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**Fausto Cavalli, Alessandra Mainini, Daniela Visetti**

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**Department of Economics, Management and Statistics**
**University of Milano – Bicocca**
**Piazza Ateneo Nuovo 1 – 2016 Milan, Italy**
**http://dems.unimib.it/**

# The role of taxation in an integrated economic-environmental model: a dynamical analysis

Fausto Cavalli[1], Alessandra Mainini[2], and Daniela Visetti[1]

[1]Department of Economics, Management and Statistics, University of Milano Bicocca, Milan, Italy, email: fausto.cavalli@unimib.it, daniela.visetti@unimib.it
[2]Department of Economics and Social Science, Catholic University, Piacenza, Italy, email: alessandra.mainini@unicatt.it

**Abstract**

We propose a model with economic and environmental domains that interact with each other. The economic sphere is described by a Solow growth model, in which productivity is not exogenous but negatively affected by the stock of pollution that stems from the production process. A regulator can charge a tax on production, and the resources collected from taxation are used to reduce pollution. The resulting model consists of a two dimensional discrete dynamical system, and we study the role of taxation from both a static and a dynamical point of view. The focus is on the determination of the conditions under which taxation has a positive effect on the environment and leads to economic growth. Moreover, we show that a suitable environmental policy can allow recovering both local and global stability of the steady states. On the contrary, we show that, if the policy is not adequate, the system can exhibit endogenous oscillating and chaotic behavior and multistability phenomena.

**Keywords** Economic-environmental modelling, environmental policy, complex dynamics, multistability, nonlinear analysis.

## 1 Introduction

Capital represents one of the essential factors of production, and a primary element in every nation's economy. Its impact on the economic growth is out of the question, but, at the same time, this fundamental aspect has to deal with the need of the environmental protection. Indeed, pollution becomes a (necessary) by-product of the economic sphere, and its effects on environment cannot be ignored. Economic and environmental sphere are intrinsically interconnected. This occurs not just because production generates pollution, which is at the same time undesirable and inevitable. In fact, a more subtle but likewise unwanted effect is that also pollution, in turn, affects production and hence economic growth. Reasons are manifold. A first, apparent evidence is that labor force suffers the damages of pollution, in terms of health, and its well-being is affected by a climate-friendly or -unfriendly context. The literature concerning this topic is wide, with a special focus on production processes, industries and classes of pollutants. We limit to mentioning the contribution by (Graff Zivin and Neidell, 2012; Chang et al., 2016; Ebenstein et al., 2016) and we refer the interested reader to the review by (Aguilar-Gomez et al., 2022). However, the capital productivity itself is directly affected by the environmental pollution. The most clear evidences come from agriculture, as air and water pollution has direct effects on both the soil and the plant, thus affecting crops (see e.g. (Spash, 1997; Fare et al., 2006; Liu and Lu, 2023)), but can be extended, for example, to the efficiency of power plants, as in (Tyagi and Khan, 2010). More in general, the reduction of performance of manufacturing machinery due to pollutants is widely documented as well. For example, pollution can accelerate degradation and induce corrosion, leading to the so called environmental drag, which results in a reduction of economic growth (see the discussion in (Bruvall et al., 1999)).

It becomes evident that economic analysis cannot disregard environmental issues, which in turn are directly affected by the economic activity, in a scenario of reciprocally interacting domains. This outlines an intrinsically dynamical setup in which, from time to time, economic decisions affect the evolution of the environmental quality, and this, in turn, has an impact on the economic growth. The theoretical economic literature addressed these questions from several points of view, starting from different assumptions and focusing on different topics. A first seminal contribution is that by (John and Pecchenino, 1994), who developed an overlapping generation model in which the agent's consumption choices are affected (and, in turn, affect) the quality of the environment they live in. Environmental quality deteriorates proportionally to the consumption level of the old agents, and improves thanks to the action of young people. This results in a dynamical model that, as shown later by (Zhang, 1999), can give rise to complex dynamics and chaos. Subsequently, (Seegmuller and Verchère, 2004) and (Fodha and Seegmuller, 2013) reconsidered the model in (John and Pecchenino, 1994), by assuming, with respect to the original setting, that the emission of pollutants is a consequence of the production rather than of consumption. (Menuet et al., 2020) showed that different patterns and links in pollution and economic growth can arise, including poverty traps, multiple equilibria, cyclical and/or complex dynamics, still taking into account threats to the environment caused by the emissions of the production processes. This aspect has been investigated in a prolific literature strand, we limit to mentioning the contribution by (Brock and Scott Taylor, 2010), who studied a Solow model in which a growth path is sustainable only when is combined with a suitable progress in reducing pollution, and (Constant and Davin, 2019), who studied, in an overlapping generations economy, the effectiveness of allocating resources for the financing of specific funds to education devoted to the improvement of the environmental awareness. Finally, along with the aforementioned macroeconomic literature, we recall some microeconomic contributions, like those by (Matsumoto and Szidarovszky, 2020) and (Matsumoto et al., 2022), which focus on the dynamical effects of environmental policies in oligopolies, and the evolutionary approaches to the green transition problem, like those by (Zeppini, 2015) and (Cavalli et al., 2023).

Common elements that can be inferred from this literature are the relevance of dynamical aspects in addressing the economic-environmental problem and the crucial role of suitable policies for pollution control. We cannot be surprised if environmental protection has become a subject of policy discussion, and one of the key objectives of the regulator intervention. A typical environmental policy can take the form of a tax on the production, and this has multiple aims. It acts as a sort of price of the permission to pollute, and can be used by the policy maker to implement solutions in order to improve the environmental quality. Indeed, taxation has a direct impact on the accumulation of capital, so the discussion regards the opportunity of introducing such an instrument, and its effects on economic growth and on the evolution of the environmental quality.

The present contribution falls into the macroeconomic literature strand regarding the effects of pollution that results from the production process. We investigate the effectiveness of an environmental policy in providing a sustainable economic growth, with a particular focus on its effects on local and global dynamical aspects. The model we propose is inspired by that of (Matsumoto and Szidarovszky, 2011), which, in turn, is based on two seminal researches by Day ((Day, 1982, 1983)). In all these works the economic side is described by a Solow growth model in which the production function takes into account the negative effects of pollution, which is a by-product of capital. The idea is that the higher the capital level, the larger the pollution level in the environment, and this negatively affects the production process. Despite the model by (Matsumoto and Szidarovszky, 2011) consists of differential equations, and that by (Day, 1982, 1983) of difference equations, they both conclude about the occurrence of unstable dynamics. We modify the models in (Matsumoto and Szidarovszky, 2011; Day, 1982, 1983) in two directions. Firstly, we explicitly model the dynamics of the environmental side, with the introduction of a state variable that describes the stock of pollution in the environment, and which evolves over time as a result of new emissions, natural decay and abatement policies. To take into account the well-known negative effects of pollution on both the well-being of workers and the efficiency of capital, described empirically in the specialized literature, we assume that pollution reduces the total factor productivity. Moreover, we introduce the possibility for a regulator to charge an environmental tax on production, and we study the effects of this policy choice on the environment and its repercussions on the economic growth.

The key research questions regard the possibility to encourage, through a suitable policy, an economic growth that is also sustainable from an environmental point of view. We show that, to address this issue, the evolution of the environmental side cannot be disregarded but, on the contrary, must be explicitly taken into account. In particular, we show that the effectiveness of the abatement technology is crucial for achieving this goal. Moreover, we confirm that the interaction between the economic and the environmental spheres can be the source of unstable dynamics, which prevents the convergence toward a long run equilibrium. This occurs, in particular, when the production process is significantly pollutant. Finally, we prove that an appropriate environmental policy is a key tool in stabilizing dynamics, both from a local and global point of view[1].

[1]We remark that, even if some of these results can be found in the existing literature, this simple model provides a novel contribution that allows focusing on the roles of emissions and abatement on the effectiveness of the environmental policy. In

The remainder of the manuscript is organized as follows. In Section 2 we describe the model, which is then analyzed from a general perspective in Section 3. In Section 4 and 5 we study the effects of two particular shapes of the total factor productivity, provide specialized analytical results for these functions and perform some numerical simulations. In Section 6 we conclude and offer some insights for future research. We report in the Appendix the proofs of all the propositions.

# 2 The baseline model

We consider a model in which we describe two reciprocally interacting spheres, namely the economic and the environmental ones. In particular, we study the evolution of the amount of (per-capita) capital $k_t$ and the amount of pollution $p_t$, where time $t$ is discrete.

*Economic domain*

For the reader's sake, we briefly summarize how the classic Solow model is obtained. The economy consists of a single sector in which labor $L_t > 0$ and capital $K_t > 0$ are the two essential factors for the production of a good. The output level $Y_t$ is described by means of production function $F : (0, +\infty)^2 \to (0, +\infty), (L_t, K_t) \mapsto Y_t = F(L_t, K_t)$, with constant returns to scale. Labor grows at a constant rate $n \geq 0$, i.e. $L_{t+1} = (1 + n)L_t$, whereas capital depreciates at a constant rate $\mu \in (0, 1]$, so that its law of motion is

$$K_{t+1} = K_t + I_t - \mu K_t, \tag{1}$$

where $I_t$ represents the investments. Agents have an average propensity to save represented by $s \in [0, 1]$ and consume a constant fraction $1 - s$ of the net income $Y_t - \mu K_t$, i.e. consumption is $C_t = (1 - s)(Y_t - \mu K_t)$. The economy is in equilibrium when supply equals demand, i.e. $Y_t = C_t + I_t$. If we substitute $I_t$ with its expression obtained from (1), as well as the definition of $C_t$, we obtain

$$F(L_t, K_t) = (1 - s)(F(L_t, K_t) - \mu K_t) + K_{t+1} - K_t + \mu K_t,$$

which can be rephrased as

$$K_{t+1} = K_t + s(F(L_t, K_t) - \mu K_t). \tag{2}$$

According to the classic Solow model, we assume a Cobb-Douglas production function $F(L_t, K_t) = AK_t^\alpha L_t^{1-\alpha}$, where $\alpha \in (0, 1)$ is the output elasticity of capital and $A > 0$ is the total factor productivity. If we replace this formulation of $F$ in (2), divide both sides by $L_t$, set $k_t = K_t/L_t$ and recall that $L_{t+1} = (1 + n)L_t$, we obtain

$$k_{t+1} = \frac{1}{n+1}(k_t + s(Ak_t^\alpha - \mu k_t)) = \frac{1}{n+1}(sAk_t^\alpha + (1 - s\mu)k_t). \tag{3}$$

In the present model, we take into account the effect of pollution on production, by assuming that the pollution level $p_t \geq 0$ negatively affects the total factor productivity. To do that, we assume that $A$ is not constant but described by a decreasing function $A : [0, +\infty) \to [0, +\infty),\ \ p \mapsto A(p)$. In particular, we assume that $A$ is strictly decreasing and strictly positive on $[0, p_A)$ for either $p_A \in (0, +\infty)$ or $p_A = +\infty$, continuous and twice differentiable for any $p \neq p_A$ and such that $\lim_{p \to p_A} A(p) = 0$. In addition, we assume that the regulator charges an environmental tax on the output at the rate $\tau$. To do this, we replace $Y_t$ with $(1 - \tau)Y_t$ in the original Solow growth model. This implies that the law of motion for $k$ becomes

$$k_{t+1} = \frac{1}{n+1}(s(1 - \tau)A(p_t)k_t^\alpha + (1 - s\mu)k_t). \tag{4}$$

We observe that the first term in brackets differs from the corresponding one in equation (3), and, in particular, we emphasize the presence of the coefficient $1 - \tau$. [2]

addition to the basic structural differences from (Matsumoto and Szidarovszky, 2011; Day, 1982, 1983), the present research diverts from (John and Pecchenino, 1994; Zhang, 1999) both for the macroeconomic setting and the source of pollution, which in these works is a consequence of the consumption of goods, and from (Seegmuller and Verchère, 2004; Fodha and Seegmuller, 2013) because pollution affects consumer preferences and not the production function. Moreover, public expenditure per worker for pollution abatement is exogenous. Finally, (Brock and Scott Taylor, 2010) do not explicitly consider dynamics for the environmental side, and the analysis is merely static, while in the work by (Constant and Davin, 2019) only convergent dynamics can arise. For this reasons, in the remainder of the paper we do not relate our results to those in the literature, as they could not be compared from an interpretative point of view.

[2]The equation for $k$ recalls the law of motion considered by (Matsumoto and Szidarovszky, 2011), where the total factor productivity decreases in $k_t$. Therefore $A$ is endogenous in equation (4), unlike the original Solow model, as happens in the corresponding equation in the work by (Matsumoto and Szidarovszky, 2011). The main difference is that here $A$ depends explicitly on another state variable, with its own dynamics, which describes the environmental domain. Further, we recall that (Matsumoto and Szidarovszky, 2011) do not take into account the role of taxation.

Note that $\tau = 0$ corresponds to the situation in which no environmental taxation is charged, while $\tau = 1$ means that production is fully taxed. It is straightforward to see that $\tau = 1$ would imply $k_t \to 0$, for which reason we do not further investigate this scenario. Hence, in what follows, we assume $\tau \in [0, 1)$.

*Environmental domain*

We assume that the law of motion of the stock of pollution is given by

$$p_{t+1} = \max\{p_t - \delta p_t + \theta y_t - \gamma\tau y_t,\, 0\} = \max\{(1-\delta)p_t + (\theta - \gamma\tau)A(p_t)k_t^\alpha,\, 0\} \tag{5}$$

where $y_t$ is the output per capita. According to (5), the stock of pollution at time $t+1$ depends on the last period pollution level $p_t$, whereas term $-\delta p_t$ describes the natural absorption of pollution, which decays at the rate $\delta \in (0, 1)$. Thus, the limit situation $\delta \to 0$ describes the scenario in which nature alone is not capable to absorb any quantity of pollution whereas $\delta = 1$ is the situation in which nature, at each time period, has the possibility to eliminate the entire pre-existing stock of pollution even without an external intervention. Pollution increases at the rate $\theta > 0$ and emissions are proportional to the produced output $y_t$. Finally, the evolution of $p_t$ is regulated, as indicated in the last term, also by the efficiency $\gamma > 0$ of the resources (i.e., the amount of taxation) invested for the protection and the improvement of the environmental situation. Parameter $\gamma$ represents the exogenous effectiveness of the technology of abatement. In particular, $\gamma$ describes the stock of pollution that is removed from the environment for each unit of collected resources. The limit case $\gamma \to 0$ thus describes a completely ineffective intervention in favor of the environment, while the greater $\gamma$, the larger the stock of pollution that is removed from the environment. In line with the literature (see e.g. (Fodha and Seegmuller, 2013)), we assume a direct, linear dependence of the abated pollution level on the amount of collected resources, i.e. $\tau y_t = \tau A(p_t)k_t^\alpha$. As a consequence, $\gamma\tau$ represents the amount of pollution that is removed as a result of the taxation of each unit of output produced. This means that $\theta - \gamma\tau$ represents the balance between emissions and abated pollution for each unit of output. Hence, if $\theta - \gamma\tau$ is positive (respectively, negative), the pollution level in the environment, net of natural decay, increases (respectively, decreases) from $t$ to $t+1$.

Note that the minimal level of pollution, or *virgin state*, representing the situation in which there is no contamination in the environment, is set to be equal to zero. Indeed, the stock of pollution that is abated and/or naturally absorbed cannot be larger than the amount of pollution present at time $t$ in the environment. If the stock of pollution that could be potentially removed were larger than $p_t$, then at time $t+1$ we would recover the virgin state. This explains the presence of $\max\{\cdot, .\}$ in (5).

The model is thus represented by the two-dimensional discrete dynamical system $M : [0, +\infty)^2 \to [0, +\infty)^2, (k_t, p_t) \mapsto M(k_t, p_t)$, where

$$\begin{cases} k_{t+1} = \dfrac{1}{n+1}(s(1-\tau)A(p_t)k_t^\alpha + (1 - s\mu)k_t) \\ p_{t+1} = \max\{(1-\delta)p_t + (\theta - \gamma\tau)A(p_t)k_t^\alpha, 0\} \end{cases} \tag{6}$$

In what follows, we first analyze this model in a general setting, and then we explore these results more in depth by assuming two particular, and significant, shapes for function $A$.

# 3 Analysis of the baseline model

We first determine the steady states, and state some of their properties. Later on, we study their dynamical stability.

## Static analysis

We do not take into consideration a steady state with null capital level, as it does not have economic relevance. Further, it results to be repelling, and hence it does not have any dynamical role.

**Proposition 1.** *Model* (6) *has a unique steady state* $\boldsymbol{\xi}^* = (k^*, p^*)$ *characterized by a positive level of capital. If* $0 \le \tau < \theta/\gamma$ *we have*

$$p^* \in (0, p_A), \quad k^* = \left(\frac{s(1-\tau)A(p^*)}{n + s\mu}\right)^{\frac{1}{1-\alpha}}, \tag{7}$$

*whereas if* $\theta/\gamma \le \tau < 1$, *we have*

$$p^* = 0, \quad k^* = \left(\frac{s(1-\tau)A(0)}{n + s\mu}\right)^{\frac{1}{1-\alpha}}. \tag{8}$$

Proposition 1 states that the steady state exists, and is unique, for each level of $\tau$. However, on the basis of $\tau$, it could take the form (7) or (8), which differ from each other since they describe an environment characterized by the presence (7) or the absence (8) of pollutants. Since $\tau < 1$, this latter scenario is possible only when $\gamma > \theta$, which implies a suitably effective technology of abatement.

The threshold of $\tau$ that separates these two situations is represented by $\tau_v = \theta/\gamma$, which is the relative strength of the rate of emission of pollution with respect to the effectiveness of the resources invested for the protection of the environment. The particular case $\tau = \tau_v$ represents the situation in which the amount $\theta$ of new emissions for each unit of output and the amount $\gamma\tau_v$ of pollutants that are eliminated for each unit of output are perfectly balanced. In this case, the new emissions would be completely absorbed, and the natural decay would then allow recovering the virgin state in the long run. Taxation rate $\tau_v$ is the minimum tax rate that guarantees this convergence.

Clearly, threshold $\tau_v$ is meaningful if it ranges between 0 and 1, i.e. when $\theta \leq \gamma$, conversely, if the emissions overtake the abatement, the virgin state cannot be recovered. We start discussing in what situations it would be possible to achieve the goal of recovering the virgin state, and what this entails, so we consider $\theta \leq \gamma$ and we focus on the two limit scenarios for Proposition 1, i.e. when $\tau_v$ is close to either 0 or 1.

The worst situation occurs when the threshold $\tau_v$ is very high, in the limit case, close to 1, which corresponds to close values of $\theta$ and $\gamma$, namely emissions of new pollutants and their abatement take place roughly at the same rates. In this case, a policy designed with the purpose of eliminating pollution and leading the system towards the virgin state, that is towards (8), would be necessarily consistent with an extremely high level of taxation. This scenario results to be more likely consistent with the presence of pollution, and with the corresponding level of capital described in (7). In this case, if on one hand the economic sphere benefits from a higher productivity, on the other hand may be harmed by poor investments.

On the other side, in presence of both a very small emission of pollutants and very effective technologies for the protection of the environment, $\tau_v$ can be very small, in the limit case, close to 0. In this case, a public policy with the objective of eliminating pollution is consistent with a wide range of tax rates $\tau > \tau_v$ and the reduction of pollution becomes plausible even in presence of a relatively small level of $\tau$.

As a consequence, these two limit situations are somehow unrealistic and extreme, and in real situations the long run objective of the policy maker is that of sustainability, that is, counteracting pollution and preserving the environment, with a corresponding target $p^*$ that is not necessarily consistent with the virgin state. This means that the steady state could be even compatible with the presence of pollutants, what is important is that the policy maker is aware of the direct consequences of this choice, in terms of taxation, on the productivity, and thus on the economic sphere. The general content of Proposition 1 is that this can be achieved only in case of a suitable level of taxation, but this has a twofold effect on the capital level, since from (7) the value of $k^*$ directly depends on $\tau$ and indirectly on the effect of taxation on $A(p^*)$.

This is due to, and makes evident, the interdependence between the economic and the environmental sphere. Indeed, when choosing the level of $\tau$, the regulator has to counterbalance two conflicting, but connected, needs. The need, on the production side, of making investments, which results in the request of lowering taxation, since $\tau$ has a direct negative effect on the economic sphere. And the need, on the environmental side, of collecting resources in favor of the environment, that pushes the tax rate $\tau$ up. However, this could also have the indirect effect of increasing production, as a consequence of the positive effect of the environmental quality on the productivity.

To conclude, what appears evident from Proposition 1 is that taxation can represent an effective tool in counteracting pollution, but this choice has to be informed, made responsibly and carefully, by considering not only the direct, and negative, effects that $\tau$ has on the investments, but also its indirect, and positive, consequences on productivity. Indeed, both these dimensions have to do with the production side, and only their comparison may lead to an adequate choice in favor of a level of tax rate or another.

To go into depth, it is worth noting, in Proposition 1, the lack of an explicit expression of the steady state[3]. The equilibrium level of the capital depends on the equilibrium level of the pollution which, on its turn, affects the productivity. However, this dependence is not evident at this stage.

In what follows, the analysis will focus above all on the scenario described by (7), i.e. for $\theta - \gamma\tau > 0$. The reason of this is simply the ineffectiveness of raising $\tau$ once the virgin state is reached. Indeed, the achievement of the ideal situation of a clean and sustainable environment, characterized by the complete lack of pollutants, makes increasing $\tau$ above $\tau_v$ worthless, and in some sense harmful. In this case, if $\theta$ decreases or $\gamma$ and $\tau$ increase, we have that $p^*$ remains null. Recalling (8), when $\theta - \gamma\tau < 0$, it is straightforward that $k^*$ is not affected by a change in $\gamma$ or $\theta$, while it decreases as $\tau$ increases. So this policy, actually, cannot reduce pollutants, but becomes uniquely detrimental to the economic sphere. This explains our choice to investigate

[3]In particular, it is related to the implicit definition of $p^*$, given by expression (17) in the proof of Proposition 1.

further only the scenario characterized by the presence of pollutants.

Accordingly, in the next proposition we just focus on the case in which the parameters fulfill $\theta - \gamma\tau > 0$. To this end, we introduce the pollution elasticity of the total factor productivity that, for those values of $p$ for which it is well defined, it is given by the expression

$$E_A(p) = \frac{pA'(p)}{A(p)}.$$

In what follows, we refer to $E_A$ simply as elasticity of $A$ or elasticity of the total factor productivity.

**Proposition 2.** *Let us consider a parameter configuration for which $\theta - \gamma\tau > 0$. We have that $p^*$ increases and $k^*$ decreases with respect to $\theta$, while $p^*$ decreases and $k^*$ increases with respect to $\gamma$.*

*On increasing $\tau$, we have that $p^*$ decreases, while $k^*$*

*a) increases if $\theta < \gamma$ and*

$$E_A(p^*) < \frac{\theta - \gamma\tau}{\theta - \gamma} \tag{9}$$

*b) decreases, if either $\theta \geq \gamma$, or if $\theta < \gamma$ and (9) holds true with the opposite inequality $>$.*

From Proposition 2, the behavior of the equilibrium level of pollution is quite natural. Indeed, as one expects, the quantity of pollutants $p^*$ increases in its rate of emission, $\theta$, and decreases in the effectiveness of the abatement technology $\gamma$ and in the level of resources addressed to the environment, which are measured by $\tau$. On the contrary, the behavior of the equilibrium level of capital $k^*$ is not univocal. On the one hand, $k^*$ decreases in the rate of emission $\theta$ and increases in the effectiveness of the environmental policies, $\gamma$, and this is easily explained by the corresponding effects on the quantity of pollution, $p^*$, and their consequences on productivity, which decreases in $p$. On the other hand, the effect of $\tau$ on $k^*$ is twofold. Indeed, the negative effect of the taxation on $k^*$, represented by the reduction of the resources devoted to investments, is counterbalanced by the positive effect on $k^*$, represented by a lower amount of pollutants, which results in a more healthy environment, and thus in an increase in productivity. The overall effect on the behavior of $k^*$ depends on which of these two opposite effects prevails.

In this play of forces, an important role is that of elasticity of the total factor productivity, which can be above or below a certain threshold, represented by $\hat{E} = (\theta - \gamma\tau)/(\theta - \gamma)$. Note that, when $\theta < \gamma$, $\hat{E}$ is clearly negative, as well as $E_A(p^*)$, because of the monotonicity of $A$.

Therefore, when $\theta - \gamma < 0$ and $|E_A(p^*)| > |\hat{E}|$, total factor productivity $A$ is very sensitive to a variation in the amount of pollution. Consequently, after an increase of $\tau$ which determines a 1% decrease in pollution, the result is a more than 1% rise in productivity. This impact on productivity is strong enough to compensate the negative effect of the increase of $\tau$ on the investments, thus leading eventually to an increase on the capital $k^*$. On the contrary, when $\theta - \gamma < 0$ and $|E_A(p^*)| \leq |\hat{E}|$, the total factor productivity is relatively insensitive to a variation in the level of pollution, in other words, productivity changes proportionally less than pollution. This effect cannot counterbalance the variation in the amount of resources invested for the production, and the overall effect on $k^*$ is negative. It is worth noting that what has been said so far is consistent with $\theta < \gamma$, that is, with a rate of emission of pollutants less than the effectiveness of environmental policies. Otherwise, as we already noted, the effect of a rise in $\tau$ always results in a decrease of capital $k^*$.

We recap the previous comments in the next remark.

**Outcome 1.** *Capital level raises as taxation increases if the total factor productivity is suitably sensitive to a variation in the level of pollution, so that the gain in productivity compensates the negative effect of the increase of taxation.*

A more detailed discussion of the behaviors described in Proposition 2 requires to take into account the particular shape of $A$, and its exact dependence on $p$. We refer to Sections 4 and 5 for this further investigation.

## Dynamical analysis

Now we turn our attention toward the study of the stability of $\boldsymbol{\xi}^*$. We stress that in what follows we avoid to detail what happens when a stability condition is violated for just one parameter value.

**Proposition 3.** *If $\tau \in [0, \theta/\gamma)$, then $\boldsymbol{\xi}^*$ is locally asymptotically stable provided that*

$$E_A(p^*) > -\tilde{E} = -\frac{2-\delta}{\delta} \cdot \frac{n + 2 - \mu s + \alpha(n + \mu s)}{n + 2 - \mu s} \tag{10}$$

*while if $\tau \in [\theta/\gamma, 1]$ then $\boldsymbol{\xi}^*$ is locally asymptotically stable. When condition (10) is violated, instability can occur just by means of a flip bifurcation.*

*If function $A(p)$ is such that for $p \in [0, p_A)$ we have*

$$A''(p) \leq \frac{(A'(p))^2}{A(p)} - \frac{A'(p)}{p}, \tag{11}$$

*increasing $\tau$ has a stabilizing effect, and we can either have that $\boldsymbol{\xi}^*$ is locally asymptotically stable for any $\tau \in [0,1]$ or there exists $\tau_f \in (0, \gamma/\theta)$ such that $\boldsymbol{\xi}^*$ is unstable on $[0, \tau_f)$ and locally asymptotically stable on $(\tau_f, 1]$.*

Proposition 3 shows that also for stability, a key role is played by the elasticity of the total factor productivity.

Consider a sufficiently high level of $\tau$, that is, the case in which $\tau$ is above the threshold $\tau_v$. According to Proposition 1, the steady state takes the form (8) and the environment is clean and free from pollution. Proposition 3 states that this equilibrium is dynamically stable. High taxation not only guarantees a desirable scenario from an environment point of view, but also allows for the convergence, in the long period, to this equilibrium.

On the other side, consider a level of $\tau$ below the threshold $\tau_v$. According to Proposition 1, the steady state takes the form (7) and the environment is characterized by the presence of pollutants. In this case, Proposition 3 states that it is not necessarily stable, and the dimension that could make the difference is the elasticity of the total factor productivity. Indeed, the steady state turns out to be asymptotically stable when productivity is not very sensitive to a variation in the level of pollution, being $|E_A(p^*)| < \tilde{E}$. On the contrary, when productivity is sensitive enough to a variation in pollution, the steady state becomes unstable. We will return on the explanation of this point in Sections 4 and 5, but, basically, large elasticity for $A$ can induce strong variations in the total factor productivity as the pollution level changes. The capital level inherits these fluctuations as well, and, through taxation, this reflects on the amount of abated pollution, which can jump from time to time. This process then self-sustains and gives rise to persistent oscillations that prevent convergence.

Conversely, if $|E_A(p^*)|$ is small, a variation in the amount of pollutants does not affect production significantly; therefore, the resources obtained from taxation, and devoted to lowering pollution, changes except in minimal part, and so does the level of pollution. In this case, any possible oscillating behavior is softened.

At a first sight, it may seem that stability does not depend on the taxation rate, as well as on the emission rate and on the effectiveness of pollution abatement. However, parameters $\tau, \theta$ and $\gamma$ affect $p^*$ and hence $E_A(p^*)$ so their influence on stability is linked to their role on determining the steady state pollution level. If we assume that the total factor productivity is concave or not too convex, so that (11) holds true, the steady state may be either stable for any level of $\tau$, or unstable for low levels of $\tau$ and stable for sufficiently high levels of $\tau$. If (11) is valid, taxation has then a stabilizing effect.

Finally we note that condition (10) also shows the role of parameter $\delta$, that is, the rate at which environment alone reduces pollution, in the stability of the steady state. Indeed, if $\delta$ is close to zero and nature is not in any way able to eliminate pollutants, the right hand side of the inequality diverges to $-\infty$, that is, the condition is, ceteris paribus, more likely satisfied. In this case, dynamics may be stable even if no policy is adopted, or independently on $\theta$ and $\gamma$. We avoid to discuss further this aspect, as well as the roles of $\mu, \alpha, s$ and $n$ on stability.

# 4 A case of study with a concave function $A$

We consider function

$$A(p) = \begin{cases} A_0(p_A - p)^\beta & p \leq p_A, \\ 0 & p > p_A, \end{cases} \tag{12}$$

with $A_0 > 0$ and $\beta < 1$. According to Proposition 1, the steady state belongs to the interval $[0, p_A)$, where function $A$ is strictly concave. Function[4] (12) attains its maximum value $A_0 p_A^\beta$ when the environment reaches the virgin state, i.e. when $p = 0$. Moreover, when $p < p_A$, the elasticity of $A$ at the steady state, which plays a key role for both comparative statics and stability, is equal to

$$E_A(p^*) = -\frac{\beta p^*}{p_A - p^*}, \tag{13}$$

and $E_A(p)$ is then strictly decreasing on $[0, p_A)$.

[4] The functional shape of the total factor productivity in (12) is in line with the per-capita production function in (Matsumoto and Szidarovszky, 2011), where a power-like correction of the original Solow production function is used.

To discuss the results presented in this section we use some numerical examples, which are obtained by setting $n = 0$, $s = 0.9$, $\mu = 0.8, \alpha = 0.2$, $\delta = 0.2$, $A_0 = 1$, $p_A = 1$ and $\beta = 0.3$. Note that, in this case, the maximum total factor productivity is $A(0) = 1$. In panel (a) of Figure 1 we report the graph of the function $A(p)$ obtained with the parameter setting above. Since $A(p)$ is concave, the marginal effect on the total factor productivity increases with $p$. If the pollution level is small enough, the increment in productivity has only a slight effect on the production, but this growth becomes significant as $p$ increases further, with a sharp fall when $p$ is close enough to $p_A$.

## Static analysis

We study comparative statics for the level of capital when $A$ is represented by (12), thus making condition (9) in Proposition 2 explicit. Since $k^*$ is decreasing for $\theta \geq \gamma$ or $\tau \geq \theta/\gamma$ (see the remark before Proposition 2 , we limit ourselves only to the discussion of the case $\theta < \gamma$ or $\tau < \theta/\gamma$.

**Proposition 4.** *Let function A be defined as in* (12)*, with* $\beta \in (0, 1)$ *and* $\tau$ *increasing on the interval* $[0, \theta/\gamma)$. *Then*

*(i)* $k^*$ *is strictly decreasing when*

$$\delta^{1-\alpha} p_A^{1-\alpha-\beta}(n + \mu s)^\alpha - A_0\beta^\beta s^\alpha(\gamma - \theta)^\beta(\theta + \beta\gamma - \beta\theta)^{1-\alpha-\beta} \geq 0;$$

*(ii) there exists a threshold* $\bar{\tau}$ *such that* $k^*$ *is strictly increasing on the interval* $[0, \bar{\tau})$ *and strictly decreasing on the interval* $(\bar{\tau}, \theta/\gamma)$ *when*

$$\begin{cases} \delta^{1-\alpha} p_A^{1-\alpha-\beta}(n + \mu s)^\alpha - A_0\beta^\beta s^\alpha(\gamma - \theta)^\beta(\theta + \beta\gamma - \beta\theta)^{1-\alpha-\beta} < 0, \\ \delta^{1-\alpha}\gamma^\alpha p_A^{1-\alpha-\beta}(n + \mu s)^\alpha - A_0\beta^{1-\alpha} s^\alpha(\gamma - \theta) > 0; \end{cases}$$

*(iii)* $k^*$ *is strictly increasing when*

$$\delta^{1-\alpha}\gamma^\alpha p_A^{1-\alpha-\beta}(n + \mu s)^\alpha - A_0\beta^{1-\alpha} s^\alpha(\gamma - \theta) \leq 0.$$

According to Proposition 4, the concavity of the total factor productivity (12) on $[0, p_A)$, i.e. the assumption $\beta < 1$, leads to three possible scenarios.

In scenario (i) any environmental policy has a depressing effect on the economic sphere, as any increase in the tax rate results in a reduction of the amount of capital. As said before, this means that the benefit of taxation on the level of production due to the reduction of pollution, and thus to the higher productivity, is more than offset by the harm caused by taxation on the investments. In scenario (ii) when the level of the tax rate is small enough, increasing taxation is beneficial for the level of capital; however, for sufficiently large values of $\tau$, a further increase in the tax rate becomes detrimental for the economy. Therefore, the largest amount of capital corresponding to the steady state is achieved when imposing $\tau = \bar{\tau} \in (0, \theta/\gamma)$. Finally, in scenario (iii) the positive effect of taxation in enhancing the environmental conditions and, consequently, raising the total factor productivity always prevails on the negative effect of draining resources for taxation. Hence, in this case the maximum capital level is obtained when $\tau = \tau_v = \theta/\gamma$.

To better understand how the three scenarios evolve as the economic and environmental situation change, in the next corollary we put in evidence the role of the effectiveness of the abatement $\gamma$.

As it is evident from Proposition 4, the conditions that permit to identify the three scenarios are quite hard to be interpreted. In the next corollary we stress the role of parameter $\gamma$, which measures the effectiveness of the abatement, in understanding what determines the transition from one case to the other.

**Corollary 1.** *Let function A be defined as in* (12)*, with* $\beta \in (0, 1)$ *and* $\tau$ *increasing on the interval* $[0, \theta/\gamma)$. *Then there exist two thresholds* $\gamma_1$ *and* $\gamma_2$*, which depend on* $\theta, \delta, n, \mu, s, \alpha, A_0, p_A$ *and* $\beta$*, such that*

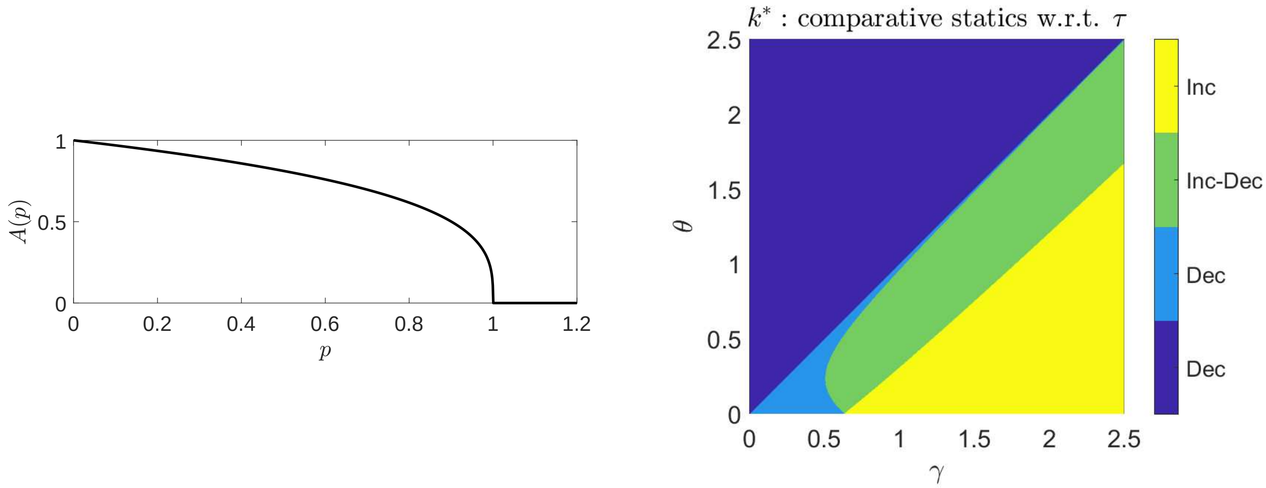
*(i)* $k^*$ *is strictly decreasing when* $\gamma \leq \gamma_1$*;*

*(ii) there exists* $\bar{\tau} \in [0, \theta/\gamma)$ *such that* $k^*$ *is strictly increasing on* $[0, \bar{\tau})$ *and strictly decreasing on* $(\bar{\tau}, \theta/\gamma)$ *when* $\gamma_1 < \gamma < \gamma_2$*;*

*(iii)* $k^*$ *is strictly increasing when* $\gamma \geq \gamma_2$.

According to Corollary 1, the behavior of $k^*$ as $\tau$ increases is strictly connected to the efficacy of the technology for the abatement.[5] Consider, for example, panel (b) in Figure 1. Each point of the diagram

[5]Thresholds $\gamma_1$ and $\gamma_2$ are implicitly defined, respectively, in equations (31) and (32), which can be found in the proof of Corollary 1 provided in the Appendix.




(a) (b)

Figure 1: (a) Graph of function (12) for $\beta < 1$. (b) Regions of the parameter space $(\gamma, \theta)$ in which we have different monotonicity behaviors of $k^*$ as $\tau$ increases. Light blue, green and yellow colors respectively represent scenarios (i),(ii) and (iii) of Corollary 1, while dark blue region corresponds to $\gamma \leq \theta$.

corresponds to a pair $(\gamma, \theta)$ and its color depends on the monotonicity of $k^*$ with respect to $\tau$. In particular, a light blue point corresponds to scenario (i) in Corollary 1, where $k^*$ is strictly decreasing on $[0, \theta/\gamma)$, a green point to scenario (ii), where $k^*$ changes its monotonicity on $[0, \theta/\gamma)$, and a yellow point to scenario (iii), where $k^*$ is strictly increasing on $[0, \theta/\gamma)$. Finally, a dark blue point represents a pair $(\gamma, \theta)$ with $\gamma \leq \theta$: in this case $k^*$ is strictly decreasing. If we fix a value for $\theta$ and cut horizontally the diagram in Figure 1, then, according to Corollary 1, lower values of $\gamma$ are consistent with those scenarios in which the level of capital $k^*$ is decreasing (dark blue and light blue regions); an increase in $\gamma$ moves afterwards towards a situation in which $k^*$ increases for small values of $\tau$ (green region); finally, a further increase in $\gamma$ becomes consistent to the scenario in which $k^*$ increases for any $\tau \in [0, \theta/\gamma)$ (yellow region).

In Figure 2 we set $\theta = 0.5$ (panel (a)) and $\theta = 1.5$ (panel (b)), and report the graphs of $p^*$ and $k^*$ for $\tau \in [0, 1]$ for three different values of the abatement rate. Consider for example panel (a). If $\gamma$ is small (black curve), then $p^*$ decreases slowly when $\tau$ increases. This means that the marginal benefits on total factor productivity resulting from the reduction of the pollution level require a significant increase of taxation to reduce pollution in an effective way, and this drags down the economic sphere. When the technology of abatement is quite ineffective, it becomes hard for the regulator to plan a policy that results to be adequate in dealing with both economic and environmental issues. The reason is that an environmental policy always turns out to be convenient for the environmental domain. As a consequence, one of the two aspects, either the economy or the environment, is bound to be disregarded, with the result of being compelled to live either in a polluted environment or with the consequences of a depressed economic sphere.

Consider now an intermediate value of $\gamma$ (blue curve). In this case $p^*$ decreases at a rate faster if compared to the situation in which $\gamma$ takes a small value (black curve). The resources from taxation improve the environment more than in the previous case, so, recalling Outcome 1 in the Static Analysis , the benefits on total factor productivity are able to counterbalance the loss of resources from taxation and therefore $k^*$ grows, at least initially. However, the concavity of $A$ implies that the benefits on the production function are decreasing, therefore the negative effects of an increase in taxation become dominant if compared to the benefits in the quality of life due to the decrease in pollution. Although the ranking of the possible outcomes in terms of $\boldsymbol{\xi}^* = (k^*, p^*)$ and the resulting optimal policy for the regulator go beyond the scopes of the present model, it is straightforward to observe that in scenario (ii) the optimal tax rate belongs to the interval $[\bar{\tau}, \theta/\gamma]$. The policy adopted depends on the weight given in social utility to the environmental issue, with respect to which the optimal choice would be $\tau = \theta/\gamma$, and to the economic condition, with respect to which the optimal choice would be $\tau = \bar{\tau}$. Finally, if the level of $\gamma$ is very high (red curve) , $p^*$ decreases quickly as taxation increases, and this allows for a significant benefit on the total factor productivity, which fully compensates the negative effect of taxation. As a consequence, the steady state level of capital increases as long as the level of emissions reduces, i.e. until the virgin state is reached. In this case, focusing on the static analysis,

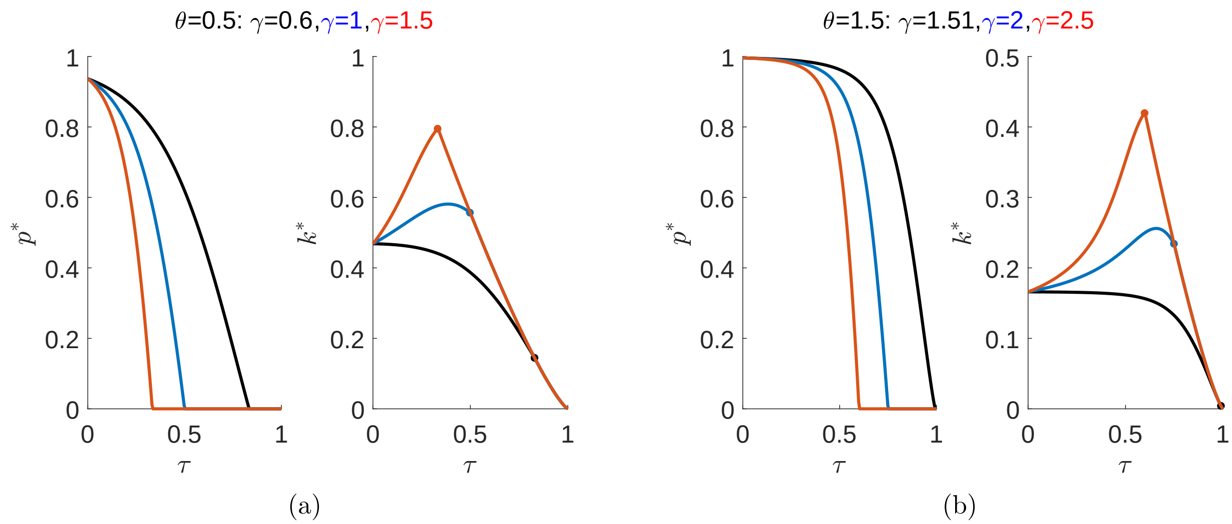



Figure 2: Comparative statics of $p^*$ and $k^*$ with $\beta < 1$ for small (panel a) and large (panel b) values of $\theta$. Related to Corollary 1, black, blue and red colors are respectively used for scenario (i), (ii) and (iii). The small circles in each graph of $k^*$ represent points at which $\tau = \theta/\gamma$.

the optimal taxation level would be $\tau = \theta/\gamma$.

What said above also applies to the example displayed in panel (b) of Figure 2, where we set $\theta = 1.5$. The main difference is that the larger emission of pollutants (1.5 rather than 0.5) results in more pollution in absence of an intervention of the policy maker, and a decrease of pollution at a slower rate. This leads to a lower level of capital at the steady state, even if its increase is significant when the technology of abatement is effective.

We summarize the previous discussion in the following outcome.

**Outcome 2.** *From a static point of view, an environmental policy can provide favorable results for both environmental and economic sides only in presence of an effective abatement technology. Indeed in this case it is possible to significantly mitigate the negative impact of pollution on production and have positive spillovers on the economic sphere, with an increment of the capital level even when the amount of taxation charged by the policy maker is particularly high.*

## Dynamical analysis

In what follows we discuss the main results on stability. Function (12) fulfills condition (11), therefore, when $\boldsymbol{\xi}^*$ is stable for $\tau = 0$, it remains stable for any $\tau > 0$, whereas, if $\boldsymbol{\xi}^*$ is unstable for $\tau = 0$, it becomes stable at some $\tau_f \in (0, \theta/\gamma)$. For the function in (12), we provide the condition that has to be satisfied by the parameters to guarantee unconditional stability for any $\tau$.

**Corollary 2.** *Let $A$ be the function defined in* (12)*, with $\beta \in (0,1)$ and $\tilde{E}$ defined in* (10)*. Then $\boldsymbol{\xi}^*$ is locally asymptotically stable for any $\tau$ provided that*

$$\theta < \bar{\theta} = \delta \frac{p_A \tilde{E}}{A_0^{\frac{1}{1-\alpha}} (\beta + \tilde{E}) \left(\frac{s}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} \left(\frac{p_A \beta}{\beta + \tilde{E}}\right)^{\frac{\beta}{1-\alpha}}}. \tag{14}$$

Stability condition (14) shows that $\boldsymbol{\xi}^*$ might be locally asymptotically stable, regardless of $\tau$, only for sufficiently low levels of the emission rate. Once we set a value of $\gamma$, and choose a level of $\theta$ larger than the threshold $\bar{\theta}$, dynamics become unstable in presence of a non-adequate environmental policy. As a consequence, only a suitable level of taxation may allow recovering stability: the reason is that the level of pollution $p^*$ decreases when $\gamma$ increases. Further, the higher the level of $\gamma$, the smaller the tax rate $\tau$ required for the stabilization of the dynamics. Indeed, the elasticity of $A$ is strictly decreasing (see equation (13)), therefore, ceteris paribus, condition (10) is more likely fulfilled as $\gamma$ increases.

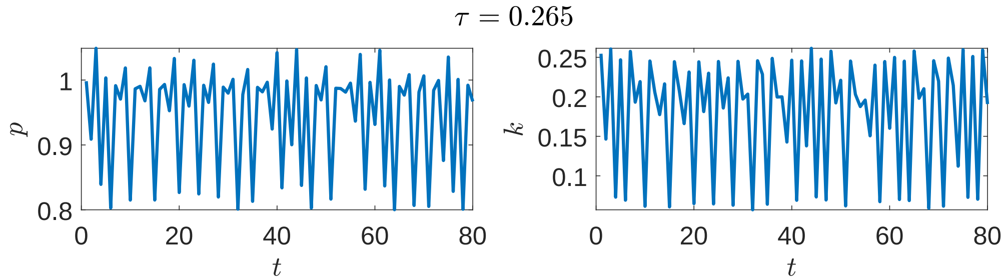



Figure 3: Time series obtained for $\theta = 1.5$ and $\gamma = 2$, showing chaotic endogenous oscillations.

The economic rationale of the destabilizing role of the emission rate has to do with the reciprocal interaction between the environmental and the economic sides. We refer to Figure 3, where we report the time series for both $p$ and $k$ corresponding to $\theta = 1.5, \gamma = 2$ and $\tau = 0.265$, for the explanation of this latter. With the parameter configuration used in this figure, the threshold of $\theta$ in (14) is $\bar{\theta} \sim 0.619$, consistent with the scenario in which dynamics are unstable when the level of tax rate is small enough. If we assume for example a high level of capital, firstly the production, and hence the emission of pollutants, is large. In the presence of a combination of an inadequate environmental policy and an ineffective abatement technology, the result is the deterioration of the environment, with a raise of the pollution level and the consequent decline of productivity. The consequence is a reduction of the level of capital and output. However, less production means lower emissions, and hence the amount of pollution $p$ decreases. Total factor productivity benefits from a reduction in the level of pollution, and the output level raises again, giving rise to endogenous self-sustained fluctuations. Small deviations from the steady state values of $p$ and $k$ initially give rise to small oscillations that quickly increase, with the possibility of sudden jumps in the levels of pollutants and of the production . If the rate of emissions $\theta$ and of abatement $\gamma\tau$ are such to lead to a large variation of the stock of pollution in the environment, the consequence is represented by endogenous fluctuations which appear nervous and chaotic like those reported in Figure 3. If the policy maker raises the tax rate $\tau$, or the abatement technologies are improved (thus raising $\gamma$), so that $\gamma\tau$ and $\theta$ are closer than in the previous case, the result is a scenario consistent with more regular oscillations, which exhibit a cyclical behavior. If $\theta - \gamma\tau$ becomes even smaller, such oscillations die out and disappear, and the system may converge toward the steady state. We remark that oscillations around $\boldsymbol{\xi}^*$, in case of instability, are quite significant. Even when the average pollution and capital level are quite close to the steady state values, a peak for $p$ or a fall for $k$ can give rise to irreversible phenomena, that we do not take into account in the present model, as a consequence of social and/or healthcare emergencies. This suggests, once again, that static analysis alone is not enough if the aim is the implementation of an adequate policy: this requires, necessarily, a dynamical perspective too.

The situation is further complicated by the nonlinear interactions between the two spheres. The red and black bifurcation diagrams reported in Figure 4, which are obtained for different initial data $(k_0, p_0)$ [6], show the possibility of coexistence among different attractors. In particular, panels (c) and (d) display a blow-up of the region in which multistability occurs. In particular, depending on $\tau$, a period-3 cycle can coexist with the steady state, with another periodic attractor or with a complex chaotic attractor. For this reason, the implementation of an adequate policy becomes quite complicated even with a stable steady state. Indeed, if we rely only on the static analysis, there is the possibility of convergence toward an unexpected outcome. In addition, the shapes of the basins of attraction, reported in Figure 5, are quite complicated. A small deviation from an initial configuration can drive trajectories toward different attractors. Moreover, the basins of attraction change with $\tau$ , which means that the same initial situation can evolve in different ways according to the policy adopted. At least, the numerical evidences, collected by considering a number of parameter settings, show that an adequate policy $\tau$ may turn out to be beneficial also from a global stability point of view. The reason is that coexisting attractors seem to exist only for small values of $\tau$ while they disappear as the tax rate increases.

[6] Both the bifurcation diagrams are obtained "following the attractor", that is, we choose the initial datum of the simulation carried out for $\tau_{i+1}$ suitably close to a point of the attractor reached with the simulation performed for $\tau_i$. The black bifurcation diagrams are obtained consistently to the values of $\tau$ decreasing from 1 to 0, whereas the red ones to the values of $\tau$ increasing from 0 to 1. Although the choice of the initial data for the starting bifurcation parameters are basically irrelevant, we set $p_0 = 1$ and $k_0 = 0.15$.

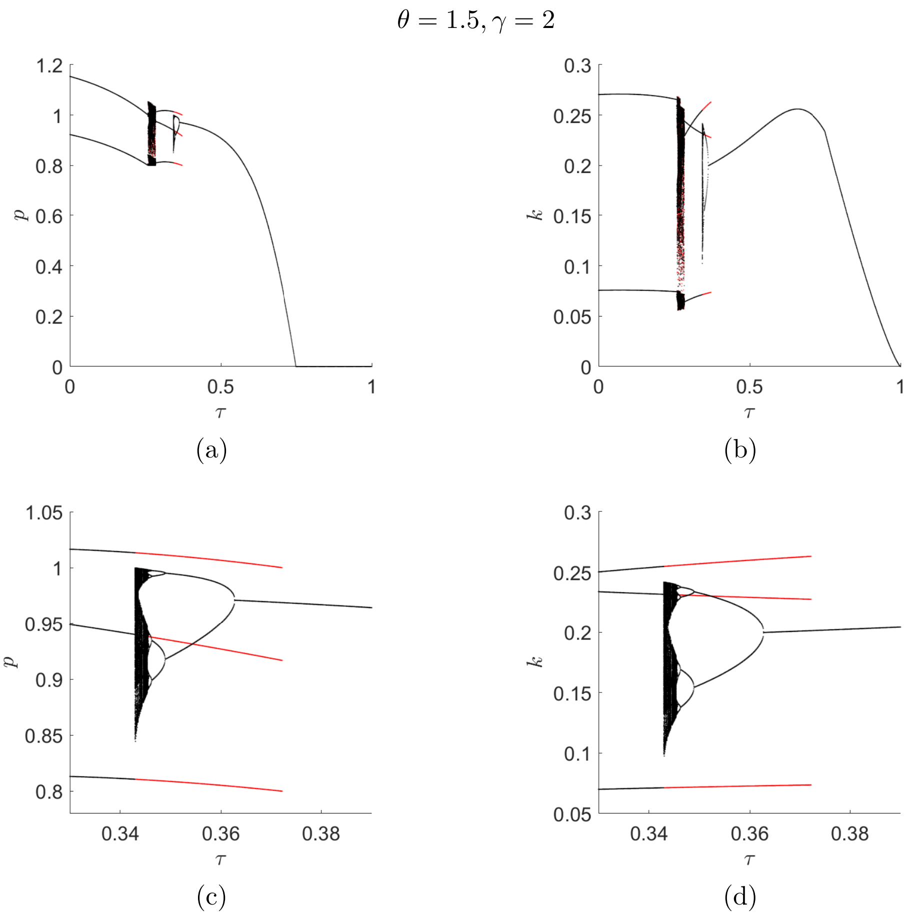



Figure 4: Bifurcation diagrams for variable $p$ (left column) and $k$ (right column). Different initial data are considered, showing the coexistence of distinct attractors (red and black bifurcation diagrams). In the top row $\tau$ ranges over the entire interval $[0, 1)$, whereas in the bottom row we report a blow up on the interval $(0.33, 0.39)$.

We summarize the various issues concerning the dynamical aspects, and emerged above, in the following outcome.

**Outcome 3.** *Large rates of emission enhance instability, which can be mitigated by adopting an appropriate environmental policy. When the rate of abatement increases, stability is recovered by reducing the tax rate. An adequate level of taxation also allows eliminating coexistence between stable attractors.*

# 5 A case of study with a convex function $A$

In this section we assume again that total factor productivity is defined as in (12), but now $\beta > 1$, i.e. function (12) is convex on $[0, +\infty)$. The numerical simulations are carried out with the same parameter setting reported in Section 4, with the only exception of $\beta$, which is now set equal to 1.25. Panel (a) of Figure 6 displays the graph of function $A(p)$. The convexity of the total factor productivity ensures that an increase in $p$ has decreasing marginal effects. This means that the benefits on production of removing a certain amount of pollutants from a polluted environment are smaller than those of removing the same quantity of pollutants from a less polluted environment. Note that even if we assume a "slightly" convex function $A(p)$ the effects of switching from a concave to a convex function are still significant.

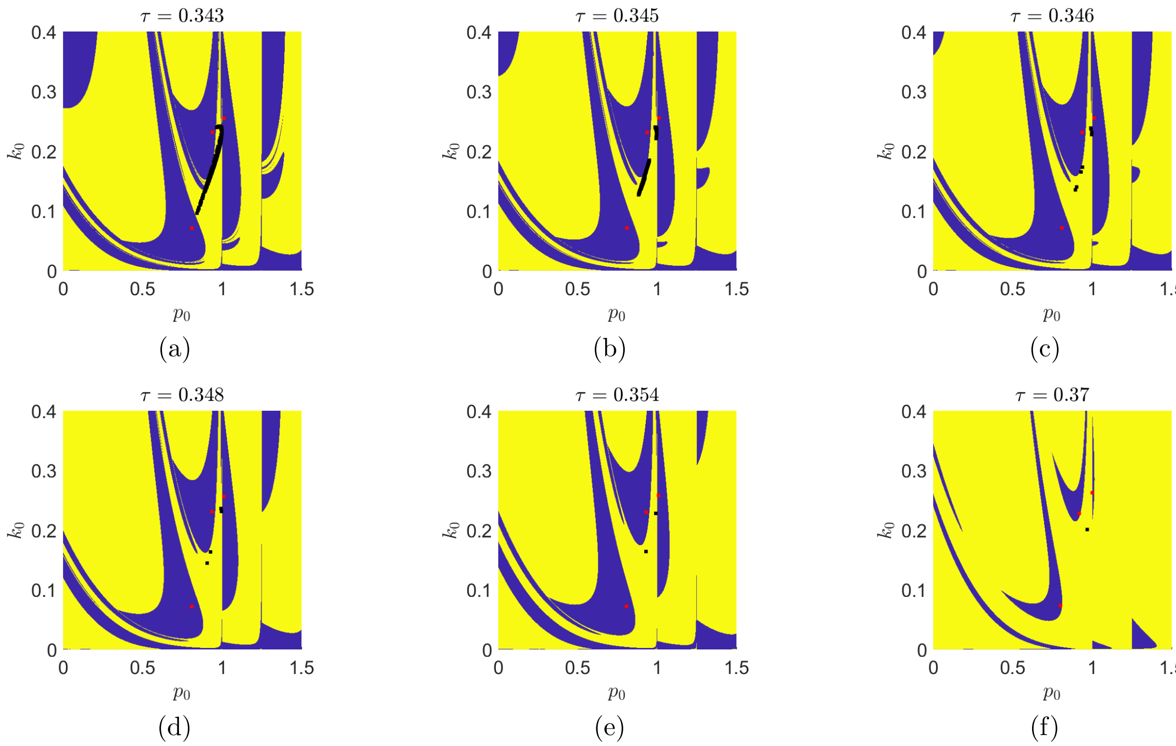



Figure 5: Basins of attraction related to the attractors shown in the bifurcation diagrams of Figure 4, for different values of $\tau$. The blue region is the basin of the period-3 cycle (depicted in red), while the yellow region is related to the black attractor.

## Static analysis

We start focusing on different comparative statics scenarios for $k^*$.

**Proposition 5.** *Let function $A$ be defined as in* (12)*, with $\beta > 1$ and $\tau$ increasing on the interval $[0, \theta/\gamma)$. Then*

*(i) $k^*$ is strictly decreasing when*

$$\begin{cases} \delta^{1-\alpha} p_A^{1-\alpha-\beta}(n+\mu s)^\alpha - A_0 \beta^\beta s^\alpha (\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta} > 0 \\ \delta^{1-\alpha}\gamma^\alpha p_A^{1-\alpha-\beta}(n+\mu s)^\alpha - A_0\beta^{1-\alpha}s^\alpha(\gamma-\theta) < 0; \end{cases}$$

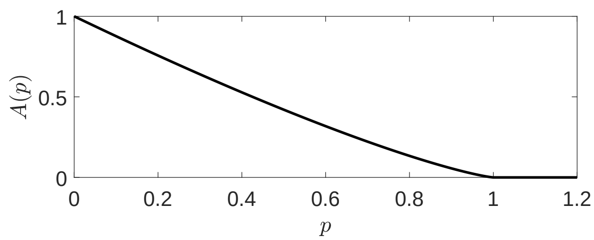
*(ii) there exists a threshold $\bar{\tau}$ such that $k^*$ is strictly decreasing on the interval $[0, \bar{\tau})$ and strictly increasing on the interval $(\bar{\tau}, \theta/\gamma)$ when*

$$\begin{cases} \delta^{1-\alpha}\gamma^\alpha (n+\mu s)^\alpha - A_0 \beta^{1-\alpha} s^\alpha (\gamma-\theta) p_A^{\alpha+\beta-1} < 0 \\ \delta^{1-\alpha}(n+\mu s)^\alpha (\theta+\beta(\gamma-\theta))^{\alpha+\beta-1} - A_0\beta^\beta s^\alpha (\gamma-\theta)^\beta p_A^{\alpha+\beta-1} > 0; \end{cases}$$

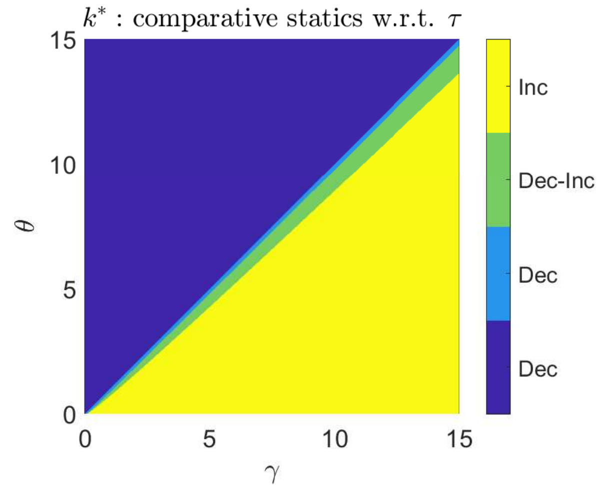
*(iii) $k^*$ is strictly increasing when*

$$\begin{cases} \delta^{1-\alpha} p_A^{1-\alpha-\beta}(n+\mu s)^\alpha - A_0 \beta^\beta s^\alpha (\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta} \leq 0 \\ \delta^{1-\alpha}\gamma^\alpha p_A^{1-\alpha-\beta}(n+\mu s)^\alpha - A_0\beta^{1-\alpha}s^\alpha(\gamma-\theta) \leq 0. \end{cases}$$

According to Proposition 5, and likewise Proposition 4, three possible scenarios may occur even under the assumption of convexity. The main difference lies in the fact that the non monotonic scenario now consists of a level of capital $k^*$ that decreases for small values of $\tau$ and then increases if $\tau$ is sufficiently large. Just like scenario (ii) in Proposition 4 follows from the assumption of concavity of $A$, here it is a consequence of the assumption of convexity. Further, since a key role for comparative statics under the assumption of concavity is played by the rate of abatement, we now make explicit the role of $\gamma$ in case of convexity.




(a)




(b)

Figure 6: (a) Graph of function (12) for $\beta > 1$. (b) Regions of the parameter space $(\gamma, \theta)$ in which we have different monotonicity behaviors of $k^*$ as $\tau$ increases. Light blue, green and yellow colors respectively represent scenarios (i),(ii) and (iii) of Corollary 3, while dark blue region corresponds to $\gamma \leq \theta$.

**Corollary 3.** *Let function $A$ be defined as in* (12)*, and assume $\beta > 1$. If $\tau$ increases on $[0, \theta/\gamma)$, then there exist two thresholds $\gamma_1$ and $\gamma_2$, which depend on $\theta, \delta, A_0, p_A, n, \mu, s, \alpha$ and $\beta$, such that*

*(i) $k^*$ is strictly decreasing when $\gamma \leq \gamma_1$;*

*(ii) there exists $\bar{\tau} \in [0, \theta/\gamma)$ such that $k^*$ is strictly decreasing on $[0, \bar{\tau})$ and strictly increasing on $(\bar{\tau}, \theta/\gamma)$ when $\gamma_1 < \gamma < \gamma_2$;*

*(iii) $k^*$ is strictly increasing when $\gamma \geq \max\{\gamma_1, \gamma_2\}$.*

Once again, the transition from a scenario to another one is driven by parameter $\gamma$.[7] In panel (b) of Figure 6 we report a diagram similar to that in panel (b) of Figure 1, the unique difference lying in the fact that a point $(\gamma, \theta)$ in the green region, and corresponding to scenario (ii), represents a level of capital $k^*$ which is decreasing for small values of $\tau$, and then increasing.

As in Corollary 1, the level of capital is strictly decreasing for small values of $\gamma$ and strictly increasing for large values of $\gamma$.

Recalling Outcome 1, the reason of that is the same as in the concave case. When $\gamma$ is small then pollution decreases slowly as $\tau$ increases, and the positive spillovers on the production resulting from the improvement of the environmental quality are too poor to counterbalance the negative effect of taxation.

On the contrary, if $\gamma$ is large, then pollution decreases rapidly as $\tau$ increases, and the direct negative effect of taxation vanishes as a consequence of the improved total factor productivity.

In scenarios (i) and (iii), the level of capital attains its maximum when the tax rate is, respectively, $\tau = 0$ and $\tau = \theta/\gamma$, that is exactly what happens under the assumption of concavity.

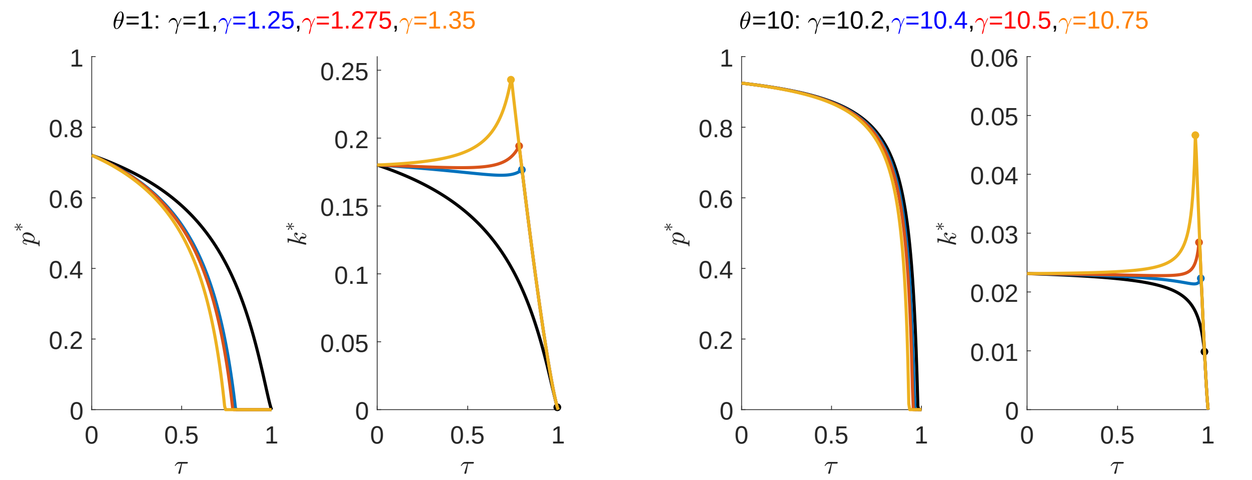
In Figure 7 we report the numerical simulations for $p^*$ and $k^*$, with $\tau \in [0, 1]$. In particular, we set $\theta = 1$ (panel (a)) and $\theta = 10$ (panel (b)) and, in both the situations, $\gamma$ varies over four possible values.[8]

The black and yellow curves represent, respectively, scenario (i) and (iii), and are discussed along the lines of the concave case. On the other hand, the numerical simulations of scenario (ii) are represented by the blue and red curves.

The convexity of $A$ means that the effect on productivity of adopting an effective environmental policy, when the tax rate is small and the pollution level is high, is not significant at the very beginning. Therefore, the direct effect of taxation more than compensates the benefit on the production of the improved environmental quality.

[7] Thresholds $\gamma_1$ and $\gamma_2$ are implicitly defined right after System (36), reported in the proof of Corollary 3 in Appendix.

[8] With regard to these numerical simulations, the values of the emission rates $\theta$ are larger than those considered under the assumption of concavity. This has less to do with the static analysis, with respect to which the next results may be obtained even if $\theta$ is smaller, as with the dynamical analysis. As explained later, instability is triggered by large values of $\theta$ if $A$ is assumed to be convex.




(a) (b)

Figure 7: Comparative statics of $p^*$ and $k^*$ with $\beta > 1$ for small (panel a) and large (panel b) values of $\theta$. Related to Corollary 3, black and yellow colors are respectively used for scenario (i) and (iii), while blue and red colors are used for two different situations related to scenario (ii). The small circles in each graph of $k^*$ represent points at which $\tau = \theta/\gamma$.

The increasing marginal benefits on the total factor productivity of the reduced pollution becomes dominant as the taxation rate increases, to such an extent that $k^*$ starts to grow. The main difference between scenarios (ii) in the two Propositions 1 and 3 is that in Proposition 1 the maximum level of the steady state capital is reached at some value of $\tau$ belonging to the interval $(0, \theta/\gamma)$, whereas in Proposition 3 the maximum is attained at either $\tau = 0$ (see the blue curve) or $\tau = \theta/\gamma$ (see the red curve).

This means that the tax rate that provides the optimal level of capital does not depend continuously on the abatement rate $\gamma$, and therefore, even a small change of this parameter could have a significant effect.

Moreover, we observe that also the behavior of $p^*$ is quite similar in the four graphs: for instance, the red and the blue curves in the left picture of panel (b) are almost coincident. This, basically, has to do with the small changes in the values of $\gamma$ used for the simulations. However, it is worth noting that this is not true for $k^*$: the corresponding graphs are significantly different, and the reason lies in the high sensitivity of the environmental and economic spheres to the level of $\gamma$.

This is a consequence of the convexity of $A$: indeed, the effect is that the progressive decrease of $p^*$ significantly accelerates the increase of total factor productivity.

We remark that the content of Outcome 2 is basically the same even under the assumption of convexity.

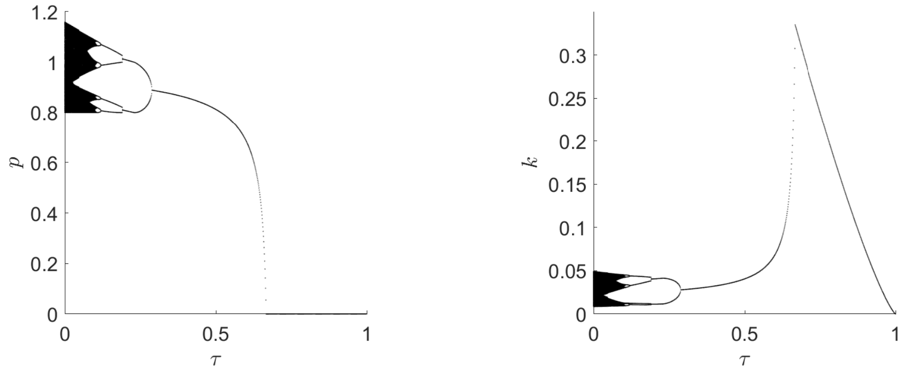
### Dynamical analysis

We note that Corollary 2 is still valid for a convex function $A$ because the proof of (14) holds true both for $\beta < 1$ and $\beta > 1$. As a consequence, what has been said for the concave case applies again, and the results related to the effects of $\tau$ on stability are robust. About the parameter setting, here the choice is $\bar{\theta} \approx 5.22$ in condition (14), along with a stability threshold larger than in the concave case, so that instability occurs only for large values of $\theta$. In Figure 7 we report two bifurcation diagrams, from which it is evident the stabilizing effect of $\tau$. Finally, we note that we can observe multistability phenomena also with the assumption of convexity for $A$, in particular for large values of $\theta$ and $\gamma$, and the coexistence of attractors occurs for those values of $\tau$ that make $\boldsymbol{\xi}^*$ unstable. To conclude, Outcome 3 is still true.

## 6 Conclusions

The effects of the environmental issues on the economy of a nation necessarily require an integrated approach. Production choices have direct impact on the environmental quality, and this damage could compress the economy's production capabilities, as a result of the degradation of the production factors. The static analysis

$\theta = 10.5, \gamma = 15$




(a) (b)

Figure 8: Bifurcation diagrams for variable $p$ (left column) and $k$ (right column).

highlights how the role of the regulator is crucial; as a matter of fact, an inaccurate environmental policy choice may have negative effects on the economy without producing significant benefits on the environment. On the contrary, an adequate policy can combine environmental improvement and a sustainable growth. However, such an integrated approach cannot disregard dynamical aspects, as the reciprocal influence between the economic and environmental spheres is the source of non convergent dynamics. Static analysis alone can become misleading in the presence of multistability phenomena, in which case it may result hard to plan an appropriate policy.

The present contribution represents a first step into the analysis of integrated economic-environmental problems. A natural evolution is to explicitly take into account health issues, in order to consider the role of epidemiological dynamics on the labor productivity. In this extended setting the regulator should allocate the resources collected from taxation between healthcare and environmental protection. Moreover, a policy is adequate if can react promptly to different scenarios that can be encountered, and should be designed in order to adapt endogenously the share of resources to be devoted to each urgency. In this perspective, the regulator should not take a decision myopically, but has to take into account, and evaluate carefully, the history and the evolution of the environmental and epidemiological situations. However, a framework like this latter would require more refined analytical and statistical tools, relying on comparative dynamics and on the development of suitable indicators that allow discriminating against the different possible attractors along which dynamics occur.

# Statements and Declarations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# Appendix

In what follows, we report the proofs of the proposition.

*Proof of Prop. 1.* We set $p_{t+1} = p_t = p$ and $k_{t+1} = k_t = k$. From the former equation in (6) we have

$$k = \frac{1}{1+n}(s(1-\tau)A(p)k^{\alpha} + (1-s\mu)k) \Leftrightarrow 1 = \frac{1}{1+n}(s(1-\tau)A(p)k^{\alpha-1} + (1-s\mu))$$

which is obtained after removing solution $k^* = 0$. Solving for $k$ we find

$$A(p)k^{\alpha-1} = \frac{n+s\mu}{s(1-\tau)} \tag{15}$$

which has no solution if $A(p)=0$ and hence $p \in [0, p_A)$. We can rewrite (15) as

$$k = \left(\frac{s(1-\tau)A(p)}{n+s\mu}\right)^{\frac{1}{1-\alpha}} \tag{16}$$

Replacing the previous expression of $k$ in the latter equation in (6) we obtain

$$p = \max\left\{p - \delta p + (\theta - \gamma\tau)A(p)\left(\frac{s(1-\tau)A(p)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}}, 0\right\}.$$

Let us introduce function $g : [0, p_A) \to \mathbb{R}, p \mapsto g(p)$ defined by

$$g(p) = p(1-\delta) + (\theta - \gamma\tau)A(p)\left(\frac{s(1-\tau)A(p)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}}$$

We start considering the case of $\theta - \gamma\tau \leq 0$. If $g(p) < 0$, we have that $\max\{g(p), 0\} = 0$, while if $g(p) > 0$, we have

$$g(p) = (1-\delta)p + (\theta - \gamma\tau)A(p)\left(\frac{s(1-\tau)A(p)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} \leq (1-\delta)p < p,$$

so the unique solution to $\max\{g(p), 0\} = p$ is $p^* = 0$.

Let us now consider $\theta - \gamma\tau > 0$. In this case $g(p) > 0$ and hence $\max\{g(p), 0\} = g(p)$. Equation $p = g(p)$ can be rephrased into

$$p = \frac{\theta - \gamma\tau}{\delta} A(p)^{\frac{1}{1-\alpha}} \left(\frac{s(1-\tau)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} \tag{17}$$

where the right hand side is a strictly decreasing function for which $\frac{\theta-\gamma\tau}{\delta} A(0)^{\frac{1}{1-\alpha}} \left(\frac{s(1-\tau)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} > 0$ and

$$\lim_{p \to p_A} \frac{\theta - \gamma\tau}{\delta} A(p)^{\frac{1}{1-\alpha}} \left(\frac{s(1-\tau)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} = 0$$

so $p = g(p)$ always has a unique solution $p^* \in (0, p_A)$.

Note that if $\theta - \gamma\tau > 0$, replacing $p^*$ in (16) we find (7). □

*Proof of Prop. 2.* To study the monotonicity of $p^*$ we write equation (17) as

$$f(p) = \varphi(\theta, \gamma, \tau) \tag{18}$$

where $f : [0, p_A) \to \mathbb{R}$ and $\varphi : (0, +\infty) \times (0, +\infty) \times (0, 1) \to \mathbb{R}$ are defined by

$$f(p) = \frac{p}{A(p)^{\frac{1}{1-\alpha}}} \qquad \text{and} \qquad \varphi(\theta, \gamma, \tau) = \frac{\theta - \gamma\tau}{\delta}\left(\frac{s(1-\tau)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}}. \tag{19}$$

It is immediate that $f$ is increasing with respect to $p$ and that $\varphi$ is increasing with respect to $\theta$, and decreasing both with respect to $\gamma$ and $\tau$. Since $p^*$ is implicitly defined by (18), $p^*$ is increasing with respect to $\theta$, decreasing both with respect to $\gamma$ and $\tau$.

The monotonicity for $k^*$ with respect to $\gamma$ or $\theta$ is the opposite of that of $p^*$ as evident from (16), in which the right hand side does not directly depend on $\gamma$ or $\theta$ while $A(p)$ is decreasing.

To study the behavior of $k^*$ with respect to $\tau$, we note that from (16) the monotonicity of $k^*$ is the same of that of $(1-\tau)A(p^*)$, for which we have

$$\frac{d}{d\tau}[(1-\tau)A(p^*)] = -A(p^*) + (1-\tau)\frac{dp^*}{d\tau}A'(p^*)$$

and we can study the sign of $-A(p) + (1-\tau)\frac{dp}{d\tau}A'(p)$.

Note that $\frac{dp^*}{d\tau}$ is implicitly defined by (17) and results

$$\frac{dp^*}{d\tau} = -\frac{(\gamma - \alpha\gamma + \alpha\theta - \gamma\tau)\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A(p^*)^{\frac{1}{1-\alpha}}}{\left[\delta(1-\alpha) - \left(\theta - \gamma\tau(p^*)^{\frac{\alpha}{1-\alpha}}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A'(p^*)\right]}(1-\tau)$$

Replacing it in $-A(p^*) + (1-\tau)\frac{dp^*}{d\tau}A'(p^*)$ we find that it is positive when

$$-A(p^*) - \frac{(\gamma - \alpha\gamma + \alpha\theta - \gamma\tau)\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A(p^*)^{\frac{1}{1-\alpha}}}{\left[\delta(1-\alpha) - (\theta-\gamma\tau)A(p^*)^{\frac{\alpha}{1-\alpha}}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A'(p^*)\right](1-\tau)}(1-\tau)A'(p^*) > 0$$

or, equivalently, when

$$-\delta + A(p^*)^{\frac{\alpha}{1-\alpha}}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A'(p^*)(\theta-\gamma) > 0. \tag{20}$$

If $\theta \geq \gamma$, the latter addend is negative, so the inequality cannot be fulfilled, and this provides the first condition of case b).

From (17), we have

$$\left(\frac{s(1-\tau)}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} = \frac{\delta p^*}{(\theta-\gamma\tau)A(p^*)^{\frac{1}{1-\alpha}}}$$

and hence (20) becomes

$$-\delta + A(p^*)^{\frac{\alpha}{1-\alpha}}\frac{\delta p^*}{(\theta-\gamma\tau)A(p^*)^{\frac{1}{1-\alpha}}}A'(p^*)(\theta-\gamma) > 0 \Leftrightarrow -1 + \frac{p^*}{(\theta-\gamma\tau)A(p^*)}A'(p^*)(\theta-\gamma) > 0$$

which provides (9) and the last condition in case b). □

*Proof of Prop. 3.* Let $\tau \in (\theta/\gamma, 1)$, in which on a suitable neighborhood of the steady state model (6) is described by

$$\begin{cases} k_{t+1} = \frac{1}{1+n}(s(1-\tau)A(0)k_t^\alpha + (1-s\mu)k_t) \\ p_{t+1} = 0 \end{cases}$$

and stability is inferred from the analysis of the derivative of the one dimensional map describing the dynamics of $k$. A direct check shows that $k^*$ is locally asymptotically stable for $\tau \in (\theta/\gamma, 1)$. Note that $k^*$ is locally asymptotically stable also if we the map is stable also if the map is extended to $\tau = \theta/\gamma$ and trajectories are monotonic.

Let $\tau \in [0, \theta/\gamma)$. In this case we have $\theta - \gamma\tau > 0$ and $p^* > 0$. The Jacobian matrix is the following one

$$J = \begin{pmatrix} \frac{(1-\mu s)k^{1-\alpha} + \alpha s(1-\tau)A(p)}{(n+1)k^{1-\alpha}} & \frac{s(1-\tau)k^\alpha A'(p)}{n+1} \\ \frac{\alpha(\theta-\gamma\tau)A(p)}{k^{1-\alpha}} & 1-\delta+(\theta-\gamma\tau)k^\alpha A'(p) \end{pmatrix}.$$

We remark that since $p^* < p_A$, function $A$ is differentiable at $p^*$. Replacing $k^*$ using (7) we find

$$J^* = \begin{pmatrix} \frac{1-\mu s+\alpha(n+\mu s)}{n+1} & \frac{s(1-\tau)\left(\frac{s(1-\tau)A(p^*)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}A'(p^*)}{n+1} \\ \frac{\alpha(n+\mu s)(\theta-\gamma\tau)}{s(1-\tau)} & 1-\delta+(\theta-\gamma\tau)\left(\frac{s(1-\tau)A(p^*)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}A'(p^*) \end{pmatrix}$$

Stability conditions (see e.g. (Elaydi, 2007)) are

$$\begin{cases} 1-\mathrm{tr}(J^*)+\det(J^*) > 0 \\ 1+\mathrm{tr}(J^*)+\det(J^*) > 0 \\ 1-\det(J^*) > 0 \end{cases}$$

Since there holds:

$$\mathrm{tr}(J^*) = 1-\delta+\frac{1-\mu s+\alpha(n+\mu s)}{n+1} + (\theta-\gamma\tau)(k^*)^\alpha A'(p^*)$$
$$\det(J^*) = \frac{(1-\delta)[1-\mu s+\alpha(n+\mu s)]}{n+1} + \frac{(\theta-\gamma\tau)(1-\mu s)(k^*)^\alpha A'(p^*)}{n+1}$$

we obtain

$$\begin{cases} \frac{(n+\mu s)\left[(1-\alpha)\delta-(\theta-\gamma\tau)(k^*)^{\frac{\alpha}{1-\alpha}}A'(p^*)\right]}{n+1} > 0 \\ (2-\delta)\frac{n+2-\mu s+\alpha(n+\mu s)}{n+1} + (\theta-\gamma\tau)\frac{n+2-\mu s}{n+1}(k^*)^\alpha A'(p^*) > 0 \\ 1-\alpha(1-\delta) - \frac{(1-\mu s)\left[(1-\alpha)(1-\delta)+(k^*)^{\frac{\alpha}{1-\alpha}}A'(p^*)(\theta-\gamma\tau)\right]}{n+1} > 0 \end{cases}$$

The first condition is always fulfilled since $\alpha \in (0,1)$ and $A'(p^*) \leq 0$.

In the last condition we have

$$1-\alpha(1-\delta)-\frac{(1-\mu s)(1-\alpha)(1-\delta)}{n+1} > 1-\alpha(1-\delta)-(1-\alpha)(1-\delta)=\delta>0$$

and, recalling the expression of $k^*$ given by (7), from $A'(p^*) \leq 0$ we have

$$-\left(\frac{sA(p^*)(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} A'(p^*)(\theta-\gamma\tau) \geq 0$$

In the second condition, which is the one related to the emergence of a flip bifurcation, if we replace $k^*$ with its expression in (7), we have

$$\left(\frac{sA(p^*)(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} (\theta-\gamma\tau)(n+2-\mu s)A'(p^*)+(2-\delta)(n+2-\mu s+\alpha(n+s\mu))>0,$$

from which

$$(\theta-\gamma\tau)(A(p^*))^{\frac{1}{1-\alpha}}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}\frac{A'(p^*)}{A(p^*)} > -\frac{(2-\delta)(n+2-\mu s+\alpha(n+s\mu))}{n+2-\mu s}.$$

Using (17), we obtain (10).

Finally we note that for $\tau \to (\theta/\gamma)^+$ we find

$$\begin{cases} \frac{(n+\mu s)(1-\alpha)\delta}{n+1} > 0 \\ (2-\delta)\frac{n+2-\mu s+\alpha(n+\mu s)}{n+1} > 0 \\ 1-\alpha(1-\delta)-\frac{(1-\mu s)(1-\alpha)(1-\delta)}{n+1} > 0 \end{cases}$$

which are all fulfilled. This, together with the considerations for $\tau > \theta/\gamma$ allows concluding that $\boldsymbol{\xi}^*$ is locally asymptotically stable for $\tau = \theta/\gamma$.

Finally, since

$$(E_A(p))' = \frac{pA(p)A''(p)-p(A'(p))^2+A(p)A'(p)}{A(p)^2}$$

and $p^*$ decreases with respect to $\tau$, under condition (11) we have that $E_A(p(\tau))$ is strictly increasing with respect to $\tau$, and this allows concluding. □

*Proof of Prop. 4 and Cor. 1.* From (17) we have

$$p = \frac{\theta-\gamma\tau}{\delta(A_0(p_A-p)^\beta)^{\frac{1}{\alpha-1}}\left(-\frac{s(\tau-1)}{n+\mu s}\right)^{\frac{\alpha}{\alpha-1}}} \Leftrightarrow \frac{p}{(A_0(p_A-p)^\beta)^{\frac{1}{1-\alpha}}} = \frac{\theta-\gamma\tau}{\delta}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}$$

Let us define function $f:[0,p_m)\to\mathbb{R}, p\mapsto f(p)$ defined by $f(p)=\frac{p}{(A_0(p_A-p)^\beta)^{\frac{1}{1-\alpha}}}$, which is strictly increasing. We have that $p^*$

$$p^* = f^{-1}\left(\frac{\theta-\gamma\tau}{\delta}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}\right) \tag{21}$$

Let $\theta<\gamma$ and $\theta-\gamma\tau>0$. Condition (9), since $E_A(p)=-\frac{\beta p}{p_m-p}$, since $E_A(p)=-\frac{\beta p}{p_A-p}$, becomes

$$p(-\beta(\theta-\gamma)+\theta-\gamma\tau) > (\theta-\gamma\tau)p_A$$

Note that since $-\beta(\theta-\gamma)+\theta-\gamma\tau \geq 0$ the previous inequality is fulfilled provided that

$$p > \frac{(\theta-\gamma\tau)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}$$

Using (21) we find

$$p > \frac{(\theta-\gamma\tau)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau} \Leftrightarrow f^{-1}\left(\frac{\theta-\gamma\tau}{\delta}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}}\right) > \frac{(\theta-\gamma\tau)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}$$

which can be rewritten as

$$\frac{\theta-\gamma\tau}{\delta}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} > \frac{\frac{(\theta-\gamma\tau)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}}{A_0^{\frac{1}{1-\alpha}}\left(p_A-\frac{(\theta-\gamma\tau)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}\right)^{\frac{\beta}{1-\alpha}}}$$

and hence

$$\frac{A_0^{\frac{1}{1-\alpha}}}{\delta}\left(\frac{s(1-\tau)}{n+\mu s}\right)^{\frac{\alpha}{1-\alpha}} > \frac{\frac{p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}}{\left(\frac{-\beta(\theta-\gamma)p_A}{-\beta(\theta-\gamma)+\theta-\gamma\tau}\right)^{\frac{\beta}{1-\alpha}}}$$

from which

$$\rho(\tau)=\frac{s(1-\tau)}{n+\mu s}-\frac{(-\beta(\theta-\gamma)+\theta-\gamma\tau)^{\frac{\alpha+\beta-1}{\alpha}}\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\beta^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\beta/\alpha}}>0$$

in which we have defined function $\rho:[0,\theta/\gamma]\to\mathbb{R},\tau\mapsto\rho(\tau)$.

So $k^*$ is strictly increasing when $\rho(\tau)>0$ and strictly decreasing when $\rho(\tau)<0$, and a monotonicity change can occur if $\rho(\tau)=0$.

Direct computation shows that

$$\rho'(\tau)=\frac{\gamma(\alpha+\beta-1)\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\alpha\beta^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\beta/\alpha}(-\beta(\theta-\gamma)+\theta-\gamma\tau)^{\frac{1-\beta}{\alpha}}}-\frac{s}{n+\mu s}$$

and

$$\rho''(\tau)=\frac{\gamma^2(1-\beta)(\alpha+\beta-1)\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\alpha^2\beta^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\beta/\alpha}(-\beta(\theta-\gamma)+\theta-\gamma\tau)^{\frac{\alpha-\beta+1}{\alpha}}} \tag{22}$$

so, depending on the sign of $(\alpha+\beta-1)$, $\rho(\tau)$ is either convex, a line or concave and $\rho(\tau)=0$ has at most two solutions.

Moreover we have

$$\rho(0)=\frac{s}{n+\mu s}-\frac{(\theta+\beta\gamma-\beta\theta)^{\frac{\alpha+\beta-1}{\alpha}}\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\beta^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\beta/\alpha}}, \tag{23}$$

$$\rho(\theta/\gamma)=\frac{s(\gamma-\theta)}{\gamma(n+\mu s)}-\frac{\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\beta^{\frac{1-\alpha}{\alpha}}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\frac{1-\alpha}{\alpha}}} \tag{24}$$

and

$$\rho'(0)=\frac{\gamma(\alpha+\beta-1)\delta^{\frac{1-\alpha}{\alpha}}}{A_0^{1/\alpha}\alpha\beta^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\gamma-\theta)^{\beta/\alpha}(\theta+\beta\gamma-\beta\theta)^{\frac{1-\beta}{\alpha}}}-\frac{s}{n+\mu s} \tag{25}$$

Note that $\rho(0)>0$ if and only if

$$A_0>\frac{\delta^{1-\alpha}p_A^{1-\beta-\alpha}(n+\mu s)^\alpha}{\beta^\beta s^\alpha(\gamma-\theta)^\beta(\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta}} \tag{26}$$

while $\rho(\theta/\gamma)>0$ if and only if

$$A_0>\frac{\delta^{1-\alpha}\gamma^\alpha p_A^{1-\alpha-\beta}(n+\mu s)^\alpha}{\beta^{1-\alpha}s^\alpha(\gamma-\theta)} \tag{27}$$

moreover $\rho'(0)>0$ if and only if

$$A_0^{1/\alpha}<\frac{\delta^{\frac{1-\alpha}{\alpha}}\gamma(n+\mu s)(\alpha+\beta-1)}{\alpha\beta^{\beta/\alpha}s(\gamma-\theta)^{\beta/\alpha}p_A^{\frac{\alpha+\beta-1}{\alpha}}(\theta+\beta\gamma-\beta\theta)^{\frac{1-\beta}{\alpha}}} \tag{28}$$

and $\rho'(\tau)=0$ is solved only at

$$\tau_e=\frac{\theta}{\gamma}+\frac{\beta(\gamma-\theta)}{\gamma}-\frac{\gamma^{\frac{\alpha+\beta-1}{1-\beta}}(n+\mu s)^{\frac{\alpha}{1-\beta}}(\alpha+\beta-1)^{\frac{\alpha}{1-\beta}}\delta^{\frac{1-\alpha}{1-\beta}}}{A_0^{\frac{1}{1-\beta}}\alpha^{\frac{\alpha}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}p_A^{\frac{\alpha+\beta-1}{1-\beta}}s^{\frac{\alpha}{1-\beta}}(\gamma-\theta)^{\frac{\beta}{1-\beta}}}. \tag{29}$$

In what follows we study the sign of $\rho$ (and its zeros) on the open interval $(0,\theta/\gamma)$, as the cases in which it vanishes at (some of) the ending points do not affect monotonicity behavior of $k^*$.

We distinguish two cases

• $\alpha + \beta - 1 \leq 0$

From (22) we have that $\rho$ is concave (it is a line for $\alpha + \beta - 1 = 0$). Since from (25) we have $\rho'(0) < 0$, $\rho$ is strictly decreasing for any $\tau$ and hence $\rho(\tau) = 0$ has at most a unique solution. Exactly one solution $\tau \in (0, \theta/\gamma)$ exists if and only if $\rho(0) > 0$ and $\rho(\theta/\gamma) < 0$, i.e. from (23) and (24) if

$$\begin{cases} A_0 > \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{\beta^\beta s^\alpha (\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta}} \\ A_0 < \frac{\delta^{1-\alpha} \gamma^\alpha p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{\beta^{1-\alpha} s^\alpha (\gamma-\theta)} \end{cases} \tag{30}$$

This provides case (ii) in Proposition 4. To obtain the same case in Corollary 1, condition (30) can be rewritten as

$$\begin{cases} (\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta} > \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^\beta s^\alpha} \\ \frac{\gamma-\theta}{\gamma^\alpha} < \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^{1-\alpha} s^\alpha} \end{cases}$$

We note that $(\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta}$ is strictly increasing for $\gamma \in (\theta, +\infty)$, vanishes as $\gamma \to \theta^+$ and diverges as $\gamma \to +\infty$. Since $\frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^\beta s^\alpha}$ is strictly positive, equation

$$(\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta} = \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^\beta s^\alpha} \tag{31}$$

can be uniquely solved for $\gamma$ for any parameter configuration. This allows introducing a function $\gamma_1$, taking values on the set in which parameters $\theta, \delta, A_0, p_A, n, \mu, s, \alpha$ and $\beta$ can vary, which is implicitly defined by the last equality. Since the very same considerations hold true of equation

$$\frac{\gamma-\theta}{\gamma^\alpha} = \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^{1-\alpha} s^\alpha}, \tag{32}$$

we can similarly introduce function $\gamma_2$.

Conversely, if $\rho(0) \leq 0$, $\rho$ is negative on $(0, \theta/\gamma)$ and this provides case (i) in Proposition 4. From (26), we have that this situation corresponds to

$$(\gamma-\theta)^\beta (\theta+\beta\gamma-\beta\theta)^{1-\alpha-\beta} \leq \frac{\delta^{1-\alpha} p_A^{1-\beta-\alpha} (n+\mu s)^\alpha}{A_0 \beta^\beta s^\alpha},$$

which, recalling the discussion about the solutions to equation (31), provides case (i) in Corollary 1. Finally, if $\rho(\theta/\gamma) \geq 0$, $\rho$ is positive on $(0, \theta/\gamma)$ this provides case (iii) in Proposition 4. From (27), we have that this case corresponds to

$$\frac{\gamma-\theta}{\gamma^\alpha} \geq \frac{\delta^{1-\alpha} p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{A_0 \beta^{1-\alpha} s^\alpha}$$

which, recalling the discussion about the solutions to equation (32), provides case (iii) in Corollary 1. This concludes the first part of the proof.

• $\alpha + \beta - 1 > 0$

In this case $\rho$ is convex and it can be either increasing or decreasing at $\tau = 0$. We distinguish these two situations.

Case 1: $\rho'(0) \geq 0$

Recalling (28), this occurs if

$$A_0 \leq \frac{\delta^{1-\alpha} \gamma^\alpha p_A^{1-\alpha-\beta} (n+\mu s)^\alpha (\alpha+\beta-1)^\alpha}{\alpha^\alpha \beta^\beta s^\alpha (\theta+\beta\gamma-\beta\theta)^{1-\beta} (\gamma-\theta)^\beta}$$

Since $\rho$ is convex, this means that it is strictly increasing for any $\tau$ and hence $\rho(\tau) = 0$ has at most a unique solution. Exactly one solution $\tau \in (0, \theta/\gamma)$ would exist if $\rho(0) < 0$ and $\rho(\theta/\gamma) > 0$. Actually, it is not possible that $\rho'(0) \geq 0$ and $\rho(\theta/\gamma) > 0$. In fact, this results in the inequalities

$$\frac{\delta^{1-\alpha} \gamma^\alpha p_A^{1-\alpha-\beta} (n+\mu s)^\alpha}{\beta^{1-\alpha} s^\alpha (\gamma-\theta)} < A_0 \leq \frac{\delta^{1-\alpha} \gamma^\alpha p_A^{1-\alpha-\beta} (n+\mu s)^\alpha (\alpha+\beta-1)^\alpha}{\alpha^\alpha \beta^\beta s^\alpha (\theta+\beta\gamma-\beta\theta)^{1-\beta} (\gamma-\theta)^\beta} \tag{33}$$

which imply that

$$\left[\frac{\beta(\gamma-\theta)}{\theta+\beta(\gamma-\theta)}\right]^{1-\beta} > \left(\frac{\alpha\beta}{\alpha+\beta-1}\right)^{\alpha}$$

and also that $\frac{\alpha\beta}{\alpha+\beta-1} < 1$, but this is equivalent to $(\alpha-1)(1-\beta) > 0$, that is false. Then the only possibility is that there is no zero of $\rho$ and the function is negative, from which we obtain case (i).

Case 2: $\rho'(0) < 0$

This occurs if

$$A_0 > \frac{\delta^{1-\alpha}\gamma^{\alpha}p_A^{1-\alpha-\beta}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{\alpha^{\alpha}\beta^{\beta}s^{\alpha}(\theta+\beta\gamma-\beta\theta)^{1-\beta}(\gamma-\theta)^{\beta}}$$

In this case it is possible to have two solutions to $\rho(\tau)=0$ if and only if $\rho(0)>0$, $\rho(\theta/\gamma)>0$ and $\tau_e \in (0,\theta/\gamma)$, with $\rho(\tau_e)<0$. However,

$$\left\{\begin{array}{l}\rho(\theta/\gamma)>0\\ \tau_e<\frac{\theta}{\gamma}\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}A_0 > \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}}{\beta^{1-\alpha}p_A^{\alpha+\beta-1}s^{\alpha}(\gamma-\theta)}\\ A_0 < \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{p_A^{\alpha+\beta-1}\alpha^{\alpha}\beta s^{\alpha}(\gamma-\theta)}\end{array}\right. \tag{34}$$

requires

$$\frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}}{\beta^{1-\alpha}p_A^{\alpha+\beta-1}s^{\alpha}(\gamma-\theta)} < \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{p_A^{\alpha+\beta-1}\alpha^{\alpha}\beta s^{\alpha}(\gamma-\theta)}$$

which, after some algebraic manipulations, provides $\frac{\alpha^{\alpha}\beta^{\alpha}}{(\alpha+\beta-1)^{\alpha}} < 1$ and hence $(1-\alpha)(1-\beta) < 0$, which is impossible.

The existence or not of a solution of $\rho(\tau)=0$ on $(0,\theta/\gamma)$ is then determined by the signs of $\rho(0)$ and $\rho(\theta/\gamma)$. A unique solution exists if

$$\left\{\begin{array}{l}\rho(0)>0\\ \rho(\theta/\gamma)<0\\ \rho'(0)<0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}A_0 > \frac{\delta^{1-\alpha}(n+\mu s)^{\alpha}(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{p_A^{\alpha+\beta-1}\beta^{\beta}s^{\alpha}(\gamma-\theta)^{\beta}}\\ A_0 < \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}}{\beta^{1-\alpha}p_A^{\alpha+\beta-1}s^{\alpha}(\gamma-\theta)}\\ A_0 > \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{\alpha^{\alpha}\beta^{\beta}s^{\alpha}(\gamma-\theta)^{\beta}p_A^{\alpha+\beta-1}(\theta+\beta\gamma-\beta\theta)^{1-\beta}}\end{array}\right. \Leftrightarrow \left\{\begin{array}{l}A_0 > \frac{\delta^{1-\alpha}(n+\mu s)^{\alpha}(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{p_A^{\alpha+\beta-1}\beta^{\beta}s^{\alpha}(\gamma-\theta)^{\beta}}\\ A_0 < \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}}{\beta^{1-\alpha}p_A^{\alpha+\beta-1}s^{\alpha}(\gamma-\theta)}\end{array}\right.$$

since

$$\frac{\delta^{1-\alpha}(n+\mu s)^{\alpha}(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{p_A^{\alpha+\beta-1}\beta^{\beta}s^{\alpha}(\gamma-\theta)^{\beta}} > \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{\alpha^{\alpha}\beta^{\beta}s^{\alpha}(\gamma-\theta)^{\beta}p_A^{\alpha+\beta-1}(\theta+\beta\gamma-\beta\theta)^{1-\beta}}$$

can be simplified as $(\theta+\beta\gamma-\beta\theta)^{\alpha}\alpha^{\alpha} > \gamma^{\alpha}(\alpha+\beta-1)^{\alpha}$, which is equivalent to $(1-\beta)(\gamma(1-\alpha)+\alpha\theta) > 0$ and hence it is true.

This gives (ii). The solution would be unique also if $\rho(0)<0$ and $\rho(\theta/\gamma)>0$, but this would happen only if $\tau_e<\theta/\gamma$, which is not possible. The only other possibility is that there are no zeros and $\rho$ is positive, which falls in case (iii).

Since the scenarios occurring for $\alpha+\beta-1>0$ overlap with those we already studied for $\alpha+\beta-1\le 0$, we can conclude. □

*Proof of Prop. 5 and Cor. 3.* In this proof we make use of the same function $\rho$ defined in the proof of Proposition 4, as well as of its derivatives and related properties. Also in this case we focus on $\tau>0$, as $\tau=0$ does not affect the results.

From (22), function $\rho$ is concave. Let us consider two cases.

**Case 1:** $\rho'(0)>0$.

It is possible to have two zeros on $(0,\theta/\gamma)$ if and only if $\rho(0)<0$, $\rho(\theta/\gamma)<0$, $0<\tau_e<\theta/\gamma$ and $\rho(\tau_e)>0$, for $\tau_e$ defined in (29). We prove that it is not possible to have $\tau_e<\theta/\gamma$ and $\rho(\theta/\gamma)<0$. Since $\beta>1$, $\tau_e<\theta/\gamma$ now results in the opposite inequality with respect to the latter one in (34), which combined with $\rho(\theta/\gamma)<0$ requires

$$\frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}(\alpha+\beta-1)^{\alpha}}{\alpha^{\alpha}\beta s^{\alpha}(\gamma-\theta)p_A^{\alpha+\beta-1}} < A_0 < \frac{\delta^{1-\alpha}\gamma^{\alpha}(n+\mu s)^{\alpha}}{\beta^{1-\alpha}s^{\alpha}(\gamma-\theta)p_A^{\alpha+\beta-1}} \tag{35}$$

and this implies $\alpha+\beta-1<\alpha\beta$, which is equivalent to $(\beta-1)(1-\alpha)<0$, and hence it is impossible.

It is possible to have only one zero on $(0,\theta/\gamma)$ if and only if $\rho(0)<0$ and $\rho(\theta/\gamma)>0$. Note that the case of $\rho(0)>0$ and $\rho(\theta/\gamma)<0$ cannot occur since we just showed that the latter condition would require that either $\tau_e\ge\theta/\gamma$ or it does not exist, hence $\rho$ would be increasing on $(0,\theta/\gamma)$, which could not provide $\rho(0)>0$ and $\rho(\theta/\gamma)<0$.

We have that $\rho(0) < 0$ implies $\rho'(0) > 0$ since

$$A_0 < \frac{\delta^{1-\alpha}(n+\mu s)^\alpha(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{p_A^{\alpha+\beta-1}\beta^\beta s^\alpha(\gamma-\theta)^\beta} < \frac{\delta^{1-\alpha}\gamma^\alpha(n+\mu s)^\alpha(\alpha+\beta-1)^\alpha}{\alpha^\alpha\beta^\beta s^\alpha(\gamma-\theta)^\beta p_A^{\alpha+\beta-1}(\theta+\beta\gamma-\beta\theta)^{1-\beta}}$$

and the rightmost inequality can be simplified into $(1-\beta)(\gamma(1-\alpha)+\alpha\theta) < 0$, which is true for $\beta > 1$.

This means that we have $\rho(0) < 0$ and $\rho(\theta/\gamma) > 0$ if and only if

$$\frac{\delta^{1-\alpha}\gamma^\alpha(n+\mu s)^\alpha}{\beta^{1-\alpha}s^\alpha(\gamma-\theta)p_A^{\alpha+\beta-1}} < A_0 < \frac{\delta^{1-\alpha}(n+\mu s)^\alpha(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{\beta^\beta s^\alpha(\gamma-\theta)^\beta p_A^{\alpha+\beta-1}}$$

which provides case (ii) of Proposition 5.

Note that we can rewrite the previous inequalities as

$$\begin{cases} \frac{\gamma^\alpha}{\gamma-\theta} < \frac{A_0\beta^{1-\alpha}s^\alpha p_A^{\alpha+\beta-1}}{\delta^{1-\alpha}(n+\mu s)^\alpha} \\ \frac{(\theta+\beta\gamma-\beta\theta)^{\alpha+\beta-1}}{(\gamma-\theta)^\beta} > \frac{A_0\beta^\beta s^\alpha p_A^{\alpha+\beta-1}}{\delta^{1-\alpha}(n+\mu s)^\alpha} \end{cases} \tag{36}$$

in which it is easy to check that the left hand sides are strictly decreasing functions, positively diverging as $\gamma \to \theta^+$ and vanishing as $\gamma \to +\infty$. As in the proof of Corollary 1, this allows introducing functions $\gamma_1$ and $\gamma_2$ implicitly defined by the corresponding equalities in (36), and this provides case (ii) of Corollary 3.

There are no solutions and $\rho$ is negative if $\rho(0)$ and $\rho(\theta/\gamma)$ are negative. Since $\rho(\theta/\gamma) < 0$, recalling (35), either $\tau_e \geq \theta/\gamma$ or it does not exist, function $\rho$ is increasing and it is negative only if $\rho(\theta/\gamma) < 0$.

Note that if $\rho(\theta/\gamma) < 0$ we have $\rho'(0) > 0$, since

$$A_0 < \frac{\delta^{1-\alpha}\gamma^\alpha(n+\mu s)^\alpha}{\beta^{1-\alpha}p_A^{\alpha+\beta-1}s^\alpha(\gamma-\theta)} < \frac{\delta^{1-\alpha}\gamma^\alpha(n+\mu s)^\alpha(\alpha+\beta-1)^\alpha}{\alpha^\alpha\beta^\beta s^\alpha(\gamma-\theta)^\beta p_A^{\alpha+\beta-1}(\theta+\beta\gamma-\beta\theta)^{1-\beta}}$$

and we have already shown in (33) that the rightmost inequality is equivalent to $(\alpha-1)(1-\beta) > 0$, which is true for $\beta > 1$.

Conditions $\rho(0) < 0$ and $\rho(\theta/\gamma) < 0$ provide case (i) of Proposition 5, from which we can obtain the corresponding case of Corollary 3.

Finally, there are no solutions and $\rho$ is positive if both $\rho(0)$ and $\rho(\theta/\gamma)$ are positive. This case corresponds to case (iii) of Proposition 5, from which we can obtain the corresponding case of Corollary 3.

**Case 2:** $\rho'(0) \leq 0$.

In this case $\rho$ is decreasing. There is one solution if $\rho(0) > 0$ and $\rho(\theta/\gamma) < 0$, but this is not possible because if $\rho(\theta/\gamma) < 0$ then $\rho'(0)$ must be positive. For the same reason it is not possible that $\rho$ is always negative. Finally, there are no zeros and $\rho$ is positive if $\rho(\theta/\gamma) > 0$ (in fact, $\rho$ is decreasing) and this corresponds to case (iii). □

*Proof of Cor. 2.* Let $\tau = 0$. If we replace the expression (12) of $A$ in (19) we can write

$$p^* = f^{-1}\left(\frac{\theta}{\delta}\left(\frac{s}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}}\right) \tag{37}$$

where $f$ is defined by the right hand side of the former identity in (19). From (10) and (13) we can write that the steady state is stable provided that

$$p^* < \frac{p_A\tilde{E}}{\beta+\tilde{E}}$$

Using (37) in the last inequality and recalling that $f$ is an increasing function we obtain

$$\frac{\theta}{\delta}\left(\frac{s}{n+s\mu}\right)^{\frac{\alpha}{1-\alpha}} < f\left(\frac{p_A\tilde{E}}{\beta+\tilde{E}}\right)$$

from which we have (14). □

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