DEPARTMENT OF ECONOMICS

UNIVERSITY OF MILAN - BICOCCA

WORKING PAPER SERIES

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No. 53 – October 2002

Dipartimento di Economia Politica
Università degli Studi di Milano - Bicocca
http://dipeco.economia.unimib.it

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August 2002

Abstract

In nonatomic games, anonymity must be assumed in order to prove the existence of a Nash equilibrium in pure strategies. This can be formalized by making payoffs dependent either on the players' distribution on the action set or on the strategy mean. An extension of Rath's (1992) proof to the case of limited anonymity is proposed: the aggregate behavior of several groups of individuals, rather than the behavior of the population as a whole, is shown sufficient to get equilibrium existence in pure strategies.

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1 Introduction

Game theory studies the multipersonal decision problems, where a player's action has an effect on the other player's payoff and vice versa, and it influences the other's decision. This framework seems intuitively well working in all the cases where the number of agents is not too large, such as in the oligopoly, in the war games, in the bargaining games and so on. What happens with many players, i.e. in large games?

For instance, as the number of firms in the Cournot game increases, the equilibrium tends, under very general conditions, to the competitive one (at the limit). The typical outcome features of the Cournot game progressively disappear: the aggregate and individual productions are less and less restricted and the price converges to its competitive level. Then, we obtain (at the limit) the competitive equilibrium.

Indeed, when the (countable) number of players increases, the strategic interaction effect decreases and, at the limit, it becomes negligible. Nevertheless, it remains true that the aggregate behavior of a subset of players large enough (at the limit, an infinite subset) has an effect on the others' choice, at equilibrium.

This is particularly evident, in a game with positive externalities (resp. with partial rivalry) like in Konishi, Le Breton, and Weber, 1997b, and Konishi, Le Breton, and Weber, 1997a, i.e. a game where the payoff of player i depends positively (resp. negatively) on the number of agents playing in the same manner of i.

A typical example of a game with partial rivalry is the problem of the city traffic: when one has to decide what road to intake with his car, he estimates how much traffic is present over any alternative. Traffic is the distribution of all other drivers on the road network at the same time. Many other examples can be found in the context of the use of a congestioned good, such as internet, roads, electricity networks, when agents take into account congestion. Congestion, as traffic, can be measured by the agents' distribution on the networks at a given instant.

With countably many players, we can define a sequence of n-players games. At any point of the sequence, the strategic interaction effect exists and it is non negligeable. Only at the limit it disappears. Even for a very large but finite n, any rational player is able to distinguish and to evaluate it, playing accordingly. Therefore, in a sense, at the limit (and only here), the game becomes "odd".

Nevertheless, there are situations in the real world where the number of players is naturally very high and each player actually knows that he has no effect on the others. In this case it seems not conceptually correct to consider a n-player game and then computing the limit of the finite game outcome to approach the real world situation we are studying. What we wish to underline is that the game is naturally "odd". Examples can be the voting decision, the determination of an equilibrium price in a market, the topic choice for the whole set of young researchers in economics. In all these cases, the player's choice effect on the others is negligible if taken isolately.

When the situation we wish to represent is such that the strategic interaction between individuals is extremely poor because of the great number of players, we can use the so called nonatomic games. The class of the nonatomic games allows to deal with problems where there is a continuum (i.e. uncountably many) of players. More precisely, a nonatomic game is a game where the set of players is endowed with a nonatomic measure. Indeed, nonatomic games allow us to model several situations where the individual weight in the "competition" is almost nihil, but where the aggregate choice of a "large number", a mass, of players is relevant. In an election a single vote is normally not relevant to determine the winner, and so the voters' utility. On the contrary, the vote of the subset of young voters or that of the old voters may have a great impact on the voting result.

Nonatomic games are worth of being studied for several reasons, already highlighted by Aumann, although in a different context (Aumann, 1964).

Firstly, they well approximate real life situations, otherwise misrepresented by finite settings. Secondly, they solely represent the economic idea of negligibility: only in the continuum setting some results arise (such as the perfect equivalence between CORE and competitive equilibria of an economy; see Aumann, 1964, and Dubey and Shapley, 1994). Thirdly, the continuous approximation allows to use the powerful tools and methods of "analysis". Finally, the existence of a Nash equilibrium, even in pure strategies, does not require strong hypothesis (concavity) on the payoff functions. Only continuity with respect to (the Lebesgue integral or the induced distribution of) the strategy profile is required. This is because of the convexification properties, arising when we consider a continuum of agents (see Aumann, 1964, Aumann, 1965, and Aumann, 1966).

In this paper (section 4) we wish to make a critical survey over two different approaches of proving Nash equilibrium existence in pure strategy, one based on the concept of strategies average (Schmeidler, 1973) and the other based on the players' distribution (Mas Colell, 1984). Furthermore, we discuss the Rath's work which represents, at least conceptually, a mixture of the two (Rath, 1992).

Thereafter (section 5), starting from a remark of Schmeidler, following the Rath's approach, we extend the existence result (in pure strategies) to the class of games where payoffs depend on the average strategy of a finite number of disjoint subsets of players, defined *a priori*. This generalization allows to greatly extend the domain of applicability of the nonatomic games to a quite large set of situations at little mathematical cost. It allows to reduce the anonymity in these games: belonging to a particular subset may be a feature that gives a different relative weight to different players.

One may imagine several examples: for instance, the "common" researcher's topic choice may depend on the distribution of the "stars" and on the distribution of the other "common" researchers; yet, the driver's route choice may depend on the distribution of several groups that are traveling on the same road network, such as car drivers and heavy truck drivers.

Since the partition of the players' set, unique for all the players, has to be defined *a priori*, we cannot deal, for instance, with games where the player's choice depends primarily on the choices of the players close to him[1], i.e., where the individual t's choice depends on the actions of those belonging to a neighborhood of t.

In the following, we will refer to the nonatomic games belonging to the generalized class defined above, as nonatomic games with "limited anonymity".

2 The relevant and related literature

In the General Economic Equilibrium literature there are several papers that model the price formation and the trading in a large economy as a non atomic game. In particular, Dubey and Shapley, 1994 give some interesting extensions, in the non atomic framework, of the finite market game proposed by Shapley and Shubik, 1977, considering two different ways of payment (by paper money and by a valuable commodity).

The fundamental feature of this approach is that players do not take the prices as given. They have an initial endowment of goods, they place it on the goods markets, receiving the right to an amount of money when the

[1] We think to a sort of influence with a limited extent of diffusion.

prices will be fixed. Then, they demand an amount of each commodity (this is the players' strategy), depending on their preferences, bidding an amount of money.

Prices are formed in such a way of clearing all the markets, i.e. the price of each good equals the ratio between the aggregate bid for that good and the aggregate endowment of it. Since the obtained bundle will depend on the formed prices, the players' utility, i.e. the players' payoff, will depend on the others' aggregate behavior. Under fairly weak conditions, Dubey and Shapley find that the Walrasian equilibria class and the strategic equilibria class are equivalent.

Furthermore, Codognato and Ghosal, 2001, extend the Forges and Minelli, 1997 paper ("Self-Fulfilling Mechanisms and Rational Expectation" in an exchange economy with private information) in the non atomic game framework.

Non atomic games have been already applied to several other economic and social issues. Constantinides and Rosenthal, 1984, study the strategic exercise of warrants and convertible bonds; Rob, 1987, considers the entry problem in a market of uncertain size. Pascoa, 1993, deals with the problem of existence of a Bertand equilibrium in a setting of monopolistic competition. Karni and Schmeidler, 1990, present a model to explain the existence of "fashions" even if preferences are fixed. Finally, Karni and Levine, 1994, analyse a restaurant pricing model.

All these papers explicitly consider and study the effects of the network and consumption externalities produced by the players' choices.

As noticed by Mas Colell, 1984, "the setting of games with a continuum of players can accommodate games with incomplete information played by a finite number of participants receiving independent signals". Khan and Sun, 1995, and Khan, Rath, and Sun, 1999 show that, only assuming at most countably many possible actions, a pure strategy Nash equilibrium of such a game exists.

In the literature, we may also find relevant contribution on dynamic non atomic games.

Many situation could be well formalized by a (infinitely) repeated non atomic game. Nevertheless, it is clear that a proof of the existence of a subgame perfect equilibrium can not be obtained simply by generalizing a standard folk-theorem. This is because in an atomless setting, it would be impossible to detect a deviator, since anyone is negligible and his actions have no effect on the others' payoff. Intuitively, the sole equilibria of a repeated game will be sequences composed of one-shot Nash equilibria. Indeed, such results are found by the so called anti-folk-theorem literature (Green, 1980,Kaneko, 1982, Dubey and Kaneko, 1984, Masso' and Rosenthal, 1989,Sabourian, 1990 and Masso', 1993, among others).

Likely, Levine and Pesendorfer, 1995, point out the fact that deviations are undetectable in a nonatomic context and use this issue to explain why equilibria can be very different using a finite players or a continuum framework[2]. They resolve this paradox by introducing, in a finite game, a noise. Indeed, a nonatomic game can be thought as a representation of a finite players game with imperfect actions observability.

Nonatomic anonymous sequential games are studied by Jovanovic and Rosenthal, 1988, and by Bergin and Bernhardt, 1992, among others. They prove the existence of a sequential equilibrium. Both assume that transition functions have an individualistic aspect; otherwise, each individual would perceive that his actions have no effects on the continuation of the game. Only the former assumes also that players' characteristics distribution evolves deterministically over time (absence of aggregate uncertainty).

The proof of the Nash equilibrium existence for the nonatomic games (general and in pure strategies), came only in 1973 with the paper of Schmeidler (Schmeidler, 1973). Thereafter, other authors gave proofs of the existence in pure strategies, following different approaches, such as Rath, 1992, and Mas Colell, 1984. Moreover, Codognato and Ghosal, 2002, assume heterogeneous individual action sets, compact subsets of a finite dimensional Euclidean space. They show that a (Pareto) undominated pure strategy Nash equilibria exists.

Refer to Kahn and Sun, 2001 for a good and comprehensive survey of the whole non atomic games theory.

3 Anonymity and nonatomicity

The existence of a Nash equilibrium in pure strategies depends on the property of anonymity. Roughly speaking, anonymity means that the single player's characteristics are not relevant in determining his impact. Only

[2]This is the case, for instance, of the free-rider problem in corporate takeovers (Grossman and Hart, 1980): whenever there is a finite number of shareholders, the rider may appropriate at least a part of the future efficiency gains; with a continuum, his appropriation level will be nihil.

actions matter, not *who* does them. Everyone is symmetric to all the others. Therefore every individual is interchangeable.

Such a property has to be differently formalized in different contexts. Here are two examples.

In the already mentioned finite games with positive externalities considered by Konishi, Le Breton, and Weber, 1997b, the strategic interaction is captured only by the fact that the player i's payoff is a (increasing) function of the number of agents that play the alternative chosen by i; considering just the number of individuals is a way to formalize the idea of anonymity: who chooses a certain alternative is irrelevant and only how many agents choose it is taken into account.

The second example comes from the social choice theory. We are interested in finding a good aggregator of preferences, a social welfare functional. Consider the majority voting rule: it is anonymous since if you permute the preference profile in any possible way, the result does not change because, again, what matters is the number of people preferring a particular alternative and not the characteristics of such people.

In the context of nonatomic games, anonymity shares part of its meaning with nonatomicity. Nonatomicity is actually the representation of the economic idea of weak strategic interaction, where a large number of players participate but each one is so small that can be neglected (as in competitive markets). Indeed, since any player, alone, is irrelevant, it follows that any two individuals are symmetric in the sense that they are both unable to influence the situation. In this view, nonatomicity, then, already embodies some of the anonymity features (this is not the case in finite games, where each player has a non negligible impact on any other)[3].

Nonatomicity has to be represented with a nonatomic measure, i.e. roughly speaking with a measure assigning a null value to any singleton. It is therefore obvious to consider the, naturally nonatomic, Lebesgue measure, at least when we model the players' set as a real interval[4].

[3] Some more insights on this dualism can be found on the anti-folk-theorem literature: there are so many players that a single deviator is undetectable. A single player looses his identity in the "ocean" of the players'set.

[4] Consider a real interval endowed with a Borel σ-algebra. In this context any subset of players can be obtained by a countable union of disjoint intervals. Then, the Lebesgue measure is appropriate, because it associates to each interval its lenght, and to each subset the lenght of the composing intervals. A singleton is a degenerated interval with zero measure.

Once the nonatomic setting has been established, requiring anonymity means that, we are interested on the aggregate behavior i.e. on the measure of the players' set which chooses a particular alternative. In this sense, none is relevant *per se*, but what matters is societal conduct.

In the nonatomic games the anonymity property is formalized making the payoff functions dependent on the players' distribution over the alternatives. More precisely, in Schmeidler, 1973, and in Rath, 1992, the payoff functions depend on the Lebesgue integral of the (vector valued) strategy profile over the players' set, while in Mas Colell, 1984, and in Khan, Rath, and Sun, 1997, the induced distribution generated by the strategy profile is directly employed.

4 Three proofs of existence

In this section we discuss about the different approaches to prove the existence of a Nash equilibrium in a normal form nonatomic game.

The point is how to formalize anonymity. Two ways have been considered in literature: we may aggregate the players behavior either taking an average of the "societal response" or dealing with the complete distribution of the players over the alternatives.

Schmeidler interprets anonymity following the first method, while Mas Colell formalizes anonymity as a distribution of the players on the alternatives. Rath makes use of the strategy profile integral, but we may interpret it as a distribution.

As Khan, Rath, and Sun, 1997, emphasize, whenever we assume the action space finite, the two approaches are perfectly equivalent, at least when we formalize pure strategies as unit vectors in the n-dimensional Euclidean space. Thus, the strategy profile will be a vector-valued function whose integral is a n-dimensional vector, with the measure of the players set choosing the *i-th* action at the *i-th* coordinate. Such vector is nothing else than the distribution induced by the strategy profile. Nevertheless, we may also interpret this integral as a mean of a vector-valued random variable. Therefore, in this case, referring to averages or distributions is a matter of interpretation.

When the action space is countably infinite it is possible to show the existence with both the distribution and the average approaches. Distributions of correspondences have to be considered when we use the first strategy (see Khan and Sun, 1995) and adequate integration tools (respectively Lebesgue integral in the finite dimensional Euclidean space and Gel'fand and Bochner integral in infinite dimensional general spaces) when averages are employed (Khan, Rath, and Sun, 1997).

A new problem arises when we deal with an uncountable action set: there is no difficulty in building a distribution, but it is impossible to prove the existence of a pure strategy Nash equilibrium (Khan, Rath, and Sun, 1997). On the other hand, how to formalize a notion of average of a vector random variable taking uncountably infinite possible values on an infinite dimensional space?

The only positive result is that of Rath, 1992, (Theorem 2), who assumes the action space as a compact (uncountable) subset of $\Re^n$ and, working with Lebesgue integrals, gets the proof of existence in pure strategies.

Here, we wish to focus mainly on the simplest case of finite actions, because it contains the principal insights of the nonatomic game theory and because it will be sufficient to understand the applications of Chapters 2 and 3 (as well as all applications mentioned in the previous subsection).

Then, we first consider Schmeidler's proof, pointing out as he finds a pure strategy Nash equilibrium through a purification process of the mixed strategy equilibria. He uses anonymity, in the sense of average, only to make use of the Lebesgue integral properties in the purification. In principle, any linear operator having the same properties of the correspondence Lebesgue integral (Aumann, 1965) could be used.

Secondly, following Rath, we show an application of the explained equivalence between integrals of vector valued functions and distributions. This author exploits the property of a subset of the Euclidean space (the set of the strategy profiles integrals) from the beginning of his proof, making use neither of function spaces nor of weak topology. Moreover, no purification is required.

Finally, we show an extreme use of the distribution formulation, proposed by Mas Colell, 1984, although infinite actions are assumed in this formulation. Mas Colell is the first to fully recognize the power of the distribution formulation. He defines a game as a distribution on the space of payoffs, i.e. there are no individual names. Each individual payoff depends on the players' distribution on the alternatives and a Nash equilibrium distribution is such that, given the players' distribution, almost every player obtains his highest payoff.

We mention that Khan and Sun, 1994, provide a synthesis of the two approaches.

The common framework and notation for the three proofs are the following. Specifications and particularities will be remarked for each one.

$T = [0, 1]$ is, without loss of generality, the set of players endowed with the Lebesgue measure λ. We consider this measure because it is atomless, indeed well representing the notion of negligibility[5].

There are n possible alternatives (actions), each of them represented by a unit vectors in $\Re^n$, that is, the vector e_i is the unit vector with 1 as i-th coordinate and zero otherwise and it is associated with the i-th alternative. Therefore, the set of alternatives is $E = \{e_1, ..., e_n\}$.

The convex hull $\widehat{E} = conv(\{e_1, ..., e_n\})$ is the set of the all possible mixed strategies, with elements denoted $\widehat{e} = (\pi_1, ...\pi_n)$, where π_i is the probability of playing the alternative i. Therefore, the pure strategy e_i is a particular mixed strategy, as usual.

A mixed strategy profile is a measurable function $\widehat{f} : T \to \widehat{E}$ that associates to each player $t \in T$ an element of $\widehat{E}$ denoted as the n-vector $(\widehat{f}^1, ..., \widehat{f}^n)$, where $\widehat{f}^i$ is the real valued component from T to $[0, 1]$. In other words, we associate to each t the probability of playing each alternative. $\widehat{F}$ is the set of all mixed strategy profiles.

A pure strategy profile is a measurable function $f : T \to E$ that associates to each $t \in T$ an element of E denoted as the n-vector $(f^1, ..., f^n)$, where f^i is the integer valued component from T to $\{0, 1\}$. F is the set of all possible strategy profiles. Obviously, since $E \subset \widehat{E}$, it follows that $F \subset \widehat{F}$.

4.1 Schmeidler (1973)

Schmeidler proves two results: the existence of a Nash equilibrium and the existence of a Nash equilibrium in pure strategies for the nonatomic games.

He first determines the equilibrium existence in mixed strategies. He considers n possible actions, a mixed strategy as a vector in $\Re^n$, where the *i-th* coordinate value is the probability assigned to the *i-th* alternative. The (mixed) strategy profile is then an n-vector valued function that associates an n-vector to each player $t \in T$. The payoff depends on the individual strategy and on the strategy profile, and, therefore, it is defined over a function

[5] We recall that a measure is a real function defined on a family of subsets. It has the properties of positivity and σ-additivity (i.e. the measure of a countable union of disjoint sets is the sum of the sets measures).

Whenever the space of use is real, the Lebesgue measure is the natural atomless measure to employ.

space. Schmeidler uses the Fan Glicksberg fixed point (a generalization of Kakutani's fixed point) to show the existence of a Nash equilibrium. The existence of a pure strategy Nash equilibrium is obtained through a process of purification of the mixed strategy profiles, using the convexity properties of the Lebesgue integral. This process is possible only under the assumption that the individual payoff depends on the integral of the strategy profile. Explicitly, the author interprets the strategy profile integral as a strategy average.

Nevertheless, the individual payoff continues to depend on the function space of the strategy profiles, while it is the payoff function that embodies the integral operator. Formally, anonymity is represented as $u_{Schmeidler} = u \circ \int : T \times \widehat{E} \times \widehat{F} \to \Re$ [and not as $u' : T \times \widehat{E} \times \left\{ \int_T f d\lambda : f \in \widehat{F} \right\} \to \Re$ like one may misunderstand given the oblique formulation of Schmeidler].

The set $\widehat{F}$, the set of all mixed strategy profiles, is endowed with the L_1 weak topology. This set is a compact, convex subset of a locally convex linear topological space.

Now, we define the payoff function as

$$u : T \times \widehat{E} \times \widehat{F} \to \Re$$

and we require $u(t, \cdot, \widehat{f})$ to be a Von Neumann-Morgenstern utility function, i.e. $u(t, \widehat{e}, \widehat{f}) = \sum_{i=1}^{n} \pi_i u(t, e_i, \widehat{f})$.

Each player t is then endowed with a specific payoff function, dependent on his mixed strategy $\widehat{e} \in \widehat{E}$ and on the mixed strategy profile $\widehat{f} \in \widehat{F}$[6].

By now we have described the normal form of the game.

We need two assumptions.

[6] Schmeidler first defines an auxiliary function $v(\cdot, \cdot) : T \times \widehat{F} \to \Re^n$. Its component $v^i(t, \widehat{f})$,for $i = 1, ..., n$, describes the utility of player $t \in T$ playing e_i when almost every player chooses $\widehat{f}$, i.e. each player plays his mixed strategy, and player t chooses the pure strategy e_i. Then the payoff of player t is defined as

$$u_t(\widehat{f}) = \widehat{f}(t) \cdot v(t, \widehat{f})$$

or the inner product in $\Re^n$. Thus, the payoff, when one playes a mixed strategy, is expected. It is obtained using the probability distribution represented by the mixed strategy.

To be consistent with Shmeidler's formulation, here we assume $u(t, \cdot, \widehat{f})$ a Von Neumann-Morgenstern (VNM) function.

Assumption 1 (Schmeidler) For all $t \in T$, $u(t, \cdot, \cdot)$ is continuous on $\widehat{E} \times \widehat{F}$ w.r.t. the weak topology[7].

Assumption 2 (Schmeidler) For all $\widehat{f}$ in $\widehat{F}$ and $i, j = 1, ..., n$ the set $\{t \in T | u(t, e_i, \widehat{f}) > u(t, e_j, \widehat{f})\}$ is measurable. In words, this is the set of all those players preferring the pure strategy e_i to the pure strategy e_j, given $\widehat{f}$ is measurable.

Both assumptions have just a technical meaning, as we will indicate in the sketch of proof. Here it is sufficient to notice that they are quite mild. Similar versions of them will be substantially present in any proof we discuss.

Definition 1 *A mixed strategy profile $\widehat{f}$ is a Nash equilibrium iff*
$\forall m \in \widehat{E} \qquad u(t, \widehat{f}(t), \widehat{f}) \geq u(t, m, \widehat{f}) \qquad \lambda-$ *a.e.*

At equilibrium, given the strategies played by all players, summarized by the function $\widehat{f}$, almost everyone (a.e.) estimates unprofitable to deviate. When we say "almost everyone" we mean that all agents, except, eventually, a zero-measure set, choose the equilibrium strategy. We content with this condition, weaker than the requirement we used in finite games, because in nonatomic games a zero-measure set of players has no effect on the strategic interaction. Intuitively, since any individual is negligible, also a small enough set (i.e. of zero-measure) is uninfluential.

Now, we can state the first result of Schmeidler.

Theorem 1 *A nonatomic game in normal form fulfilling assumptions (1) and (2) has a Nash equilibrium.*

The proof of the Theorem 1 is, as usual, based on a fixed point argument.

Schmeidler first defines the best reply correspondence for the player t and given the strategy profile $\widehat{f}$ as

$$B(t, \widehat{f}) = \{m \in \widehat{E} | \forall m' \in \widehat{E} : u(t, m, \widehat{f}) \geq u(t, m', \widehat{f})\}$$

[7] This assumption is equivalent to that of Schmeidler (he assumes just continuity of $v(t, \cdot)$ on $\widehat{F}$), because we have assumed $u(\cdot)$ a VNM function. Indeed, continuity on $\widehat{E}$ is not a restiction, given the kind of linearity required.

The best reply correspondence is nonempty (because of continuity of $u(t,\cdot)$) and convex valued (i.e. if any two strategies belong to it, then all their convex combinations are best responses, given the linearity with respect to the second argument). Furthermore, since $u(t,\cdot)$ is continuous on $\widehat{E}\times\widehat{F}$, it is non empty and has closed graph. The main interest of the proof lies in the following correspondence $\alpha:\widehat{F}\to\widehat{F}$ defined as:

$$\alpha(\widehat{f})=\{\widehat{g}\in\widehat{F}|\text{ a.e. }\widehat{g}(t)\in B(t,\widehat{f})\}$$

Such a function associates to each mixed strategy profile the set of the mixed strategy profiles with the property that for almost every player, the mixed strategy played by player t belongs to the best reply correspondence of t, given the strategy profile $\widehat{f}$. That is $\alpha(\widehat{f})$ is the set of the best responses profile of all agents facing the profile $\widehat{f}$. It is clear from now that if the best response profile to $\widehat{f}$ is $\widehat{f}$, then $\widehat{f}$ is a Nash equilibrium. Hence, we want the function $\alpha(\widehat{f})$ having a fixed point. Indeed, Schmeidler shows that $\alpha(\widehat{f})$ is non empty, convex and it has closed graph. Therefore, by the Fan-Glicksberg fixed point theorem a fixed point exists and the proof is done.

The main result, as emphasized by Schmeidler himself, is:

Theorem 2 *If, in addition to the conditions of Theorem 1, a.e., $u(t,\widehat{f}(t),\widehat{f})$ depends only on $\int_T\widehat{f}$, then there is a Nash equilibrium in pure strategies.*

Theorem 2 is a corollary of Theorem 1. We have to show that there exists a pure strategy profile p having the same Lebesgue integral of the mixed strategy equilibrium $\widehat{f}^*$ and belonging, for almost all players, to the respective best reply correspondence, given $\widehat{f}^*$. In other words, we need the effect of the two strategies on the payoff to be the same, since we have assumed that the $u(\cdot)$ depends on $\int_T\widehat{f}$, and that the player is indifferent between the pure or the mixed strategy of equilibrium, because both p and $\widehat{f}^*$ belong to $B(t,\widehat{f}^*)$. Roughly speaking, p is an "alternative" as good as $\widehat{f}^*$ for almost all players.

Since $B(t,\widehat{f}^*)$ is convex valued, when more than one pure strategy belongs to $B(\cdot,\cdot)$, all the mixed strategies (=convex combinations) that assign positive probabilities only to these alternatives, belong to $B(\cdot,\cdot)$. More formally,

$$B(t,\widehat{f}^*)=conv(\{e_i|e_i\in B(t,\widehat{f}^*)\})$$

Moreover, we have:

$$\int_T B(t, \widehat{f^*}) = \int_T \{e_i | e_i \in B(t, \widehat{f^*})\}$$

To show this, we use an intuitive argument. Suppose that $B(t, \widehat{f^*}) = \{e_1, e_2\}$. By the definition of integration of correspondence, we know that $\int_T \{e_1, e_2\} d\lambda = \{(a, 1-a, 0, ..., 0) \text{ for all } a \in [0,1]\}$. This is noting else that the convex hull of $\{e_1, e_2\}$. Now, if we integrate the set $\{(a, 1-a, 0, ..., 0) \text{ for all } a \in [0,1]\}$, we obtain the same set, because a linear combination of two elements of another linear combination belongs to the last.

By definition, the Lebesgue integral of a correspondence is the set of Lebesgue integrals of all integrable selections belonging to the correspondence. Therefore we have that:

$$\int_T B(t, \widehat{f^*}) = \left\{ \int_T p \;\mid\; p \in \widehat{F} \text{ and a.e. } p(t) \in B(t, \widehat{f^*}) \right\}$$

$$\int_T \{e_i | e_i \in B(t, \widehat{f^*})\} = \left\{ \int_T p \;\mid\; p \in \widehat{F} \text{ and a.e. } p(t) \in \{e_i | e_i \in B(t, \widehat{f^*})\} \right\}$$

Clearly, a selection of the set $\{e_i | e_i \in B(t, \widehat{f^*})\}$ is a profile of pure strategies. Recall that $\int_T \widehat{f^*} \in \int_T B(t, \widehat{f^*})$, since $\widehat{f^*}$ is a selection of $B(t, \widehat{f^*})$. Combining the two equalities above, we have that $\int_T \widehat{f^*} \in \left\{ \int_T p \;\mid\; p \in \widehat{F} \text{ and a.e. } p(t) \in \{e_i | e_i \in B(t, \widehat{f^*})\} \right\}$ and, therefore, there exists a pure strategy profile with the same integral of $\widehat{f^*}$ that also belongs to $B(t, \widehat{f^*})$ for almost all t.

The measurability condition (2) ensures that everything is integrable.

Since the relevant result, in our perspective, is surely the existence in pure strategies (theorem 2), we have focused on the purification process. It is based on Aumann, 1965, (Theorem 3), which makes use of the convexity property of the Lyapunov's Theorem.

Notice that the linearity assumed for the payoff functions determines the starting point of this process: it enables us to represent, for each t, the best replay correspondence as the convex hull of some pure strategies[8]. But

[8] Recall that, with a VNM function, if a mixed strategy is chosen, then it will mean that the player is indifferent among all pure strategies receiving a positive probability.

then, any player has at least one pure strategy in his best reply. Hence, what remains is just to pick the "good" pure strategy from any player best response set.

4.2 Rath (1992)

Rath proves the existence of a pure strategy Nash equilibrium, assuming that the payoff functions depend on the Lebesgue integral of the strategy profile.

He uses the same setting as Schmeidler to formalize the individual pure strategy; moreover, the agents are supposed to play only pure strategies. Contrary to Schmeidler, he directly exploits the properties of the Lebesgue integral of the vector valued strategy profile. Hence, this yields a distribution of the players on the alternatives because its outcome is a n-vector with in each coordinate $i = 1, ...n$ the measure of the set of players choosing the i-th action.

Given this interpretation, we claim that the Rath, conceptually, matches the average and the distributional approach.

We denote by $S = \{\int_T f d\lambda | f \in F\}$ the set of all Lebesgue integrals of the strategy profile, defined as $(\int_T f^1 d\lambda, ..., \int_T f^n d\lambda)$, or, in words, the Lebesgue integral of the vector f is the integral of all its coordinates. Finally, we denote with s an element of S. It is important to remark that S can be identified by the simplex in $\Re^n$.

Then, the individual payoff directly depends on the players distribution, i.e. $u_{Rath} : T \times E \times \left\{\int_T f d\lambda : f \in F\right\} \rightarrow \Re$.

What differs from the Schmeidler's procedure is the fact that, here, the existence in pure strategies is proved directly, i.e., there is no process of purification of the equilibrium in mixed strategies: this is important from a conceptual point of view because mixed strategies "have a limited appeal in many practical situations, and it thus seems paradoxical to show the existence of a pure strategy equilibrium by focusing on equilibrium in [mixed] strategies first and then purifying it later."(Khan and Sun, 1995, p.637). Furthermore, "On a more technical level, the direct proof requires that the players search only among their set of pure strategies, thus leading to the computation of the fixed point of a correspondence coming only from their pure strategies, and hence free from all of the objections against using [mixed] strategies."(*ibidem*).

Given the simplification implied by the use of an Euclidean space, rather than a function space, the proof calls "only" for the classic Kakutani's fixed point theorem.

Assumption 1 (Rath) We denote $u : T \times E \times S \rightarrow \Re$ the payoff function; $u(\cdot)$ is real valued and continuous on $E \times S$.

This assumption parallels Schmeidler assumption 1.

Remark 1 *Under anonymity, Schmeidler's assumption 1 is equivalent to Rath's assumption 1.*

Proof. Since $u_{Schmeidler}$ is continuous on $\widehat{E} \times \widehat{F}$, then it is continuous also on $E \times F$, being $E \times F \subset \widehat{E} \times \widehat{F}$. Then we focus on $E \times F$. Under anonymity, $u_{Schmeidler} = u \circ \int : T \times E \times F$. This is equivalent to $u_{Schmeidler} = u : T \times E \times \left\{ \int_T f d\lambda : f \in F \right\}$. Since $S = \left\{ \int_T f d\lambda : f \in F \right\}$, we conclude that $u_{Scmeidler}$ is continuous on $E \times S$. ■

Notice that the Rath's payoff functions are defined on S, the set of the Lebesgue integrals of the strategy profiles[9]. This definition really means that distributional nature of the Lebesgue integral is considered, rather than its meaning of average.

Here is the definition of a Nash equilibrium (in pure strategies) in the Rath setting. What differs from Schmeidler, is, obviously, the third argument of the payoff function and the focus on pure, rather than mixed, strategies.

Definition 2 *A (pure strategy) Nash equilibrium of a game is a pure strategy profile $f \in F$ such that for almost every t, $u(t, f(t), \int_T f) \geq u(t, e_i, \int_T f)$ $\forall e_i \in E$.*

We need the following assumption, equivalent, under anonymity, to the Schmeidler's assumption 2.

Assumption 2 (Rath) For any $s \in S$ and $e_i, e_j \in E$, the set $\{t \in T \mid u(t, e_i, s) > u(t, e_j, s)\}$ is measurable[10].

Given this framework, the main result is:

[9]Notice also that the set of Lebesgue integrals of mixed strategy profiles coincides with the set of Lebesgue integrals of pure strategy profiles.

[10]Equivalence is easily verified representig Schmeidler payoff not as $u \circ \int (t, f(t), f)$, but as $u(t, f(t), \int f)$.

Theorem 3 *Under Rath's assumptions 1 and 2, every nonatomic game has a pure strategy Nash equilibrium.*

As mentioned above, the proof is based on a fixed point argument and calls for the Kakutani's fixed point theorem. Since from the beginning only the anonymous nonatomic games are considered, the proof is greatly simplified. The identification of S with the simplex in $\Re^n$ allows us to consider only a finite dimension Euclidean space.

The first step is to define the best reply correspondence $B : T \times S \to E$ by

$$B(t, s) = \{a \in E | u(t, a, s) \geq u(t, e_i, s) \; \forall e_i \in E\}$$

Notice that s, representing the Lebesgue integral of a strategy profile, is the distribution of the players over the alternatives. Indeed, $B(t, s)$ is the set of the best responses for t.

Given the continuity of the utility function, this correspondence is non empty and has closed graph.

The second and interesting step of the proof is to define the correspondence $\Gamma : S \to S$ by

$$\Gamma(s) = \int_T B(t, s) d\lambda$$

If the correspondence Γ had a fixed point, then it would exist a pure strategy profile f^* such that $\int_T f^* d\lambda = s^* \in \int_T B(t, s^*) d\lambda$. This would mean that $f^*$is a selection of $B(t, s^*)$ or that for almost all t, the strategy $f^*(t)$ belongs to the best reply of t. Then, the profile f^* would be a Nash equilibrium. In fact, Rath shows that $\Gamma(s)$ is non empty and convex and has closed graph. Therefore, by the Kakutani's theorem, $\Gamma(s)$ has a fixed point.

We will use this kind of proof to demonstrate the extension mentioned in the introduction and discussed in the next section.

Note that, here, nothing guarantees that $B(t, s)$ is convex (because we have not assumed any concavity of u, unlike in finite games). Convexification is the integral role in Γ.

4.3 Mas-Colell (1984)

The Mas-Colell's work is quite different from the previous two because he explicitly considers the (probability) distribution of the players rather than the Lebesgue integral of the strategy profile as an argument of the payoff functions. Therefore, the proof is not based on the properties of the Lebesgue integration. As Rath, he directly shows the existence of an equilibrium in pure strategies.

Moreover, the game itself is intended as a distribution. We have seen that in Schmeidler and in Rath there is an, implicit, rule that associates to each player a payoff function. Therefore, we have to name any individual (T is the set of players' names) and to give to each name a specific utility. However, in a nonatomic and anonymous context this is, actually, unnecessary. What really matters is how many players have the same preferences and not who these individuals are. Indeed, we may consider the (induced) distribution of the players on the space of payoffs.

Let us present the framework.

E is always the set of actions[11]. We consider the set of all possible probability distribution on E. Clearly, such a set is the simplex in $\Re^n$, i.e. the set S[12].

A player is completely characterized by (or his name is) a continuous utility function

$$u : E \times S \rightarrow \Re$$

Notice that there is no more a dependence on T. In particular, given an action $e_i \in E$ and a distribution $s \in S$, $u(e_i, s)$ is the utility enjoyed by the player.

U_E is the space of all continuous utility function $u(\cdot, \cdot)$ endowed with the supremum norm. This represents also the space of players characteristics.

A game with a continuum of players is then characterized by a Borel measure μ on U_E. Notice that here the global "number" of players does not matter. What matters are the characteristics of the players or their heterogeneity. If such heterogeneity is not too large, i.e. if the characteristics set is not "dispersed", clearly μ may present some atoms: in other words, a given characteristic may have a strictly positive measure. This only means

[11] Actually, Mas-Colell is much more general, allowing for any non empty and compact metric space.

[12] Again, the author is much more general and considers the set of the Borel probability measures on the action space, endowed with the weak convergence topology. Here we make this simplification to unify our presentation of the three approaches to prove the Nash Equilibrium existence in the nonatomic games.

that a strictly positive measure set of players has the same utility function, but it is not in contradiction with the nonatomic feature of the game, i.e. it remains true that each individual is negligible.

Definition 3 *Given a game μ, a Borel measure τ on $U_E \times E$ is a Nash equilibrium distribution if, denoting τ_U, τ_E the marginals of τ on U_E and E respectively, we have*
(i) $\tau_U = \mu$
(ii) $\tau(\{(u,a)|u(a,\tau_E) \geq u(a',\tau_E)\ \forall e_i \in E\}) = 1$

Notice that, even here, we are interested on the Nash equilibrium distribution and not on the strategy profile of equilibrium. This is because, there is no actual worth in knowing what each individual plays. Mas Colell is the first to exploit this feature.

Let us discuss on this definition.

The point (i) requires simply that all player characteristics are taken into account in the equilibrium distribution. The point (ii) requires that for almost all characteristics, there is a best reply alternative $a \in E$ given the marginal distribution of the players over the alternative set. We can also read such a condition as "the probability of the set of the pairs (u,a) such that $u(a,\tau_E) \geq u(e_i,\tau_E)\ \forall e_i \in E$ is one".

The main result is:

Theorem 4 *Given a game μ on U_E there exists a Nash equilibrium distribution.*

The proof is an application of the Ky Fan fixed point theorem.

Let us denote by Ω the set of all probability measures on $U_E \times E$ with the property that $\tau_U = \mu$, i.e., the set of all the distribution that verifies the condition (i).

Given $\tau \in \Omega$, $B_\tau = \{(u,a)|u(a,\tau_E) \geq u(e_i,\tau_E)\ \forall e_i \in E\}$ is the set of all the pairs (u,a) considered above. Now, a correspondence $\Phi : \Omega \to \Omega$ is defined by

$$\Phi(\tau) = \{\tau' \in \Omega | \tau'(B_\tau) = 1\}$$

Such a correspondence draws all the joint distributions that verify the condition (i) and (ii), given the joint distribution τ. It is the equivalent in this context of the $\alpha(\cdot)$ function of Schmeidler: it associates to each distribution τ the (set of its) best reply distribution. Clearly, if a fixed point exists, the distribution τ is a Nash equilibrium distribution. In fact, such a correspondence is shown convex valued, upper hemicontinuous and compact valued. Therefore there exists a fixed point by the Ky Fan theorem.

Note how in such framework is very general and relatively low demanding to show the existence of a Nash equilibrium in pure strategies. What is determinant is the way to define the Nash equilibrium distribution. Thereafter, the procedure is quite usual.

Furthermore, if we give again a name to each player and then we make u dependent also on T, we are in the setting of Rath.

5 An extension to limited anonymity

In what precedes, it is apparent that the existence of a pure strategy equilibrium in a nonatomic game is a consequence of the payoff function dependence over the average strategy or over the distribution of the players on the alternatives. In other words, the existence of an equilibrium in pure strategies depends on the anonymity assumption[13].

Nevertheless, anonymity is not always a good requirement, if we are interested in modelling settings, where there are different groups of players having different impacts on the payoff function and so on the choice of a given player. We remember two examples: the researcher's topic choice may depend on the distribution of the stars and on the distribution of the other researchers; still, the driver's route choice may depend of the distribution of several groups that are moving at his same time, such as students, workers, employees etc.

Inside each group there is no reason to give up anonymity, but, between the groups, having complete anonymity impedes to model correctly the setting. A way to introduce a "limited anonymity" is to consider that the payoff functions depend over several average responses, one for each group. In this way is possible to consider the different weight that a particular group decision has on the choice of a given player.

[13] Actually, D'Agata, 2002, shows that a pure strategy Nash equilibrium also exists in games where the players' space exibits some atoms. This is possible when the atoms are "small enough" to make the correspondence integral star-shaped valued.

In Schmeidler, 1973, there is a remark suggesting a easy generalization of the existence proof in pure strategies in this sense. In what follows, we will state the framework and prove the existence of a such equilibrium. The proof is an extension of the Rath's proof.

Our strategy is different from Khan and Sun, 1995. They prove equilibrium existence in pure strategies in a setting of countably many possible actions. Nevertheless, their generalization requires a different formal implant, dealing directly with distributions. Moreover they develop a set of theorems on the distributions of a correspondence from a nonatomic probability space to a countable compact metric space. All this is based on a original extension of the Bollobas and Varopoulos, 1974, version of the marriage lemma.

On the contrary, as Rath, we formalize distributions by the mean of Lebesgue integrals of measurable vector-valued functions and we consider only finite alternatives. Despite this last limitation, the interest of our extension is its simplicity: since it follows the Rath's approach, it embodies its features. Only standard tools on correspondence integration and the classic Kakutany's fixed point theorem are applied. In any case, the interest of dealing with countably many alternatives is mostly mathematical.

Consider a set $T = [0, 1]$ that represents the set of all players in the game. Such a set is endowed with the atomless Lebesgue measure λ. Consider k real numbers in T, denoted as $\tau_1 < ... < \tau_k$. Let τ_0 be 0 and τ_k be 1 (the boundaries of the T interval).

Denote $T_1 = [0, \tau_1]$ and T_h the subset $]\tau_{h-1}, \tau_h]$[14]. By construction, we have that $\bigcup_{h=1}^{k} T_h = T$. Therefore, the T_h subsets represent the groups of players discussed above.

It is worth to remark that the group definition has to be a fixed partition, i.e. it cannot be player-dependent. For instance, it is not possible, in the context of this extension, to consider situations where the player t's payoff mainly depends on the strategies adopted by the players close to him, i.e. belonging to a symmetric neighborhood $N_t \subset T$ of t. In this case, we would have a binary partition $\{N_t, T \backslash N_t\}$, *for each* t.

We recall that the set of alternatives is the set of the unit vectors in $\Re^n$ where the vector e_i has one at the i-th coordinate. A pure strategy profile

[14] We can equivalently deal with a general parition of measurable sets, rather than intervals, as suggested by Schmeidler, 1973. Here we use this formulation only to unify the exposition with the applications presented in the following chapters.

is a measurable function $f : T \to E$ which associates an alternative to each player.

Since a function integrable on T is integrable on any T_h (Aliprantis and Burkinshaw, 1998), we denote with $S_h = \{\int_{T_h} f d\lambda | f \in F\}$ for $h = 1, ..., k$ the set of the Lebesgue integrals for any possible strategy profile f. We label also an element of S_h as s_h. Notice that $S_1 \times ... \times S_k$ is a compact and convex subset of $\Re^{k\times n}$ and that $S_h = \{(s_h^1, ..., s_h^n) \in \Re^+ | \sum_{i=1}^{n} s_h^i = \tau_h - \tau_{h-1}\}$ or the $(\tau_h - \tau_{h-1})-$simplex in $\Re^n$.

We define the payoff function as

$$u : T \times E \times S_1 \times ... \times S_k \to \Re$$

and we require that it is continuous on $E \times S_1 \times ... \times S_k$[15].

We assume that $\{t \in T | u(t, e_i, s_1, ..., s_k) > u(t, e_j, s_1, ..., s_k)\}$ is measurable for any $s_1, ..., s_k$ and for any $e_i, e_j \in E$[16].

Definition 4 *A non atomic game with limited anonymity in normal form is a family* $G = \{T, (\tau_1, ..., \tau_k), E, u\}$.

Now we are ready to state the appropriate Nash equilibrium definition for G.

Definition 5 *A profile* $f \in F$ *is a pure strategy Nash equilibrium of* G *if for almost all* $t \in T$, $u(t, f(t), \int_{T_1} f, ..., \int_{T_k} f) \geq u(t, e_i, \int_{T_1} f, ..., \int_{T_k} f)$ $\forall e_i \in E$.

In other words, at equilibrium, almost all players have no incentives to deviate, given the distribution of each subset of agents over the alternatives.

The result of this extension is:

Theorem 5 *G has a Nash equilibrium in pure strategies.*

Proof. Define the best reply correspondence $B : T \times S_1 \times ... \times S_k \to E$ as

$$B(t, s_1, ..., s_k) = \{a | u(t, a, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k) \ \forall e_i \in E\}$$

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[15] Continuity is a requirement always present. It is necessary for the best reply correspondence to be nonempty ("Theorem of Maximum": Berge, 1962)

[16] We require mesurability to build mesurable (an integrable) selections of the best reply correspondence. More precisely, we need that any image set of a selection $g : T \to E$ is measurable. If it is the case, g is a measurable function by definition.

For any $(t, s_1, ..., s_k)$, $B(t, s_1, ..., s_k)$ is non empty because of the finite number of alternatives and because of the continuity of the utility function. For any $t \in T$, $B(t, \cdot)$ has closed graph. Indeed, for any pair of sequences $\{s_1^m, ..., s_k^m\} \to (s_1, ..., s_k)$ and $\{a^m\} \to a$ such that $a^m \in B(t, s_1^m, ..., s_k^m)$ $\forall m$ we have that $u(t, a^m, s_1^m, ..., s_k^m) \geq u(t, e_i, s_1^m, ..., s_k^m)$ $\forall e_i \in E$. Since $u(t, \cdot)$ is continuous on $E \times S_1 \times ... \times S_k$ at the limit we have that $u(t, a, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k)$ $\forall e_i \in E$ and thus the graph is closed.

Let us define the correspondence $\Gamma : S_1 \times ... \times S_k \to S_1 \times ... \times S_k$ as

$$\Gamma(s_1, ..., s_k) = \prod_{h=1}^{k} \left[\int_{T_h} B(t, s_1, ..., s_k) \right]$$

- Γ is non empty for all $s_1, ..., s_k$.

  Fix a profile $(s_1, ..., s_k) \in S_1 \times ... \times S_k$. For any $e_i, e_j \in E$ define the set $V_{ij} = \{t \in T | u(t, e_i, s_1, ..., s_k) \geq u(t, e_i, s_1, ..., s_k)\}$ or the set of all those players that prefer e_i to e_j when facing the profile $(s_1, ..., s_k)$. Because of the assumption of measurability, such a set is measurable. Now we construct a partition of T starting from the family of set V_{ij}. $V_i = \bigcap_{j \neq i} V_{ij}$ is the set of players that prefer e_i to any other alternative. Such a set is measurable. Let $V_1' = V_1$ and $V_i' = V_i \cap (\cup_{j<i} V_j')^c$ for $i = 2, ..., n$. By construction $\{V_1', ..., V_n'\}$ is a partition of T of measurable subsets.

  Let us define the function $g : T \to E$ as $g(t) = e_i$ if $t \in V_i'$. Therefore, $g(\cdot)$ is measurable by definition and $g(t) \in B(t, s_1, ..., s_k)$ for all $t \in T$, since $g(t)$ represents the best response for t, by construction. Then, for any $(s_1, ..., s_k)$ there exists a measurable selection of $B(\cdot, s_1, ..., s_k)$ represented by $g(t)$. Finally, $\Gamma(s_1, ..., s_k)$ is non empty for any $(s_1, ..., s_k)$.

- $\Gamma(\cdot)$ is convex valued.

  Since λ is atomless $\Gamma(\cdot)$ is convex valued (this comes from the definition of Lebesgue integral of a correspondence)

- $\Gamma(\cdot)$ has closed graph (and it is upper hemicontinuous since the image set is compact).

  Let the function $H : T \to \Re^n$ defined as $h(t) = (1, ..., 1) = e$ $\forall t \in T$. Clearly, $h(\cdot)$ is bounded as well as $\int_T h(t) d\lambda$. Therefore, we have $f(t) \leq h(t)$ $\forall t \in T$ and $\forall f \in F$, because $f(t)$ is a unit vector for all t. Since we showed that $B(t,\cdot)$ has closed graph, Γ too has closed graph because any possible selection is bounded (Aumann's theorem (Aumann, 1976) or "integration preserves upper hemicontinuity").

By the Kakutani's fixed point theorem, Γ has a fixed point $(s_1^*,...,s_k^*)$. Indeed, it exists a pure strategy profile $f^* \in F$ such that $\prod_{h=1}^{k} \int_{T_h} f^* d\lambda = (s_1^*,...,s_k^*) \in \Gamma(s_1^*,...,s_k^*)$ and thus $f^* \in B(t, s_1^*,...,s_k^*)$ for almost all $t \in T$ or, equivalently, $f^*(t)$ is a selection of $B(t, s_1^*,...,s_k^*)$ for almost all $t \in T$, or, again, f^* is a pure strategy Nash equilibrium. ■

This proof closely follows the procedure found by Rath. The main extension is represented by the definition of the Γ correspondence as a Cartesian product of Lebesgue integrals, rather than a simple Lebesgue integral of the best reply correspondence.

6 Conclusions

This paper mainly analyzes the existence of a Nash equilibrium in pure strategies for the class of normal form nonatomic games.

Anonymity is the fundamental requirement to get the result. We present two different approaches to formalize anonymity, one dealing with the notion of average of the "societal response" and the other with the concept of distribution induced by the strategy profile.

Three different proofs of existence are discussed. We wish to point out the connection between the Rath's and Mas Colell's settings, at least when we limit to finite action spaces. Actually, both consider players' distribution, although the former builds it as an integral of a strategy profile taking values in $\Re^n$, while the latter directly employes a distribution.

On the other hand, Schmeidler's work makes use of anonymity, in the sense of strategy profile integral, only to exploit Lebesgue integration properties in the purification process. Indeed, he shows mixed strategy equilibrium existence without assuming anonymity.

Thereafter, following the Rath's approach, we prove, in a natural and simple way, the existence of a Nash equilibrium in pure strategies for the class of nonatomic games with limited anonymity whose payoff functions depend on the distributions of a finite number of players' subsets. This extension allows to model a much larger set of problems, given the fact that it permits to differentiate the players in groups, avoiding the limitation implied by a complete anonymity (or symmetry) among players.

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