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# GROUP SIZE AS SELECTION DEVICE

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ABSTRACT. In a coordination game with multiple Pareto ordered equilibria and population uncertainty, we show that group size helps select a unique equilibrium, for reasons reminiscent of the global games literature. A critical mass phenomenon emerges at equilibrium. Group size has an emboldening effect on participants.

KEY WORDS. POISSON GAMES; COORDINATION GAMES; EQUILIBRIUM SELECTION; GLOBAL GAMES

JEL CLASSIFICATION. C72, D82.

## 1. INTRODUCTION

Imagine you have accepted an invitation to a party, but it's unclear whether it's a costume party or not. A dilemma emerges: showing up in a costume will be fun if other guests dress up, but awkward otherwise. There is a safe but boring choice — i.e. dress mundanely— and a potentially fun but risky choice — i.e. dress up. Among the guests who may join the party, some can't wait to dress up, while the rest would be willing to dress up if enough others do but not otherwise. Nobody, except the host, knows exactly how many people have been invited to the party.

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This situation can be represented formally as a coordination game with population uncertainty.[1] Following Myerson (1998), we model the uncertainty over the actual number of guests assuming that it is drawn from a Poisson distribution with a known expected number of guests. The resulting game has a unique equilibrium when the party is expected to be very small, with all the strategic guests dressed mundanely, or very large, with all the guests dressed up, but has multiple equilibria for intermediate party sizes.

It turns out that, if also the expected number of guests is unknown, but each guest receives a slightly imprecise message from a friend of the host about how large the party is expected to be, a unique equilibrium of the underlying game can be selected, with an argument analogous to the one used in the global games literature à la Carlsson and Van Damme (1993a) and Morris and Shin (1998).[2]

Given that the expected size of the party is not common knowledge among the guests, the possibility that the party may be either small or large exerts an influence on ordinary situations through the hierarchy of mutual beliefs, helping select a unique equilibrium for any party size. Which equilibrium is selected depends on the expected size of the party. All the guests show up in costumes when the party is sufficiently large, as the presence of a few costume enthusiasts exerts a pull on the undecided guests, but not otherwise.

The social dilemma gives rise to a critical mass phenomenon of the type discussed by Schelling (1978). Critical mass phenomena have been examined in the theoretical literature under the rubric of regime change, using the terminology of Angeletos et al (2006, 2007), as coordination games in which the status quo is abandoned if enough players take action against it. As in global games, a unique

[1] Using a terminology that goes back to the XVIII century Swiss-French philosopher, Jean Jacques Rousseau (Binmore, 1994, p. 120), this type of coordination game is known as a stag hunt, see Aumann (1990).

[2] Originally, the result was proved in a two players stag hunt game. Carlsson and Van Damme (1993b) have extended it to a stag hunt game with a finite number of players. Subsequently, the result has been extended further to a continuum of players and multiple actions with strategic complementarity. See Morris and Shin (2003) for a comprehensive survey of the literature on global games.

equilibrium can be selected when the strength of the status quo is not common knowledge among the players.

In our model, it is the (actual and expected) number of participants in the game that is not common knowledge. Hence, the selection device is the overall size of the group rather than the strength of the status quo. There is well documented experimental evidence that group size matters for equilibrium selection in coordination games with multiple Pareto ordered equilibria, e.g. Van Huyck et al (1990) and Weber (2006) for experiments on the minimum effort game à la Bryant (1983) and Arifovic et al (2023) on bank runs à la Diamond and Dybvig (1983).

Below, we recast some critical mass phenomena, such as political protests of the type examined in Atkeson (2001) and Edmond (2013), as stag hunt Poisson games, finding an emboldening effect of group size. We apply the framework also to bank runs, finding that the undecided depositors withdraw their resources if the expected number of participants is sufficiently large but not otherwise.

Poisson games, whose main properties we summarize in the appendix, have been applied to model scenarios with large but finite populations, such as general elections, e.g. Myerson (2000, 2002). Battaglini (2017) contains an application to political protests.[3] In a Poisson game with strategic complementarity, Makris (2008) has shown that the equilibrium is unique for sufficiently small population. Here, instead, we show that, even when the underlying stag hunt Poisson game has multiple equilibria, a small amount of incomplete information about the expected number of players helps select a unique equilibrium for any population size.

The paper proceeds as follows. In Section 2, we present the model. In Section 3, we find the equilibria of the underlying stag hunt Poisson game. In Section 4, the selection procedure is carried out. The applications are in Section 5. Section 6 concludes. Appendix A contains a summary of Poisson games and their main properties. Appendix B contains the proof of a key Lemma.

[3] Economic applications include Makris (2009) for public good provision; Ritzberger (2009) and De Sinopoli et al (2023) for, respectively, Bertrand and Cournot competition; Lauermann and Speit (2023) for auctions.

## 2. The model

Consider a binary action Poisson game $\Gamma$. The number of agents is a Poisson random variable with parameter $n \in \mathbb{R}_+$, that we call *population size*. Given $n$, the actual number of agents is $m$ with probability

$$P(m \mid n) = \frac{e^{-n} n^m}{m!}.$$

The interaction resembles the classical "stag hunt" game. Players can choose between a safe action $S$ – corresponding to hunting the hare – which yields a certain payoff irrespective of what opponents do, and a risky action $R$ – corresponding to hunting the stag – which yields a payoff that increases in the number of agents choosing it.

Agents can be of three types. Agents of type 1 are the actual players of the game with action set $A = \{S, R\}$.[4] Agents of type 2 are automata who always play the risky action, while agents of type 3 are automata who always play the safe action. A randomly sampled agent is an actual player with probability $p$, a type 2 automaton with probability $r$, and a type 3 automaton with probability $s = 1 - p - r$.

Let $x$ be the total number of agents choosing action $R$.[5] The payoff of choosing $S$ is independent of $x$, while the payoff of choosing $R$ increases in steps in $x$, with finitely many steps. Let $k = (k_1, ..., k_I)$, where $k_i \in \mathbb{Z}_+$ for $i = 1, \ldots, I$. Players' payoffs are normalized to

$$u(S, x) = v \qquad \text{for every } x,$$

with $v \in (0,1)$, and

$$\begin{aligned} u(R,x) &= 0 && \text{if } x \le k_1, \\ u(R,x) &= u_i && \text{if } k_i < x \le k_{i+1} \quad \text{for } i = 1, \ldots, I-1, \\ u(R,x) &= 1 && \text{if } x > k_I, \end{aligned}$$

with $u_i \in [0,1]$ for $i = 1, \ldots, I-1$ and $u_{i'} \ge u_i$ for $i' > i$. We let $u_0 = 0$ and $u_I = 1$.

[4] We will often refer to type 1 players simply as *players*.

[5] In the description of Poisson games in Appendix A, $x$ denotes the entire action profile. We slightly abuse notation since only the number of agents choosing $R$ is relevant for the analysis.

Population uncertainty implies that all (type 1) players choose the same strategy $\sigma \in \Delta(A)$. A strategy $\sigma$ induces the population average behavior $\tau(\sigma) \in \Delta(A)$ which is defined by

$$\tau(\sigma)(S) = p\sigma(S) + s,$$

$$\tau(\sigma)(R) = p\sigma(R) + r = 1 - \tau(\sigma)(S).$$

When players play according to $\sigma$, $\tau(\sigma)(a)$ is the probability that a randomly sampled agent chooses action $a$. We will often avoid to specify the dependence of $\tau$ on $\sigma$, and we let $\tau_R = \tau(\sigma)(R)$. The probability that exactly $x$ agents choose the risky action is given by

$$P(x \mid n\tau_R) = e^{-n\tau_R} \frac{(n\tau_R)^x}{x!},$$

and the players' expected payoffs are given by

$$U(S, \tau \mid n) = v,$$

$$U(R, \tau \mid n) = \sum_{x=0}^{\infty} P(x \mid n\tau_R) u(R, x).$$

We recall the standard concepts of dominated strategy and Nash equilibrium.

**Definition 1.** Strategy $\sigma$ is *dominated* by strategy $\sigma'$ if $U(\sigma, \tau \mid n) \leq U(\sigma', \tau \mid n)$ for every $\tau \in \Delta(A)$ and $U(\sigma, \tau' \mid n) < U(\sigma', \tau' \mid n)$ for some $\tau' \in \Delta(A)$.

**Definition 2.** Strategy $\sigma$ is a *Nash equilibrium* of the Poisson game $\Gamma$ if $U(\sigma, \tau(\sigma) \mid n) \geq U(\sigma', \tau(\sigma) \mid n)$ for every $\sigma' \in \Delta(A)$.

We say that the average behavior $\tau$ is an *equilibrium behavior* (or, simply, an *equilibrium*) if it is induced by a Nash equilibrium.

## 3. Equilibria

The space of population sizes $\mathbb{R}_+$ can be partitioned into three intervals according to the equilibrium outcome induced in the corresponding Poisson game $\Gamma(n)$.

Let the Poisson cumulative distribution function be given by

$$F(\bar{m} \mid n) = \sum_{m=0}^{\bar{m}} P(m \mid n)$$

for every $\bar{m} \in \mathbb{Z}_+$. The expected payoff of choosing $R$ can be written as

$$U(R,\tau \mid n) = \sum_{i=1}^{I-1} \left[F(k_{i+1} \mid n\tau_R) - F(k_i \mid n\tau_R)\right]u_i + 1 - F(k_I \mid n\tau_R)$$

$$= 1 - \sum_{i=1}^{I} F(k_i \mid n\tau_R)(u_i - u_{i-1}).$$

For every $i = 1,\dots,I$, the function $F(k_i \mid n\tau_R)$ is continuous and strictly decreasing in $n\tau_R$.[6] Moreover, $F(0 \mid 0) = 1$ and $\lim_{n\to\infty} F(k_i \mid n) = 0$.

Let $\underline{n}$ be the value of the population size such that

$$1 - \sum_{i=1}^{I} F(k_i \mid \underline{n}(p+r))(u_i - u_{i-1}) = v,$$

that is, type 1 players are indifferent between $R$ and $S$ when $\tau_R = p + r$. For every $n < \underline{n}$, the players always prefer the safe action to the risky one, even when all the opponents are playing $R$. The game $\Gamma(n)$ has therefore a unique equilibrium, in which all type 1 players choose the safe action, i.e. $\tau_R = r$. If $n = \underline{n}$, type 1 players are indifferent between the safe and the risky actions when all opponents are playing $R$, while they prefer the safe action to the risky one for all other behaviors. Thus, the game $\Gamma(\underline{n})$ has two equilibria, one with $\tau_R = r$ and one with $\tau_R = p + r$. Note that the first equilibrium is strict, while the second equilibrium is dominated.

Now, let $\overline{n}$ be the value of the population size such that

$$1 - \sum_{i=1}^{I} F(k_i \mid \overline{n}r)(u_i - u_{i-1}) = v,$$

that is, type 1 players are indifferent between $R$ and $S$ when $\tau_R = r$. Note that we have $\overline{n} = \underline{n}(p+r)/r$. For every $n > \overline{n}$, the players always prefer the risky action to the safe one, even when no opponent is playing $R$. The game $\Gamma(n)$ has therefore a unique equilibrium, in which all type 1 players choose the risky action, i.e. $\tau_R = p + r$.[7] If $n = \overline{n}$, type 1 players are indifferent between the safe and the risky actions when no other type 1 player is playing $R$, while they prefer the risky action to the safe one for all other behaviors. Thus, $\Gamma(\overline{n})$ has two

[6] Given $n' > n$, the Poisson distribution with parameter $n'$ first order stochastically dominates the one with parameter $n$.

[7] We need $r > 0$ to guarantee the existence of this dominance solvable region of population sizes where every player chooses the risky action with probability 1, while $s$ could be nil.

equilibria, a dominated equilibrium with $\tau_R = r$ and a strict equilibrium with $\tau_R = p + r$.

Consider now $n \in (\underline{n}, \overline{n})$. Since

$$1 - \sum_{i=1}^{I} F(k_i \mid n(p+r))(u_i - u_{i-1}) > v$$

and

$$1 - \sum_{i=1}^{I} F(k_i \mid nr)(u_i - u_{i-1}) < v,$$

the game $\Gamma(n)$ has three equilibria. There are two strict equilibria with $\tau_R = r$ and $\tau_R = p+r$, in which type 1 players choose, respectively, the safe and the risky action with probability 1. In addition, there is a mixed strategy equilibrium with $\tau_R = \tau_R^*$ such that

$$1 - \sum_{i=1}^{I} F(k_i \mid n\tau_R^*)(u_i - u_{i-1}) = v,$$

where type 1 players choose $R$ with probability $(\tau_R^* - r)/p$. Note that, given $n$, we have

$$\tau_R^* = \frac{\overline{n}}{n} r = \frac{\underline{n}}{n}(p+r).$$

The multiplicity of equilibria in the region $(\underline{n}, \overline{n})$ cannot be addressed using standard strategic stability principles, as all the three equilibrium points are stable sets as defined in De Sinopoli et al (2014). In broad terms, a stable set of a Poisson game is a minimal subset of Nash equilibria such that every close-by game obtained through perturbations of the average behavior has a Nash equilibrium close to the stable set. As in standard games, also in Poisson games strict equilibria are robust to any possible perturbation. For the equilibrium in mixed strategy, any perturbation of the average behavior in the definition of stable set can be compensated by the players' mixed action.

In Section 4 we will see that the equilibrium selection problem can be solved by introducing some uncertainty about the parameter $n$.

### 3.1. *Example: the party dilemma*

Consider the party dilemma illustrated in the Introduction. Although, strictly speaking, only the costume enthusiasts are necessary for our result, for the sake of symmetry we will assume that there are both people who hate and people who love costume parties. Suppose the probabilities of the different personalities are $p = 5/7$ and $r = s = 1/7$. Let the payoff of dressing mundanely be equal to 0.6, and the payoff of wearing a costume be equal to 0 in case no other guest does so and to 1 if at least another guest does. If all the other guests except those who hate costume parties show up wearing costumes, the payoff of doing so is given by

$$1 - P\left(0 \,\middle|\, \frac{6}{7}n\right),$$

which is the probability that at least another guest wears a costume. The above expression is equal to 0.6 for $\underline{n} \approx 1.069$. Thus, whenever the expected number of invitees is smaller than 1.069, an undecided guest will always be better off dressing up safely, as the probability of feeling misfit wearing a costume even in the most favorable event is too high. On the other hand, in the least favorable event in which only people who love to wear costumes do so, the payoff of wearing a costume is

$$1 - P\left(0 \,\middle|\, \frac{1}{7}n\right).$$

This is equal to 0.6 for $\overline{n} \approx 6.414$. So, whenever the expected number of invitees is larger than 6.414, an undecided guest will always be better off dressing up in a costume, as the probability of having fun at the party is always high enough. For intermediate party sizes, the outcome of the party is undetermined. There is an equilibrium where all the guests in a dilemma dress normally, since

$$1 - P\left(0 \,\middle|\, \frac{1}{7}n\right) < 0.6,$$

an equilibrium where they all wear costumes, since

$$1 - P\left(0 \,\middle|\, \frac{6}{7}n\right) > 0.6,$$

and an equilibrium where they choose to wear a costume with probability $\sigma_R = \frac{\overline{n}-n}{2n}$, since

$$1 - P\left(0 \,\middle|\, \left(\frac{1}{7} + \frac{5}{7}\sigma_R\right)n\right) = 0.6.$$

At the end of the next section, it will be possible to determine a unique equilibrium outcome for every given party size.

## 4. Equilibrium Selection

### 4.1. *Uncertainty in the population size*

Assume that all the parameters of the Poisson game $\Gamma$ are common knowledge except the population size $n$, which is observed by the players only with some slight noise. The resulting incomplete information game is denoted $\tilde{\Gamma}$ and can be described as follows:

1. Nature selects the population size $n$ according to the uniform density over the interval $[n', n'']$, where $n' < \underline{n}$ and $n'' > \overline{n}$.
2. When the population size is $n$, each player observes a signal that is drawn uniformly from the interval $[n-\varepsilon, n+\varepsilon]$ for some small $\varepsilon > 0$.[8] Conditional on $n$, the signals are identical and independent across players.
3. Based on the observed signal, the players choose between the safe and the risky action. Agents of types 2 and 3 choose, respectively, actions $R$ and $S$ with probability 1 independently of their observation.
4. Payoffs are determined by $\Gamma$ and the players' choices.

A strategy for players in the game $\tilde{\Gamma}$ is a measurable function $\tilde{\sigma} : [n', n''] \to \Delta(A)$ that assigns a mixed action to each observation. Given the strategy function $\tilde{\sigma}$, we denote with $\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}, n)$ the probability for a player with signal $\tilde{n}$ that the number of other agents choosing $R$ is $x$, when the population size is $n$.[9] A player's posterior of $n$ if he observes signal $\tilde{n} \in [n' + \varepsilon, n'' - \varepsilon]$ will be uniform on $[\tilde{n} - \varepsilon, \tilde{n} + \varepsilon]$.[10] The player's conditional expected payoffs when the other players play according to $\tilde{\sigma}$ are given by

$$\tilde{U}_{\tilde{n}}(S, \tilde{\sigma}) = v,$$
$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}) = \frac{1}{2\varepsilon} \int_{\tilde{n}-\varepsilon}^{\tilde{n}+\varepsilon} \sum_{x=0}^{\infty} \tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}, n) u(R, x) \mathrm{d}n.$$

[8] We need $2\varepsilon < \min\{\underline{n} - n', n'' - \overline{n}\}$.

[9] Differently from before, the number of agents who choose a given action is not necessarily a Poisson random variable. In fact, we lose the independent actions property that is specific of Poisson games, as the numbers of agents who choose each action are not mutually independent for every strategy function.

[10] The endpoints of such interval must be appropriately adjusted if $\tilde{n} < n' + \varepsilon$ or $\tilde{n} > n'' - \varepsilon$.

**Definition 3.** The strategy function $\tilde{\sigma}$ is a *Nash equilibrium* of $\tilde{\Gamma}$ if $\tilde{U}_{\tilde{n}}(\tilde{\sigma}(\tilde{n}),\tilde{\sigma}) \geq \tilde{U}_{\tilde{n}}(\sigma',\tilde{\sigma})$ for every $\sigma' \in \Delta(A)$ and $\tilde{n} \in [n', n'']$.

We can show that the game $\tilde{\Gamma}$ has a unique equilibrium outcome.

### 4.2. *Unique equilibrium*

Suppose a player observes a signal that is sufficiently small, i.e. $\tilde{n} < \underline{n} - \varepsilon$. Then, the player's conditional payoff of choosing $R$ is always lower than $v$, as it is lower for the strategy that prescribes players to choose $R$ for every possible observation.[11] In fact, given such a strategy, we have $\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}, n) = P(x \mid n(p+r))$ for every $x$, since every realized player chooses $R$ with probability 1, and $1 - \sum_{i=1}^{I} F(k_i \mid n(p+r))(u_i - u_{i-1}) < v$ for every $n \in [\tilde{n}-\varepsilon, \tilde{n}+\varepsilon]$. It follows that $S$ is conditionally strictly dominant at $\tilde{n}$. On the other hand, suppose a player observes a signal that is sufficiently large, i.e. $\tilde{n} > \overline{n} + \varepsilon$. Then action $R$ is strictly dominant at that observation, as the payoff of choosing $R$ is higher than $v$ for every $n$ in the region. Now, consider a strategy that prescribes players to play $S$ for every observation below a given signal and to play $R$ for every observation above it. We can show that the payoff of choosing $R$ for a player at the margin of switching from $S$ to $R$ increases strictly and continuously in the cutoff signal. Since that payoff is lower than $v$ for sufficiently small signals and higher than $v$ for sufficiently large ones, this implies that there is a unique cutoff signal such that a marginal player is indifferent between the two actions. We can show that this signal characterizes the equilibrium of the game $\tilde{\Gamma}$.

Given the signal $\tilde{n}$, let $\tilde{\sigma}_{\tilde{n}}$ be the strategy function that prescribes players to choose action $S$ for all observations smaller than $\tilde{n}$ and action $R$ for all observations larger than $\tilde{n}$. We call $\tilde{\sigma}_{\tilde{n}}$ the *cutpoint strategy* at $\tilde{n}$, and $\tilde{n}$ the *cutpoint*.

**Lemma 1.** *$\tilde{U}_{\tilde{n}}(R,\tilde{\sigma}_{\tilde{n}})$ is continuous and strictly increasing in $\tilde{n}$.*

The formal proof of this result is in Appendix B. The intuition is the following. For a player with marginal signal equal to the cutpoint, the probability

[11] It is clear that the probability that, given a strategy function, the number of agents choosing $R$ is above a given threshold increases in the set of signals for which the strategy function prescribes action $R$.

that exactly $x$ other players choose $R$ is equal to the probability that exactly $x$ other players have made larger observations. Uniform priors imply that, for any realization of the number of other players, the probability that $x$ of them have made larger observations depends only *on the difference* between the player's signal and the true population size, independently of the signal. As the signal increases, the values that the player assigns to the true population size become larger. Consequently, for any given threshold, the probability that the population realization is above that threshold increases and, therefore, the probability for the marginal player that the number of opponents with larger observations is above that threshold also increases. Moreover, the probability that the realization of type 2 agents is above the given threshold increases as well. It follows that, the higher is the cutoff signal on the population size, the higher is the probability for the player with that signal that the total number of opponents choosing $R$ is above any given threshold and, hence, the higher is his expected payoff of choosing $R$.

We can use the above result to prove that the equilibrium outcome is unique. In fact, Lemma 1 implies that there exists essentially a unique equilibrium where players play according to a cutpoint strategy. We can show that there exists no equilibrium where players play according to a different strategy function.

**Theorem 1.** *There exists a unique $n^*$ such that, in every equilibrium of $\tilde{\Gamma}$, a player with signal $\tilde{n}$ plays action $S$ if $\tilde{n} < n^*$ and action $R$ if $\tilde{n} > n^*$.*

*Proof.* Let $\tilde{\sigma}_a$ be the strategy function that prescribes to choose action $a \in A$ with probability 1 for every observed signal. For every $\tilde{n} < \underline{n} - \varepsilon$ we have

$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) \leq \tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_R) < v.$$

On the other hand, for every $\tilde{n} > \overline{n} + \varepsilon$ we have

$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) \geq \tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_S) > v.$$

Lemma 1 implies that there exists a unique value $n^*$ such that

$$\tilde{U}_{n^*}(R, \tilde{\sigma}_{n^*}) = v.$$

Note that for every $\tilde{n} < n^*$ we have

$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{n^*}) \leq \tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) < v,$$

while for every $\tilde{n} > n^*$ we have

$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{n^*}) \geq \tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) > v,$$

so $\tilde{\sigma}_{n^*}$ is an equilibrium of $\tilde{\Gamma}$.

Suppose there exists another equilibrium $\tilde{\sigma}$ that is not a cutpoint strategy. Let $\tilde{n}_1$ be the largest signal such that $\tilde{\sigma}(\tilde{n})(S) = 1$ for all $\tilde{n} < \tilde{n}_1$ and let $\tilde{n}_2$ be the smallest signal such that $\tilde{\sigma}(\tilde{n})(R) = 1$ for all $\tilde{n} > \tilde{n}_2$. Clearly, $\tilde{n}_1 < \tilde{n}_2$. By continuity of the payoff function in the signal, we have

$$\tilde{U}_{\tilde{n}_1}(R, \tilde{\sigma}) = \tilde{U}_{\tilde{n}_2}(R, \tilde{\sigma}) = v.$$

Note that

$$\tilde{U}_{\tilde{n}_1}(R, \tilde{\sigma}) \leq \tilde{U}_{\tilde{n}_1}(R, \tilde{\sigma}_{\tilde{n}_1})$$

and

$$\tilde{U}_{\tilde{n}_2}(R, \tilde{\sigma}) \geq \tilde{U}_{\tilde{n}_2}(R, \tilde{\sigma}_{\tilde{n}_2}).$$

But, by Lemma 1, we have

$$\tilde{U}_{\tilde{n}_1}(R, \tilde{\sigma}_{\tilde{n}_1}) < \tilde{U}_{\tilde{n}_2}(R, \tilde{\sigma}_{\tilde{n}_2}),$$

which leads to a contradiction. □

*Remark* 1. In the stag hunt model we have considered, the payoff associated to the safe action is fixed. Our selection result can be extended to a different payoff function as long as two main conditions hold. First, the payoff gain of choosing the risky action rather than the safe action

$$\Pi(\tau \mid n) = U(R, \tau \mid n) - U(S, \tau \mid n)$$

must be continuous and strictly increasing in $n\tau_R$. Second, the initial class of games must be large enough to contain games with different equilibrium structures, that is, there must exist $\underline{n}$ and $\overline{n}$ such that, for every $\tau$,

$$\Pi(\tau \mid n) < 0 \qquad \text{for every } n < \underline{n}$$

and

$$\Pi(\tau \mid n) > 0 \qquad \text{for every } n > \overline{n}.$$

In particular, the payoff of choosing the safe action may depend on the number of agents choosing the risky action $x$, provided that

(1) $u(S,0) > u(R,0)$,
(2) $u(R,x) > u(S,x)$ for all $x \geq \bar{x} > 0$, and
(3) $u(R,x) - u(S,x)$ is weakly increasing in $x$.

Given that $u(R,x)$ increases with $x$, this last assumption is clearly satisfied whenever the payoff of choosing the safe action decreases with the number of agents choosing the risky action.[12] However, the payoff of choosing the safe action may also increase with $x$, as long as condition (3) holds.

*Remark* 2. Our selection result can be extended also to the case in which the payoff yielded by the risky action decreases with the number of agents choosing the safe action.[13] Formally, let $x$ denote now the number of agents choosing action $S$. The payoffs of type 1 players are given by

$$u(S,x) = v \qquad \text{for every } x,$$

with $v \in (0,1)$, and

$$\begin{aligned} u(R,x) &= 1 && \text{if } x \leq k_1, \\ u(R,x) &= u_i && \text{if } k_i < x \leq k_{i+1} \quad \text{for } i = 1,\ldots,I-1, \\ u(R,x) &= 0 && \text{if } x > k_I, \end{aligned}$$

with $u_i \in [0,1]$ for $i = 1,\ldots,I-1$, and $u_{i'} \leq u_i$ for $i' > i$. As before, in the remote regions of population sizes, players have a strictly dominant strategy and,

[12] This is analogous to Kim (1996), where, precisely, the payoff of choosing the safe action increases with the number of players choosing it. In our model with population uncertainty, that precise assumption would not guarantee the existence of the dominance solvable regions of population sizes.

[13] As we show later in the paper, this setting fits a bank run scenario à la Diamond and Dybvig (1983).

hence, the corresponding Poisson games have a unique Nash equilibrium. Differently from before, for sufficiently small values of $n$ the equilibrium is such that every player chooses action $R$, while for sufficiently large values of $n$ every player chooses action $S$.[14] For values of $n$ in the intermediate region, the Poisson games still have multiple equilibria, precisely, the two pure strategy equilibria in which the players choose each action with probability 1 and the equilibrium in which the players randomize. As above, we can solve the equilibrium selection problem adding vanishing noise about $n$ through the (properly modified) incomplete information game $\tilde{\Gamma}$. In particular, the payoffs of a player with signal $\tilde{n}$ are given by

$$\tilde{U}_{\tilde{n}}(S,\tilde{\sigma}) = v,$$
$$\tilde{U}_{\tilde{n}}(R,\tilde{\sigma}) = \frac{1}{2\varepsilon}\int_{\tilde{n}-\varepsilon}^{\tilde{n}+\varepsilon}\sum_{x=0}^{k_I}\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma},n)u(R,x)dn,$$

where $\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma},n)$ is now the probability that the number of opponents choosing $S$ is $x$, given the strategy function $\tilde{\sigma}$ and the population size $n$. Let $\tilde{\sigma}_{\tilde{n}}$ be the strategy function that prescribes to choose action $R$ for all observations smaller than $\tilde{n}$ and action $S$ for all observations larger than $\tilde{n}$. The proof of Lemma 1 implies that $\tilde{U}_{\tilde{n}}(R,\tilde{\sigma}_{\tilde{n}})$ is continuous and strictly *decreasing* in the cutpoint signal $\tilde{n}$. Then, a proof analogous to that of Theorem 1 leads to the following result.

*Corollary* 1. *There exists a unique $n^*$ such that, in every equilibrium of $\tilde{\Gamma}$, a player with signal $\tilde{n}$ plays action $R$ if $\tilde{n} < n^*$ and action $S$ if $\tilde{n} > n^*$.*

### 4.3. *Limiting behavior*

In the limit, as the noise becomes negligible, the equilibrium strategy can be characterized in a fairly simple way.

Fix the realization $y$ of the number of other players. The fact that the prior distribution of $n$ is uniform implies that a signal $\tilde{n}$ gives to a player no additional information on his ranking within the population of signals. Thus, the

[14] In this case we need $s > 0$ to guarantee the existence of the dominance solvable region of population sizes where every player chooses the safe action with probability 1.

player believes that the number of opponents with larger signals is distributed uniformly on the interval $[0, y]$. If the player expects the opponents to follow the cutpoint strategy at $\tilde{n}$, he will assign probability $\frac{1}{y+1}$ to the event that exactly $z$ opponents choose the risky action, for every $z \leq y$. This is true for every $\varepsilon$, and so it remains valid for observation errors that are infinitesimally small. As $\varepsilon$ tends to zero, the players' observations become perfectly correlated and, in the limit, they coincide with the actual value of $n$. The probability for a player that the realized number of type 1 opponents is equal to $y$ tends to $P(y \mid np)$, while the probability that the realized number of type 2 agents is equal to a given value $l$ tends to $P(l \mid nr)$. It follows that, in the limit, the probability for the marginal player that the total number of opponents choosing the risky action is equal to $x$ is approximated by

$$P_n(x \mid \tilde{\sigma}_n) = \sum_{z=0}^{x} \sum_{y=z}^{\infty} \frac{1}{y+1} P(y \mid np) P(x-z \mid nr).$$

The player's utility of choosing $R$ is given by

$$U_n(R, \tilde{\sigma}_n) = 1 - \sum_{i=1}^{I} F_n(k_i \mid \tilde{\sigma}_n)(u_i - u_{i-1}),$$

where

$$F_n(k_i \mid \tilde{\sigma}_n) = \sum_{x=0}^{k_i} P_n(x \mid \tilde{\sigma}_n).$$

Thus, the limit value of $n^*$ as $\varepsilon$ tends to zero solves the indifference condition

$$U_{n^*}(R, \tilde{\sigma}_{n^*}) = v,$$

that is,

$$\sum_{i=1}^{I} F_{n^*}(k_i \mid \tilde{\sigma}_{n^*})(u_i - u_{i-1}) = 1 - v. \tag{4.1}$$

Introducing vanishing uncertainty about the population size $n$ offers an equilibrium selection criterion for the stag hunt Poisson games $\Gamma(n)$ in the indeterminacy region $(\underline{n}, \overline{n})$. The criterion selects the equilibrium in which every player chooses the safe action if $n < n^*$ and the equilibrium in which every player chooses the risky action if $n > n^*$, where $n^*$ is implicitly given by Equation (4.1).

### 4.4. *Example (cont.): two's a party, three's a crowd*

We can now solve the indeterminacy problem of the party dilemma. Recall that the indeterminacy region goes from $\underline{n} \approx 1.069$ to $\overline{n} \approx 6.414$. Suppose that every party guest receives some noisy information about the expected party size $n$, as in the model described above. As the noise vanishes, the payoff of wearing a costume for a guest who expects all the invitees with higher signals to do so and all those with lower signals to dress normally is approximated by

$$1 - P\left(0 \,\middle|\, \frac{1}{7}n\right) \sum_{x=0}^{\infty} P\left(x \,\middle|\, \frac{5}{7}n\right) \frac{1}{x+1},$$

which is the probability that at least one other guest will show up wearing a costume in that event. The limit value of $n$ for which the guest is indifferent between dressing safely and wearing a costume solves

$$1 - P\left(0 \,\middle|\, \frac{1}{7}n\right) \sum_{x=0}^{\infty} P\left(x \,\middle|\, \frac{5}{7}n\right) \frac{1}{x+1} = 0.6$$

and is given by $n^* \approx 2$, which determines the outcome of the party. All the guests in a dilemma will dress normally if the expected number of invitees is at most 2, while they will all show up in a costume for larger party sizes.

The main freebie of a theory of equilibrium selection is the possibility to perform comparative-statics exercises. A natural exercise in our setting is to ask what are the effects on the equilibrium of variations of the safe payoff $v$, the probability $r$ of having agents who single-mindedly take the risky action, and the thresholds $k_i$.

First, suppose that $v$ rises to 0.7. Not surprisingly, the group size threshold increases to $n^* \approx 2.711$, i.e. the region in which all the guests in a dilemma will dress normally expands as the safe action becomes more attractive. This holds in general, as $U_{n^*}(R, \tilde{\sigma}_{n^*})$ increases in $v$.

Suppose, next, that $r$ rises to 2/7, while $p$ falls to 4/7. As expected, the threshold falls to $n^* \approx 1.669$, i.e. the region in which all the guests dress up expands when more costume enthusiasts are likely to show up. In fact, $U_{n^*}(R, \tilde{\sigma}_{n^*})$ increases in $r$.

Finally, suppose that having fun in a costume requires at least two other people dressed up instead of one. In this case, the threshold increases to $n^* \approx 4.437$, i.e. the region in which all the guests dress up shrinks. This is intuitive, as $U_{n^*}(R, \tilde{\sigma}_{n^*})$ decreases in the minimum number of dressed up guests that guarantees fun, everything else equal.

Overall, if dressing mundanely becomes less appealing or dressing up more appealing or having some costume enthusiasts becomes more likely, the guests will show up in costumes for relatively smaller expected party sizes.

*Remark* 3. Our result suggests a key role for communicating the information about the expected party size when this takes intermediate values. Suppose the host loves costume parties, but does not want to force her guests one way or another. The host could choose to convey information to the guests about the expected size of the party to influence indirectly their behavior.

When there is no noise, the outcome of the party is indeterminate, as the game has multiple equilibria for intermediate party sizes. Suppose that the outcome is selected as a focal point following the realization of a publicly observable random variable that is unrelated to the fundamentals, i.e. a sunspot event. For instance, the guests will wear costumes if and only if there is a clearly visible full moon on the night of the party.

With (vanishing) noise, the outcome is determinate and the guests will dress up if the party is expected to be larger than $n^* \approx 2$, but not otherwise. Given uniform priors, the former event occurs with probability $\pi^* = \frac{\overline{n}-n^*}{\overline{n}-\underline{n}} \approx 0.83$, while the latter with the complementary probability. Then, if the probability that the party will take place with the full moon is larger than $\pi^*$, the party host may decide to reveal the expected party size publicly. Otherwise, she may decide to communicate privately with the participants.

By revealing publicly the information about the expected party size, the host can create common knowledge among the guests, thus, destroying the selection procedure. This may be in the interest of the host if the expected outcome of the interaction as selected by the public focal point mechanism is more likely to cater to her preferences than the alternative selection based on the absence of common knowledge. Otherwise, making the information about the expected party size fuzzier will help the host achieve her preferred outcome.

## 5. Applications

Finally, we present some applications of our theory to economic and socio-political situations in which the assumption of population uncertainty seems natural.

### 5.1. *Brain drain*

After obtaining their university degree, some graduates are deciding whether to move out or stay in their country. Leaving the country entails a positive value net of moving costs, $v \in (0,1)$, independent of how many other graduates decide to leave the country, while staying gives a benefit that is increasing in the number of other graduates who decide to stay. This may reflect a positive externality generated by human capital as argued in the endogenous growth literature à la Romer (1986), where the growth of income per capita of the country is increasing in the number of graduates operating in the country. Nobody knows exactly how many graduates are deciding to stay or leave. The number is drawn from a Poisson distribution. Some people will move out of the country in any case and others will stay no matter what others will do, but a fraction of the graduates of the country will decide strategically what to do depending on the behavior of their peers. If, except those who would leave anyway, everybody else stays in the country, the benefit of staying is unitary, while if nobody stays, except a few sedentary die-hards, the benefit is nil. The expected number of graduates is not common knowledge among the players who receive slightly disturbed signals about its value. Our selection argument and, hence, the uniqueness result applies directly to this setting. The unique equilibrium features the critical mass property, whereby the strategic players decide to stay if the number of graduates is expected to be sufficiently large but not otherwise.

### 5.2. *Political protest*

A political demonstration has been organised to oppose an authoritarian government, as discussed in Atkeson (2001). The demonstration may turn into a full-fledged protest against the government or not. The Poisson game representing this situation has the following features. The demonstrators decide whether to actively protest against the government or not. If a person does not join the protest, gets nothing. If the person joins the protest and the government backs down, enjoys the benefit at the cost of getting hurt during the protest, with payoff $1-v \in (0,1)$, otherwise gets hurt without any benefit, with payoff $-v$. The protest succeeds and the government backs down if enough people join the protest, otherwise the measure is implemented. Nobody knows how many citizens will show up at the demonstration. The number is drawn from a Poisson distribution. The expected number of demonstrators is not common knowledge among the players who receive slightly disturbed signals about its value. There is a group of political activists who will join the protest for sure, while the rest decide strategically what to do. Adding $v$ uniformly to all payoffs, we have the same setting considered above with a single step function, i.e. $I = 1$. The uniqueness result derived above applies directly to this setting. At the unique equilibrium, the strategic participants join the protest if the demonstration is expected to be sufficiently large but not otherwise. Hence, there is an emboldening effect of group size, due to the presence of some activists who exert an indirect influence on the undecided participants.

### 5.3. *Bank run*

The previous examples were direct applications of the stag hunt Poisson game analysed earlier. The next example fits the setting described in Remark 2.

We build on the version of the bank run model of Diamond and Dybvig (1983) that appears in Morris and Shin (2001).[15] Some of the current account holders of a bank who have deposited a unit of their resources in the past are deciding whether to withdraw immediately or wait until later. If people decide to withdraw immediately, they always get their deposit back no matter what the others will do, but if they decide to wait, they get a return that is decreasing in the number of depositors who decide to withdraw immediately. Thus, withdrawing

[15] See also Goldstein and Pauzner (2005).

corresponds to the safe action while waiting corresponds to the risky one. Nobody knows exactly how many depositors are deciding what to do. The number of participants is drawn from a Poisson distribution. Also the expected number of participants is not common knowledge among the players who receive slightly disturbed signals about its value. There is a group of depositors with urgent liquidity needs who will withdraw immediately no matter what the others will do, while the rest choose strategically whether to withdraw immediately or wait until later. If only the depositors with urgent liquidity needs end up withdrawing their deposits, the bank has enough resources to reimburse all the others who have decided to wait with a gross interest payment $\mathscr{R} = v^{-1} > 1$, but if all the depositors decide to withdraw their deposits immediately the bank goes bankrupt. Multiplying by $v$ all the payoffs we obtain exactly the model in Remark 2. Hence, the selection procedure applies to this setting as well. There is a unique equilibrium in which the strategic depositors withdraw their resources if the expected number of participants is sufficiently large, but not otherwise.

## 6. Conclusion

We have shown that modelling some economic and political situations that require the participants to coordinate their decisions on whether to alter the status quo or not as Poisson games in which the expected number of players is not common knowledge helps select a unique equilibrium in a way that parallels what happens in global games, allowing to interpret these situations as critical mass phenomena in which the overall expected number of participants plays a key role in determining whether the status quo is abandoned or not. Our result suggests an important role for the information concerning the size of the group that participates in the interaction.

## Appendix A. Poisson games

This appendix summarizes the basic structure of Poisson games and their main properties.[16]

[16] We refer to Myerson (1998).

A Poisson game is given by a tuple $\Gamma = (n, \mathscr{T}, q, A, u)$. The number of players is a Poisson random variable with parameter $n$, so the actual number of players is $m$ with probability

$$P(m \mid n) = \frac{e^{-n} n^m}{m!}.$$

The set $\mathscr{T} = \{1, \ldots, T\}$ is the set of players' types. The probability that a randomly selected player is of each type is given by the vector $q = (q_1, \ldots, q_T) \in \Delta(\mathscr{T})$. That is, a player is of type $t \in \mathscr{T}$ with probability $q_t$. A type profile $y \in \mathbb{Z}_+^T$ is a vector that specifies for each type $t \in \mathscr{T}$ the number of players $y_t$ of that type. The finite set of actions is $A$. An action profile $x \in Z(A) = \mathbb{Z}_+^{|A|}$ specifies for each action $a \in A$ the number of players $x(a)$ that choose that action. Players' preferences are summarized by the vector $u = (u_1, \ldots, u_T)$, where $u_t : A \times Z(A) \to \mathbb{R}$ for every $t \in \mathscr{T}$. That is, $u_t(a, x)$ is the payoff for a type $t$ player when he chooses action $a$ and the realization resulting from the rest of the population's behavior is the action profile $x \in Z(A)$.

The set of mixed actions is $\Delta(A)$. A strategy function $\sigma$ maps each type to the set of mixed actions, and induces the average behavior $\tau(\sigma) \in \Delta(A)$ which is defined by $\tau(\sigma)(a) = \sum_{t \in \mathscr{T}} q_t \sigma_t(a)$. When players play according to $\sigma$, $\tau(\sigma)(a)$ is the probability that a randomly sampled agent chooses action $a$. Since a player's payoff depends only on the number of other players who choose each action, independently of their specific types, $\tau$ is a sufficient statistic for the analysis of players' optimal behavior.

A Poisson game is characterized by the following properties.

*Decomposition property.* Let each player be independently assigned some characteristic in a set $S$ according to the probability distribution $(\theta(s))_{s \in S}$. Let $w(s)$ be the number of players with characteristic $s$. For every $s \in S$, the random variables $w(s)$ are mutually independent, and each $w(s)$ has a Poisson distribution with mean $n\theta(s)$.

*Independent actions property.* For every strategy function $\sigma$ and action $a$, the random variables $x(a)$ are independent.

*Environmental equivalence property.* A player of any type assesses the same probability distribution for the type profile of the other players as an external observer assesses for the type profile of the whole game.

The decomposition and independent actions properties imply that, when the population's aggregate behavior is $\tau$, the number of players who choose action $a$ is a Poisson random variable with mean $n\tau(a)$ and is independent of the number of players who choose any other action. Then, the probability that the action profile $x \in Z(A)$ is realized is equal to

$$\mathbf{P}(x \mid \tau) = \prod_{a \in A} \left( e^{-n\tau(a)} \frac{(n\tau(a))^{x(a)}}{x(a)!} \right).$$

Environmental equivalence implies that $\mathbf{P}(x \mid \tau)$ is also the probability that each player assigns to the event that the action profile resulting from the other players' behavior is $x$. Hence, the expected payoff to a player of type $t$ who plays $a \in A_t$ is given by

$$U_t(a, \tau \mid n) = \sum_{x \in Z(A)} \mathbf{P}(x \mid \tau) u_t(a, x).$$

## Appendix B. Proof of Lemma 1

**Lemma 1.** *$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}})$ is continuous and strictly increasing in $\tilde{n}$.*

*Proof.* Given a cutpoint strategy $\tilde{\sigma}_{\tilde{n}}$, the probability for a player with signal $\tilde{n}$ that the number of opponents choosing $R$ is equal to $x$ is the probability for the player that the realizations of type 2 agents and of other type 1 players with signal larger than $\tilde{n}$ sum up to $x$.

Let $P_{\tilde{n}}(z \mid np)$ be the probability for the player that $z$ other (type 1) players have made larger observations, if the population size is $n$. We have

$$P_{\tilde{n}}(z \mid np) = \sum_{y=z}^{\infty} P(y \mid np) B(y, z \mid \tilde{n}, n),$$

where $P(y \mid np)$ is the probability that there are $y$ other players in the game when the population size is $n$, and $B(y, z \mid \tilde{n}, n)$ is the probability that $z$ out of $y$ realized players have observed a signal larger than $\tilde{n}$. The assumption of uniform priors on $n$ implies that, given $y$ and $z$, $B(y, z \mid \cdot)$ depends only on the difference between the player's signal and the actual population size, independently of the signal.[17] In particular, we have

$$B(y,z \mid \tilde{n},n) = B(y,z \mid \tilde{n}-n) = \binom{y}{z} \mathbb{P}(\tilde{n}-n)^{y-z} \left[1-\mathbb{P}(\tilde{n}-n)\right]^{z},$$

where

$$\mathbb{P}(\tilde{n}-n) = \frac{\tilde{n}-n+\varepsilon}{2\varepsilon}.$$

Let $\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}_{\tilde{n}}, n)$ be the probability for a player with signal $\tilde{n}$ that the number of opponents choosing $R$ is equal to $x$, given the cutoff strategy $\tilde{\sigma}_{\tilde{n}}$ and if the population size is $n$. Since types are independent, we have

$$\begin{aligned}
\tilde{P}_{\tilde{n}}(0 \mid \tilde{\sigma}_{\tilde{n}}, n) &= P_{\tilde{n}}(0 \mid np)P(0 \mid nr),\\
\tilde{P}_{\tilde{n}}(1 \mid \tilde{\sigma}_{\tilde{n}}, n) &= P_{\tilde{n}}(0 \mid np)P(1 \mid nr) + P_{\tilde{n}}(1 \mid np)P(0 \mid nr),\\
\tilde{P}_{\tilde{n}}(2 \mid \tilde{\sigma}_{\tilde{n}}, n) &= P_{\tilde{n}}(0 \mid np)P(2 \mid nr) + P_{\tilde{n}}(1 \mid np)P(1 \mid nr) + P_{\tilde{n}}(2 \mid np)P(0 \mid nr),\\
&\vdots\\
\tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}_{\tilde{n}}, n) &= \sum_{z=0}^{x} P_{\tilde{n}}(z \mid np)P(x-z \mid nr).
\end{aligned}$$

For $k_i \in \mathbb{Z}_+$, let the cumulative distribution at $k_i$ be

$$\tilde{F}_{\tilde{n}}(k_i \mid \tilde{\sigma}_{\tilde{n}}, n) = \sum_{x=0}^{k_i} \tilde{P}_{\tilde{n}}(x \mid \tilde{\sigma}_{\tilde{n}}, n).$$

We can express $\tilde{F}_{\tilde{n}}(k_i \mid \tilde{\sigma}_{\tilde{n}}, n)$ as a function of the signal $\tilde{n}$ and the difference $\delta = \tilde{n} - n$ and denote it $\mathbf{F}(k_i \mid \tilde{n}, \delta)$, obtaining

$$\mathbf{F}(k_i \mid \tilde{n},\delta) = \sum_{x=0}^{k_i} \sum_{z=0}^{x} \sum_{y=z}^{\infty} P(y \mid (\tilde{n}-\delta)p)\, B(y,z \mid \delta)\, P(x-z \mid (\tilde{n}-\delta)r).$$

The utility of choosing $R$ for a player with marginal signal $\tilde{n}$ is equal to

$$\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) = 1 - \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \sum_{i=1}^{I} \mathbf{F}(k_i \mid \tilde{n},\delta)(u_i - u_{i-1})\, \mathrm{d}\delta.$$

Fix $\delta$ and let $\tilde{n}' > \tilde{n}$. We can show that, for every $k_i \in \mathbb{Z}_+$,

$$\mathbf{F}(k_i \mid \tilde{n},\delta) > \mathbf{F}(k_i \mid \tilde{n}',\delta).$$

Since $u_i - u_{i-1} \geq 0$ for every $i = 1, \ldots, I$, this will imply that $\tilde{U}_{\tilde{n}}(R, \tilde{\sigma}_{\tilde{n}}) < \tilde{U}_{\tilde{n}'}(R, \tilde{\sigma}_{\tilde{n}'})$.

[17] At least if the signal is at least $\varepsilon$ inside the support of $n$.

Note from the above expressions that the formula of $\mathbf{F}(k_i \mid \tilde{n},\delta)$ can be expressed both as

$$P(0\mid(\tilde{n}-\delta)r)\sum_{z=0}^{k_i}P_{\tilde{n}}(z\mid(\tilde{n}-\delta)p)+P(1\mid(\tilde{n}-\delta)r)\sum_{z=0}^{k_i-1}P_{\tilde{n}}(z\mid(\tilde{n}-\delta)p)+\ldots$$
$$+P(k_i\mid(\tilde{n}-\delta)r)P_{\tilde{n}}(0\mid(\tilde{n}-\delta)p)=\sum_{x=0}^{k_i}P(x\mid(\tilde{n}-\delta)r)\sum_{z=0}^{k_i-x}P_{\tilde{n}}(z\mid(\tilde{n}-\delta)p),$$

and as

$$P_{\tilde{n}}(0\mid(\tilde{n}-\delta)p)\sum_{z=0}^{k_i}P(z\mid(\tilde{n}-\delta)r)+P_{\tilde{n}}(1\mid(\tilde{n}-\delta)p)\sum_{z=0}^{k_i-1}P(z\mid(\tilde{n}-\delta)r)+\ldots$$
$$+P_{\tilde{n}}(k_i\mid(\tilde{n}-\delta)p)P(0\mid(\tilde{n}-\delta)r)=\sum_{z=0}^{k_i}P_{\tilde{n}}(z\mid(\tilde{n}-\delta)p)\sum_{x=0}^{k_i-z}P(x\mid(\tilde{n}-\delta)r).$$

Moreover, note that $\sum_{z=0}^{k_i}P_{\tilde{n}}(z\mid(\tilde{n}-\delta)p)$ is given by

$$P(0\mid(\tilde{n}-\delta)p)B(0,0\mid\delta)+P(1\mid(\tilde{n}-\delta)p)B(1,0\mid\delta)+P(2\mid(\tilde{n}-\delta)p)B(2,0\mid\delta)+\ldots$$
$$+P(1\mid(\tilde{n}-\delta)p)B(1,1\mid\delta)+P(2\mid(\tilde{n}-\delta)p)B(2,1\mid\delta)+P(3\mid(\tilde{n}-\delta)p)B(3,1\mid\delta)+\ldots$$
$$+P(2\mid(\tilde{n}-\delta)p)B(2,2\mid\delta)+P(3\mid(\tilde{n}-\delta)p)B(3,2\mid\delta)+P(4\mid(\tilde{n}-\delta)p)B(4,2\mid\delta)+\ldots$$
$$\vdots$$
$$+P(k_i\mid(\tilde{n}-\delta)p)B(k_i,k_i\mid\delta)+P(k_i+1\mid(\tilde{n}-\delta)p)B(k_i+1,k_i\mid\delta)+$$
$$P(k_i+2\mid(\tilde{n}-\delta)p)B(k_i+2,k_i\mid\delta)+\ldots$$

$$=P(0\mid(\tilde{n}-\delta)p)+P(1\mid(\tilde{n}-\delta)p)+P(2\mid(\tilde{n}-\delta)p)+\ldots+P(k_i\mid(\tilde{n}-\delta)p)+$$
$$P(k_i+1\mid(\tilde{n}-\delta)p)\left[1-B(k_i+1,k_i+1\mid\delta)\right]+$$
$$P(k_i+2\mid(\tilde{n}-\delta)p)\left[1-B(k_i+2,k_i+1\mid\delta)-B(k_i+2,k_i+2\mid\delta)\right]+$$
$$P(k_i+3\mid(\tilde{n}-\delta)p)\left[1-B(k_i+3,k_i+1\mid\delta)-B(k_i+3,k_i+2\mid\delta)-B(k_i+3,k_i+3\mid\delta)\right]+$$
$$\vdots$$

For every $y \in \mathbb{Z}_+$ and $k_i \in \mathbb{Z}_+$, let $\mathbb{F}(y,k_i \mid \delta) = \sum_{x=0}^{k_i} B(y,x \mid \delta)$. After some algebraic manipulations, we have

$$\sum_{z=0}^{k_i} P_{\tilde{n}}(z \mid (\tilde{n}-\delta)p) = 1 - \sum_{y=k_i+1}^{\infty} P(y \mid (\tilde{n}-\delta)p)\big(1-\mathbb{F}(y,k_i \mid \delta)\big)$$

$$= 1 - \sum_{y=k_i}^{\infty} \big(\mathbb{F}(y,k_i \mid \delta) - \mathbb{F}(y+1,k_i \mid \delta)\big)\big[1 - F(y \mid (\tilde{n}-\delta)p)\big].$$

Since $\mathbb{F}(y,k_i \mid \delta) - \mathbb{F}(y+1,k_i \mid \delta) > 0$ for every $y \in \mathbb{Z}_+$ and $k_i \in \mathbb{Z}_+$, and $F(y \mid (\tilde{n}-\delta)p) > F(y \mid (\tilde{n}'-\delta)p)$ for every $y \in \mathbb{Z}_+$, we have

$$\sum_{z=0}^{k_i} P_{\tilde{n}}(z \mid (\tilde{n}-\delta)p) > \sum_{z=0}^{k_i} P_{\tilde{n}'}(z \mid (\tilde{n}'-\delta)p)$$

for every $k_i \in \mathbb{Z}_+$. Therefore, for every $k_i \in \mathbb{Z}_+$, we have

$$\mathbf{F}(k_i \mid \tilde{n},\delta) = \sum_{x=0}^{k_i} P(x \mid (\tilde{n}-\delta)r) \sum_{z=0}^{k_i-x} P_{\tilde{n}}(z \mid (\tilde{n}-\delta)p) >$$

$$\sum_{x=0}^{k_i} P(x \mid (\tilde{n}-\delta)r) \sum_{z=0}^{k_i-x} P_{\tilde{n}'}(z \mid (\tilde{n}'-\delta)p) = \sum_{z=0}^{k_i} P_{\tilde{n}'}(z \mid (\tilde{n}'-\delta)p) \sum_{x=0}^{k_i-z} P(x \mid (\tilde{n}-\delta)r) >$$

$$\sum_{z=0}^{k_i} P_{\tilde{n}'}(z \mid (\tilde{n}'-\delta)p) \sum_{x=0}^{k_i-z} P(x \mid (\tilde{n}'-\delta)r) = \mathbf{F}(k_i \mid \tilde{n}',\delta).$$

It follows that, given the cutoff strategy $\tilde{\sigma}_{\tilde{n}}$, the payoff of choosing $R$ for a player with signal $\tilde{n}$ is strictly increasing in the cutoff $\tilde{n}$. Continuity derives from the fact that the Poisson probabilities are continuous in the parameter. □

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