big(u(f) \geq i2^{-n} - k\big) - k \\
&= \int_0^k \gamma_V(u(f(k)) \geq t - k) dt - k \\
&= \int_0^k \gamma_V(u(f) \geq t) dt - \int_{-k}^0 [1 - \gamma_V(u(f) \geq t)] dt \\
&= \int_{-k}^k u(f) d\gamma_V .
\end{aligned}$$

From this we deduce (42) by letting $k \to +\infty$ and exploiting **(A3)**.

Convexity of γ_V follows from the inclusion $\Lambda \subset \mathrm{core}(\gamma_V)$, Theorem 3 and ordinary rules of Choquet integration by which

$$\int u(f) d\gamma_V \leq \inf_{\mu \in \mathrm{core}(\gamma_V)} \int u(f) d\mu \leq \inf_{\lambda \in \Lambda} \int u(f) d\lambda = \int u(f) d\gamma_V, \qquad f \in \mathbf{A}. \tag{45}$$

If, conversely, the Choquet representation (42) holds, $\theta \geq_u \eta$ and $\theta \in \Theta_1$ then $V(\theta) = \int u(\theta) d\gamma_V \geq \int u(\eta) d\gamma_V \geq V(\eta)$. □

The importance of Choquet expected utility formula (42), introduced in the theory of decisions by Schmeidler [25], justifies some historical comments. In his seminal contribution Schmeidler [24] rightly attributes a special importance to convex capacities in view of two facts: (*i*) these are the infimum of the elements of their core and (*ii*) the corresponding Choquet integral is superadditive and additive over comonotonic functions. Following Schmeidler, the credit for discovering these two important properties has thenceforth unanimously been ascribed to Shapley [27] (the paper was originally published in 1965) and to Dellacherie [8], respectively. As a matter of fact (and quite curiously) both results were already well known at the time the above references were published since they had been proved, and in more general terms, by Eisenstadt and Lorentz [12] in 1959 (see Theorem 2 in that paper which fully anticipates the main result of [24]). Even the term *convex* (or rather concave), attributed to Shapley, had already been introduced by Eisenstadt and Lorentz.

7. The structure of subjective capacities.

In the preceding sections we have repeatedly used the capacity γ_V defined in (9), which, depending on the degree of coherence assumed, may or not be additive or convex. A different utility representation of the same preference system will in general induce a capacity with different properties. The following result provides another illustration of the connection between decision theory and finance. Notice that a weaker property than **(A2)** is actually needed.

Lemma 3. *Under* **(A1)** *and* **(A2)** *there exists a utility function V_* on $\mathbf{A}$ which is equivalent to V and is associated with a subadditive capacity.*

Proof. Consider a function $H \in \mathfrak{F}(\mathbf{A})$ of the form

$$H = \sum_{n=1}^{N} t_n \{ I(y_{B_n} x, y_{A_n \cup B_n} x) - I(x, y_{A_n} x) \} \tag{46}$$

where $t_n \geq 0$, $\gamma_V(B_n) \geq \gamma_V(A_n)$ and, upon rearranging terms, $\gamma_V(A_N) \geq \ldots \geq \gamma_V(A_1) > 0$. Then,

$$\min H + \max H \leq H(y_{A_1} x) + \max H = -\sum_{n=1}^{N} t_n + \max H \leq 0.$$

This inequality applies *a fortiori* to all elements of the convex cone of functions dominated by some H as in (46) as well as to the closure of such cone in the topology of uniform distance. As in the proof of Lemma 2 we can apply Yan Theorem and obtain the existence of some $\mu \in ba(\mathbf{A})_+$ such that

$$\sup_{A \subset B} \mu\big(I(y_B x, y_{A \cup B} x) - I(x, y_A x)\big) \leq 0 < \mu(I(g, f)), \qquad f, g \in \mathbf{A},\ f \succ g. \tag{47}$$

Since $\mu \neq 0$ we can normalize μ so that $\mu\big(I(x, y)\big) = 1$. The set function $\gamma_*(A) = \mu(I(x, y_A x))$ possesses then the following properties, valid for all $A, B \subset \Omega$:

$$(a)\ \gamma_*(\varnothing) = 0, \quad (b)\ \gamma_*(A) \leq \gamma_*(B) \text{ when } A \subset B \quad \text{and} \quad (c)\ \gamma_*(A \cup B) \leq \gamma_*(A) + \gamma_*(B). \tag{48}$$

Define

$$V_*(f) = \mu\big(I(x, f)\big) - \mu\big(I(f, x)\big), \quad f \in \mathbf{A}, \tag{49}$$

so that $V_*(y_A x) = \gamma_*(A)$ and

$$V^*(f) - V^*(g) = \mu\big(I(g, f)\big) - \mu\big(I(f, g)\big), \qquad g, f \in \mathbf{A}.$$

Thus V^* is a normalized utility function equivalent to V. $\square$

A set function satisfying (48) is often called a submeasure. It is possible to reformulate the preceding result in a purely measure theoretic language.

Corollary 3. *Every capacity γ on an algebra $\mathscr{A}$ of subsets of Ω admits a submeasure γ_* on $\mathscr{A}$ such that*

$$\gamma(A) > \gamma(B) \quad \textit{if and only if} \quad \gamma_*(A) > \gamma_*(B), \qquad A, B \in \mathscr{A}. \tag{50}$$

Proof. Take the space of bounded, real valued, $\mathscr{A}$ measurable functions as the collection of acts and for each such function b let $V(b) = \int b d\gamma$ and apply Lemma 3. $\square$

One interesting property of γ_V is connected with the following axiom.

(A4). *For every set B such that $\gamma_V(B) > 0$ there exists $N \in \mathbb{N}$ with the property that any partition π of Ω of size $\geq N$ satisfies*

$$\gamma_V(B \setminus A) > \gamma_V(A) \qquad \textit{for some} \qquad A \in \pi. \tag{51}$$

It is natural to compare **(A4)** with the apparently similar axiom (**P6**) of Savage. In plain terms, **(A4)** asserts that any large partition of Ω contains *at least one* "small" set (possibly null). Savage (**P6**) requires instead the existence of a partition (necessarily of large size) *all* of whose elements are equally "small". Despite its relative weakness, assumption **(A4)** is sufficient to put in relation utility from acts with probability. Although such probability may not be unique it need not be atomless either, thus avoiding an artificial implication of Savage construction.

Lemma 4. *Assume* **(A1)** *and* **(A2)**. *Then* **(A4)** *is equivalent to the existence of $P \in \mathbb{P}(\Omega)$ such that*

$$\lim_n P(A_n) = 0 \quad \textit{if and only if} \quad \lim_n \gamma_V(A_n) = 0. \tag{52}$$

Proof. It is obvious that γ_V satisfies either **(A3)** or (52) if and only if so does the submeasure γ_* defined in Lemma 3. Fix then B such that $\gamma_*(B) > 0$ and assume **(A4)**. Then, there exists $N \in \mathbb{N}$ such that for every disjoint collection $A_1, \ldots, A_N$ of subsets of Ω we can find an index $1 \leq i \leq N$ such that $\gamma_*(B \setminus A_i) > \gamma_*(A_i)$ and thus $\gamma_*(A_i) < \gamma_*(B \setminus A_i) \leq \gamma_*(B)$. This inequality, being true for all B with $\gamma_*(B) > 0$, implies that for each $\varepsilon > 0$ any partition of sufficiently large size contains an element A such that $\gamma_*(A) < \varepsilon$. In other words, γ_* is uniformly exhaustive and it thus satisfies (52) for some $P \in \mathbb{P}(\Omega)$, by [18, Theorem 3.4].

Conversely, assume (52) and fix B such that $\gamma_V(B) > 0$. Choose $0 < 3\varepsilon < \gamma_V(B)$ and let $P(A)$ is sufficiently small so that $\gamma_V(A) \leq \varepsilon$ and $\gamma_V(B) \leq \gamma_V(B \setminus A) + \varepsilon$, which we may do by choosing appropriately A from any partition of sufficiently large size. Then, $\gamma_V(A) < \gamma_V(A) + \varepsilon < \gamma_V(A) + \gamma_V(B) - 2\varepsilon \leq \gamma_V(B) - \varepsilon \leq \gamma_V(B \setminus A)$. $\square$

8. Relation with the literature

Introducing a linear structure on the set of acts, e.g. via the embedding of acts into gambles, is, of course, not a new idea. Most of the papers which adopt it, however, end up with justifications which are mainly of an objective nature. This raises the criticism that introducing objective elements into a purely subjective framework is somewhat contradictory. The best example of this state of things is the classical work of Anscombe and Aumann [1] in which, as is well known, prizes are tickets to roulette lotteries. Scott [26] presents a totally abstract (but extremely appealing) framework to choice based on formal sums which can be applied to several problems involving the numerical representation of binary relations over a finite set, such as non transitive preferences. His paper is perhaps the first attempt to obtain a choice theoretic model from simple separating arguments, later followed by Fishburn [13] who considered the problem of choice among probability distributions. The best attempt to obtain a linear structure with purely behavioural assumptions is the paper by Ghirardato et al. [14] in which the concept of *utility mixtures* is introduced based on the assumption that the image of X under u is convex. But even in this setting one needs the assumption that acts include all simple functions.

In relatively recent times a number of papers have looked at decision problems from a financial point of view and are thus indirectly related to our interpretation of utility as the value of a gamble. Echenique et al. [10] propose to test revealed preference theory by measuring the profits that could be made in case of violations of its axioms. Echenique and Saito [11] implement a similar approach for expected utility under risk aversion. The paper that comes closer to the present one is the recent work of Gilboa and Samuelson [15]. They investigate whether the utility functions u and V satisfy the integral representation (to which they refer to as coherence) or the maxmin representation. They interpret this as a problem of an agent trying to justify his choices, based on V, to some principal who evaluates the results through u. They assume that both set X and Ω are finite (acts are in fact matrices) and that acts include all functions $\Omega \to X$. The conditions they obtain are mainly interpreted in geometric terms. However, it is the first paper that recognizes explicitly the connection with finance.

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