DEMS Department of Economics, Management and Statistics

DEMS WORKING PAPER SERIES

The Set of Equilibria in Max-Min Two Groups Contests with Binary Actions and a Private Good Prize

Mario Gilli, Andrea Sorrentino

No. 539 – June 2024

Department of Economics, Management and Statistics
University of Milano – Bicocca
Piazza Ateneo Nuovo 1 – 2016 Milan, Italy
http://dems.unimib.it/

The Set of Equilibria in Max-Min Two Groups Contests with Binary Actions and a Private Good Prize

Mario Gilli* Andrea Sorrentino†

Department of Economics, Management and Statistics

University of Milano-Bicocca

June 2024

Abstract

In this paper we consider a deterministic complete information two groups contest where the effort choices made by the teammates are aggregated into group performance by the weakest-link technology (perfect complementarity), that is a "max-min group contest", as defined by Chowdhury et al. (2016). However, instead of a continuum effort set, we employ a binary action set. Further, we consider private good prizes, so that there is a sharing issue within the winning group. Therefore, we include two stages: the first one about the setting of a sharing rule parameter and the second one about simultaneous and independent actions' choices. The binary action set allow us to innovate on the existing literature by (i) characterizing the full set of the second stage equilibrium actions; (ii) computationally characterizing in MATLAB the set of within-group symmetric subgame perfect Nash equilibria in pure strategies in the entire game. We find conditions such that the set of within-group symmetric subgame perfect Nash equilibria in pure strategies have the cardinality of the continuum. We also check whether this paper's results are due to discreteness or to binary choice, proving that in this case there are no subgame perfect Nash equilibria in pure strategies, as proved in the continuum case in Gilli and Sorrentino (2024).

Acknowledgment: The paper has greatly benefited from comments by Davide Bosco. We also thank the seminar participants at DEMS ReLunch 2024.

Keywords: Group contests, sharing rules, indeterminacy.
JEL Codes: D74, D71, C72

*mario.gilli@unimib.it
†a.sorrentino9@campus.unimib.it

1 Introduction

Competition among groups of agents is widespread in many socioeconomic activities, spanning from rent-seeking actions by firms, labour markets, investment in R&D, military conflict, electoral competition and sports. In many settings, the choice faced by teammates can be conceived as a binary decision. For instance, we could think of research groups where each member can apply or not for a grant, or the possible signature of an international agreement among countries belonging to two contrasting alliances, or again whether to vote pro or against an issue such as a group's strike. Furthermore, complementarities within groups can be a salient feature of competition among groups. For example, on many issues, EU requires unanimity, so that a dissenting vote is enough to prevent the adoption of a policy.[1]

In this paper we consider a deterministic complete information two groups contest where the effort choices made by the teammates are aggregated into group performance by the weakest-link technology (perfect complementarity), that is a "max-min group contest", as defined by Chowdhury et al. (2016). However, instead of a continuum effort set, we employ a binary action set. Further, we consider private good prizes, so that there is a sharing issue within the winning group. Therefore, we include two stages: the first one about the setting of a sharing rule parameter and the second one about simultaneous and independent actions' choices. The binary action set allow us to innovate on the existing literature by (i) characterizing the full set of the second stage equilibrium actions; (ii) computationally characterizing in MATLAB the set of within-group symmetric subgame perfect Nash equilibria in pure strategies in the entire game. Depending on the size of the private good prize with respect to groups' size, we find conditions such that the set of within-group symmetric subgame perfect Nash equilibria in pure strategies have the cardinality of the continuum, a case we call of indeterminacy.

This result is particularly interesting, because in Gilli and Sorrentino (2024), we proved that max-min group contest with continuum effort and private prize have no subgame perfect equilibria, unless the choice of the sharing rule is restricted. Thus, we check whether this paper's results are due to discreteness or to binary choice. To this aim, we expand the set of second-stage actions from the binary case to any subset of the natural numbers with cardinality at least equal to three. We find that the characterization of the full set of subgame perfect Nash equilibria in pure strategies would require hundreds of billions of iterations even in our simple model. Nonetheless, we are able to find a counterexample, proving that in this case there are no subgame perfect Nash equilibria in pure strategies, as in the continuum case. The reason is that once we introduce more than two effort choices, it is necessary to ensure that both upward and downward deviations are not profitable, not just upward or downward ones, as in the binary action setup.

The paper is organized as follows. The next subsection quickly reviews the related literature, while section 2 outlines the model. Section 3 characterizes the set of second-period equilibria for an exogenous profile of incentive schemes. Section 4 restricts the analysis to within-group symmetric (WGS) equilibria and endogenises the incentive schemes, providing the subgame perfect WGS equilibria of the entire game. Section 5 discuss the extension to K actions, while section 6 concludes the paper.

[1] For example, see Gilli and Tedeschi (2022).

1.1 Related Literature

Our reference model is Chowdhury et al. (2016). In particular, we complement their result on the existence of pure strategies Nash equilibria in a deterministic group contest with the weakest-link impact function, for the case of a private good prize, with a binary action set.

Regarding all-pay auctions, seminal contributions are certainly Hillman and Riley (1989), Baye et al. (1996), Skaperdas (1996), Clark and Riis (1998), and Baik et al. (2001). Concerning private good prizes in group contests and the key role of sharing rules, Nitzan (1991), Baik and Lee (2001) and Baik and Lee (2012) are essential references. Finally, key contributions for group contests with non-standard impact functions under both complete information and incomplete information are, in the former case, Lee (2012), Chowdhury et al. (2013), Kolmar and Rommeswinkel (2013) and Barbieri et al. (2014) and, in the latter, Barbieri and Malueg (2016), Barbieri et al. (2019) and Barbieri and Topolyan (2021), other than the definition of the best-shot and weakest-link impact functions in a public good provision model by Hirshleifer (1983).

2 A Binary Group Contest with a Finite Number of Agents

Consider a simple two-groups model that sums up the main characteristics of group contests under complete information. The model is defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;

2. each group has $n_j \geq 4$ members in each group. The total number of agents is $N = n_1 + n_2$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;

3. the **choice** of member $i \in \{1, ..., n_j\}$ in group $j \in \{1, 2\}$,to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all j-group agents' efforts, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let define the share of active players i in group j as

$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1];$$

4. a private **prize** worth v to be allocated to one of the groups;

5. the **impact function** of group j is given by the weakest-link technology

$$X_j = \min \{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\};$$

6. the **contest success function** is given by the *all-pay auction:*

$$p_j(X_1, X_2) = \begin{cases} 1 & if \;\; X_j > X_{-j} \\ \frac{1}{2} & if \;\; X_j = X_{-j} \\ 0 & if \;\; X_j < X_{-j}; \end{cases}$$

7. the **sharing rule**, such that if group $j \in \{1,2\}$ wins, then a member $i \in \{1,\ldots,n_j\}$ gets a share of the prize

$$q_{ij}\left(x_{1j},...,x_{n_j j}\right) =$$
$$= \begin{cases} \underbrace{(1-\alpha_j)}_{incentivation\ part} \frac{x_{ij}}{\sum_{i=1}^{n_j} x_{ij}} + \underbrace{\alpha_j}_{equalizing\ part} \frac{1}{n_j} & if \quad \sum_{i=1}^{n_j} x_{ij} > 0 \\ \frac{1}{n_j} & otherwise \end{cases}$$

where

- $\alpha_j \in \mathbb{R}$ is the share of the prize that the members of the winning team get independently of their effort: let us call α_j the **equalizing part** of the sharing rule, while $1-\alpha_j$ is called the **incentivation part**;

8. the individual **costs of effort** $C_{ij}\left(x_j(i)\right) = x_j(i)$;

9. the **timing**: there are two stages:

   i. in the first stage, the groups choose the optimal sharing rule within each group α_j;

   ii. in the second stage all the members of the groups observe the first stage choices (α_1, α_2) and choose simultaneously and independently their effort $x_j(i)$ and the prize is allocated to one of the two groups according to the contest success function.

As a consequence of these modelling characteristics, player ij has the expected **payoff**

$$\pi_{ij}\left(\alpha_j, \alpha_{-j}, x_{1j},...,x_{n_j j}, x_{1-j},...,x_{n_{-j}-j}\right) = p_j q_{ij} v - x_{ij} =$$
$$= \begin{cases} \left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

Now, we are able to provide a formal definition of a binary group contest.

Definition 1 *A* Binary Max-Min Group Contest *$BMMGC$ is a two stages game $BMMGC = \langle \{1,2\}, N, S_j, B_{ij}, \pi_{ij} \rangle$ defined by*

1. *the set of groups $\{1,2\}$;*

2. *the set of players $N = \{1,...,n_1+n_2\}$;*

3. *the set of first-period actions $S_j = \mathbb{R}$: for each group j, the choice of the share α_j in the sharing rule*;

4. *the set of second-period actions $B_{ij} = \{0,1\}$: for each player ij, the choice of the bid $x_j(i)$;*

5. the payoff functions for each player $ij \in N$

$$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x}) = p_j q_{ij} v - x_j(i) =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

where $\boldsymbol{\alpha}$ and $\mathbf{x}$ are, respectively, the vector of first and second period actions.

The notation used in this paper is summed up in table 1.

Variable	Meaning
ij or $j(i)$	agent i of group j
$\{1, ..., n_j\}$	set of agents in group j
$x_j(i)$ or x_{ji}	effort of agent i in group j
$X_j = \min\{x_j(i) \in \{0,1\}, i \in \{1, ..., n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C_{ij}(x_j(i)) = x_j(i)$	cost of effort for agent i of group j
$p_j(X_1, X_2)$	probability of group j of winning the contest
$q_{ij}(x_{1j}, ..., x_{n_j j})$	sharing rule for agent i of group j
$\alpha_j \in \mathbb{R}$	equalizing part of the sharing rule
α	vector of α_j for $j \in \{1,2\}$
$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x})$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in [0,1]$	share of active agents in group j

Table 1

3 The Set of Second Period Equilibria

Without loss of generality, the equilibria are presented in terms of share of active agents in each group, i.e. as pairs

$$(\gamma_1, \gamma_2) \in [0,1] \times [0,1],$$

so that geometrically they can be represented in the unit square. Then, we will indicate by

$$\pi_{ij}(\gamma_1, \gamma_2 | \boldsymbol{\alpha})$$

the second-period payoff of player ij, as a function of (γ_1, γ_2) for a given $\alpha := (\alpha_1, \alpha_2)$.

Moreover, when $\gamma_j \in (0,1)$, denote by

$$\gamma_j^+ = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} + 1\right) \in [0,1]$$

the share of active agents at a marginal increase and by

$$\gamma_j^- = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} - 1\right) \in [0,1]$$

the share of active agents at a marginal decrease.

3.1 Characterization of the Set of Pure Strategy Nash Equilibria in the Second Period.

In this subsection, we characterize the full set of second-period Nash equilibria in pure strategies, to study the interplay of strong complementarities at play within groups, which favour the alignment of effort choice by teammates, and the selective incentives induced by the sharing rules.

Proposition 1 *In the BMMGC, the set of the second period pure strategy Nash equilibria of the game is characterized as follows:*

1. *if* $v \geq 2\max\{n_1, n_2\}$, *then*

$$(\gamma_1^*, \gamma_2^*) = (1,1) \ \ for \ any \ \ (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R};$$

2. *if* $v > 0$, *then*

$$(\gamma_1^{**}, \gamma_2^{**}) = (0,0) \ \ for \ any \ \ (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1 - 1)\, v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2 - 1)\, v}, \infty\right)$$

3. *if* $v > 0$, *then*

$$\left(\gamma_j^{***}, \gamma_{-j}^{***}\right) = (1,0) \ \ for \ any \ \ (\alpha_j, \alpha_{-j}) \in \left(-\infty, 2\left(1 - \frac{n_j}{v}\right)\right] \times \mathbb{R};$$

4. *if* $v > 0$, *then*

$$\left(\gamma_j^{****a}, \gamma_{-j}^{****a}\right) \in (0,1) \times \{0\} \quad with \ \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$$

$$for \ any \quad (\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2\,(n_j\gamma_j + 1)}{v}, 1 - \frac{2n_j\gamma_j}{v}\right] \times \left[1 - \frac{2n_{-j}}{(n_{-j} - 1)\, v}, \infty\right);$$

5. *if* $v > 0$, *then*

$$\left(\gamma_j^{****b}, \gamma_{-j}^{****b}\right) \in (0,1) \times \{0\} \quad with \ \sum_{i=1}^{n_j} x_{ij} = 1$$

$$for \ any \quad (\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2\,(n_j\gamma_j + 1)}{v}, 1 - \frac{2n_j\gamma_j}{(1 - \gamma_j)\, v}\right] \times \left[1 - \frac{2n_{-j}}{(n_{-j} - 1)\, v}, \infty\right);$$

6. *if* $0 < v \leq 2$, *then*

$$\left(\gamma_j^{****c}, \gamma_{-j}^{****c}\right) \in (0,1) \times \{0\} \quad with \ \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

$$for \ any \quad (\alpha_j, \alpha_{-j}) \in \left[2\left(1 - \frac{n_j}{v}\right), 1 - \frac{2n_j\gamma_j}{v}\right] \times \left[1 - \frac{2n_{-j}}{(n_{-j} - 1)\, v}, \infty\right);$$

7. *if* $v > 0$, *then*

$$\left(\gamma_1^{*****a}, \gamma_2^{*****a}\right) \in (0,1) \times (0,1) \quad \textit{with} \quad \sum_{i=1}^{n_j} x_{ij} \in \{2, \ldots, n_j - 2\}$$

$$\textit{for any} \quad (\alpha_1, \alpha_2) \in \left[1 - \frac{2(n_1\gamma_1 + 1)}{v}, 1 - \frac{2n_1\gamma_1}{v}\right] \times \left[1 - \frac{2(n_2\gamma_2 + 1)}{v}, 1 - \frac{2n_2\gamma_2}{v}\right];$$

8. *if* $v > 0$, *then*

$$\left(\gamma_1^{*****b}, \gamma_2^{*****b}\right) \in (0,1) \times (0,1) \quad \textit{with} \quad \sum_{i=1}^{n_j} x_{ij} = 1$$

$$\textit{for any} \quad (\alpha_1, \alpha_2) \in \left[1 - \frac{2(n_1\gamma_1 + 1)}{v}, 1 - \frac{2n_1\gamma_1}{(1-\gamma_1)v}\right] \times \left[1 - \frac{2(n_2\gamma_2 + 1)}{v}, 1 - \frac{2n_2\gamma_2}{(1-\gamma_2)v}\right];$$

9. *if* $0 < v \leq 2$, *then*

$$\left(\gamma_1^{*****c}, \gamma_2^{*****c}\right) \in (0,1) \times (0,1) \quad \textit{with} \quad \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

$$\textit{for any} \quad (\alpha_1, \alpha_2) \in \left[2\left(1 - \frac{n_1}{v}\right), 1 - \frac{2n_1\gamma_1}{v}\right] \times \left[2\left(1 - \frac{n_2}{v}\right), 1 - \frac{2n_2\gamma_2}{v}\right];$$

Proof. The results are derived by direct inspection.

1. Suppose

$$(\gamma_1, \gamma_2) = (1, 1).$$

Then

$$x_j(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 1 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j$$

so that

$$\pi_{ij}(\gamma_1, \gamma_2 | \boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_j}v - 1.$$

If agent ij deviates to $x_j(i) = 0$, then

$$x_j(i) = \min\{\mathbf{x}_j\} = 0 < \min\{\mathbf{x}_{-j}\} = 1 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 1$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_1, \gamma_2 | \boldsymbol{\alpha}) = 0.$$

Hence for any player ij there is no incentive to deviate if and only if $v \geq 2n_j$.

2. Suppose

$$(\gamma_1, \gamma_2) = (0, 0).$$

Then

$$x_j(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0$$

so that

$$\pi_{ij}(\gamma_1, \gamma_2|\boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_j}v.$$

If agent ij deviates to $x_j(i) = 1$, then

$$x_j(i) = 1 > \min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = 1$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j) + \alpha_j\frac{1}{n_j}\right]v - 1.$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_j}v \geq \frac{1}{2}\left[(1-\alpha_j) + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow \left[(1-\alpha_j) + \alpha_j\frac{1}{n_j} - \frac{1}{n_j}\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_j)\left(1 - \frac{1}{n_j}\right)\right]v \leq 2 \Leftrightarrow \left[(1-\alpha_j)\left(\frac{n_j - 1}{n_j}\right)\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1 - \alpha_j \leq \frac{2n_j}{(n_j - 1)v} \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j}{(n_j - 1)v}.$$

3. Suppose

$$(\gamma_j, \gamma_{-j}) = (1, 0).$$

Then

$$x_j(i) = \min\{\mathbf{x}_j\} = 1 > \min\{\mathbf{x}_{-j}\} = x_{-j}(i) = 0, \ \sum_{i=1}^{n_j} x_j(i) = n_j \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{n_{-j}}v - 1 \text{ and } \pi_{i-j}(\gamma_j, \gamma_{-j}) = 0.$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} = 0 = x_j(i), \ \sum_{i=1}^{n_j} x_j(i) = n_j - 1 \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{1}{n_j}\alpha_j v \leq \pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{1}{n_j}v - 1 \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \leq 2\left(1 - \frac{n_j}{v}\right);$$

hence no agent of group j has an incentive to deviate if and only if $\alpha_j \leq 2\left(1 - \frac{n_j}{v}\right)$. On the other hand, if agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$\min\{\mathbf{x}_j\} = 1 > \min\{\mathbf{x}_{-j}\} = 0, \ \sum_{i=1}^{n_j} x_j(i) = n_j \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = 1$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+ | \boldsymbol{\alpha}\right) = -1 \leq \pi_{i-j}(\gamma_j, \gamma_{-j}) = 0,$$

hence no agent of group j has an incentive to deviate.

4. Suppose

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_j(i) \in \{2, \ldots, n_j - 2\} \text{ and } \gamma_{-j} = 0 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) \in \{2, \ldots, n_j - 2\}, \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0.$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - 1 \in \{1, \ldots, n_j - 3\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\alpha_j \frac{1}{n_j}\right] v;$$

hence agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \geq \frac{1}{2}\left[\alpha_j \frac{1}{n_j}\right] v \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j} v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{v}.$$

Suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}) = \frac{1}{2}\alpha_j \frac{1}{n_j} v \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j + 1 \in \{3, \ldots, n_j - 1\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j\frac{1}{n_j}\right]v - 1 \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{\gamma_j+1}v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2(n_j\gamma_j+1)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2(n_j\gamma_j+1)}{v}$$

Moreover,

$$\pi_{i-j}(\gamma_j, \gamma_{-j}) = \frac{1}{2}\frac{1}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$x_{-j}(i) = 1 > \min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = 1$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}}\right]v - 1;$$

hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}}v \geq \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}}\right]v - 1 \Leftrightarrow \left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}} - \frac{1}{n_{-j}}\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_{-j})\left(1 - \frac{1}{n_{-j}}\right)\right]v \leq 2 \Leftrightarrow \left[(1-\alpha_{-j})\left(\frac{n_{-j}-1}{n_{-j}}\right)\right]v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1 - \alpha_{-j} \leq \frac{2n_{-j}}{(n_{-j}-1)v} \Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}}{(n_{-j}-1)v} \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_j(i) \in \{2, \ldots, n_j - 2\} \text{ and } \gamma_{-j} = 0$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1 - \frac{2(n_j\gamma_j+1)}{v}, 1 - \frac{2n_j\gamma_j}{v}\right] \text{ and } \alpha_{-j} \geq 1 - \frac{2n_{-j}}{(n_{-j}-1)v} \ .$$

5. Suppose

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_j(i) = 1 \text{ and } \gamma_{-j} = 0 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 1 \ .$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^-, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right]v \Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j\gamma_j} - \alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right]v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right]v \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j\gamma_j}\right]v - 1 - \frac{1}{2}\left[\frac{1}{n_j}\right]v \geq \frac{1}{2}\left[\alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right]v \Leftrightarrow$$

$$\Leftrightarrow \frac{(1-\gamma_j)v - 2n_j\gamma_j}{2n_j\gamma_j} \geq \alpha_j\left(\frac{1-\gamma_j}{2n_j\gamma_j}\right)v \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{(1-\gamma_j)v} \ .$$

Suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\alpha_j\frac{1}{n_j}v \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 2$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j + 1} + \alpha_j\frac{1}{n_j}\right]v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j \frac{1}{n_j} v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j+1} + \alpha_j \frac{1}{n_j}\right] v - 1 \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{\gamma_j+1} v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2(n_j\gamma_j+1)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2(n_j\gamma_j+1)}{v} \ .$$

Moreover,

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_{-j}} v \ .$$

If agent ij deviates to $x_{-j}(i) = 1$, then

$$x_{-j}(i) = 1 > \min\{\mathbf{x}_j\} = 0 = \min\{\mathbf{x}_{-j}\} \ \text{ and } \ \sum_{i=1}^{n_{-j}} x_{-j}(i) = 1$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_j}\right] v - 1 \ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}} v \geq \frac{1}{2}\left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}}\right] v - 1 \Leftrightarrow \left[(1-\alpha_{-j}) + \alpha_{-j}\frac{1}{n_{-j}} - \frac{1}{n_{-j}}\right] v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \left[(1-\alpha_{-j})\left(1 - \frac{1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow \left[(1-\alpha_{-j})\left(\frac{n_{-j}-1}{n_{-j}}\right)\right] v \leq 2 \Leftrightarrow$$

$$\Leftrightarrow 1 - \alpha_{-j} \leq \frac{2n_{-j}}{(n_{-j}-1)v} \Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}}{(n_{-j}-1)v} \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) \ \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_j(i) = 1 \text{ and } \gamma_{-j} = 0$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1 - \frac{2(n_j\gamma_j+1)}{v}, 1 - \frac{2n_j\gamma_j}{(1-\gamma_j)v}\right] \ \text{ and } \alpha_{-j} \geq 1 - \frac{2n_{-j}}{(n_{-j}-1)v} \ .$$

6. Suppose

$$(\gamma_j, \gamma_{-j}) \ \text{ such that } \gamma_j \in (0,1) \text{ with } \sum_{i=1}^{n_j} x_j(i) = n_j - 1 \text{ and } \gamma_{-j} = 0 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 1, \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0 \ .$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 2$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^-, \gamma_j|\boldsymbol{\alpha}) = \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j}v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{v} \ .$$

Suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\alpha_j\frac{1}{n_j}v.$$

If agent ij deviates to $x_j(i) = 1$, then

$$\min\{\mathbf{x}_j\} = 1 > \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j})\frac{1}{n_j}v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{n_j}v - 1 \Leftrightarrow \alpha_j \geq \frac{2n_j}{v}\left(\frac{1}{n_j}v - 1\right) \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \geq 2\left(1 - \frac{n_j}{v}\right).$$

Note that $2\left(1-\frac{n_j}{v}\right)v < 1 - \frac{2n_j\gamma_j}{v}$ if and only if $v \leq 2$. Moreover,

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{1}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i)=1$, then

$$x_{-j}(i)=1>\min\{\mathbf{x}_j\}=0=\min\{\mathbf{x}_{-j}\}\text{ and }\sum_{i=1}^{n_j}x_j(i)=1$$

so that the deviation payoff is

$$\pi^D_{i-j}\left(\gamma_j,\gamma^+_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[(1-\alpha_{-j})+\alpha_{-j}\frac{1}{n_{-j}}\right]v-1;$$

hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{1}{n_{-j}}v\geq\frac{1}{2}\left[(1-\alpha_{-j})+\alpha_{-j}\frac{1}{n_{-j}}\right]v-1\Leftrightarrow\left[(1-\alpha_{-j})+\alpha_{-j}\frac{1}{n_{-j}}-\frac{1}{n_{-j}}\right]v\leq 2\Leftrightarrow$$

$$\Leftrightarrow\left[(1-\alpha_{-j})\left(1-\frac{1}{n_{-j}}\right)\right]v\leq 2\Leftrightarrow\left[(1-\alpha_{-j})\left(\frac{n_{-j}-1}{n_{-j}}\right)\right]v\leq 2\Leftrightarrow$$

$$\Leftrightarrow 1-\alpha_{-j}\leq\frac{2n_{-j}}{(n_{-j}-1)\,v}\Leftrightarrow\alpha_{-j}\geq 1-\frac{2n_{-j}}{(n_{-j}-1)\,v}\ .$$

Thus,

$$(\gamma_j,\gamma_{-j})\text{ such that }\gamma_j\in(0,1)\text{ with }\sum_{i=1}^{n_j}x_j(i)=n_j-1\text{ and }\gamma_{-j}=0$$

is a Nash equilibrium if and only if

$$\alpha_j\in\left[2\left(1-\frac{n_j}{v}\right),1-\frac{2n_j\gamma_j}{v}\right]\text{ and }\alpha_{-j}\geq 1-\frac{2n_{-j}}{(n_{-j}-1)\,v}\ \forall\ v\leq 2\ .$$

7. Suppose

$$(\gamma_1,\gamma_2)\quad\text{such that }\gamma_j\in(0,1)\text{ with}\sum_{i=1}^{n_j}x_{ij}\in\{2,\ldots,n_j-2\}\ .$$

Then

$$\min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}=0\text{ and }\sum_{i=1}^{n_j}x_j(i)\in\{2,\ldots,n_j-2\}\ .$$

Suppose $x_j(i)=1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j}+\alpha_j\frac{1}{n_j}\right]v-1\ .$$

If agent ij deviates to $x_j(i)=0$, then

$$\min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}=0\text{ and }\sum_{i=1}^{n_j}x_j(i)=n_j\gamma_j-1\in\{1,\ldots,n_j-3\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}+\alpha_j\frac{1}{n_j}\right]v-1\geq\frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v\Leftrightarrow\frac{1}{2}\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j}v\geq 1\Leftrightarrow$$

$$\Leftrightarrow\left(1-\alpha_j\right)\geq\frac{2n_j\gamma_j}{v}\Leftrightarrow\alpha_j\leq 1-\frac{2n_j\gamma_j}{v}\ .$$

Suppose $x_j\left(i\right)=0$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\alpha_j\frac{1}{n_j}v\ .$$

If agent ij deviates to $x_j\left(i\right)=1$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j+1\in\left\{3,\ldots,n_j\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{+},\gamma_j|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j+1}+\alpha_j\frac{1}{n_j}\right]v-1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v\geq\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j+1}+\alpha_j\frac{1}{n_j}\right]v-1\Leftrightarrow\frac{1}{2}\left(1-\alpha_j\right)\frac{1}{n_j\gamma_j+1}v\leq 1\Leftrightarrow$$

$$\Leftrightarrow\left(1-\alpha_j\right)\leq\frac{2\left(n_j\gamma_j+1\right)}{v}\Leftrightarrow\alpha_j\geq 1-\frac{2\left(n_j\gamma_j+1\right)}{v}\ .$$

Thus,

$$\left(\gamma_1,\gamma_2\right)\ \textit{such that}\ \ \gamma_j\in\left(0,1\right)\text{with}\sum_{i=1}^{n_j}x_{ij}\in\left\{2,\ldots,n_j-2\right\}$$

is a Nash equilibrium if and only if

$$\alpha_j\in\left[1-\frac{2\left(n_j\gamma_j+1\right)}{v},1-\frac{2n_j\gamma_j}{v}\right].$$

8. Suppose

$$\left(\gamma_1,\gamma_2\right)\ \text{ such that }\gamma_j\in\left(0,1\right)\ \text{ with }\ \sum_{i=1}^{n_j}x_{ij}=1\ .$$

Then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_j\left(i\right)=1.$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_j | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[\frac{1}{n_j}\right] v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j \frac{1}{n_j}\right] v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right] v \Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j\gamma_j} - \alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right] v - 1 \geq \frac{1}{2}\left[\frac{1}{n_j}\right] v \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{2}\left[\frac{1}{n_j\gamma_j}\right] v - 1 - \frac{1}{2}\left[\frac{1}{n_j}\right] v \geq \frac{1}{2}\left[\alpha_j\left(\frac{1-\gamma_j}{n_j\gamma_j}\right)\right] v \Leftrightarrow$$

$$\Leftrightarrow \frac{(1-\gamma_j) v - 2n_j\gamma_j}{2n_j\gamma_j} \geq \alpha_j\left(\frac{1-\gamma_j}{2n_j\gamma_j}\right) v \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{(1-\gamma_j) v}.$$

Suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j} | \boldsymbol{\alpha}) = \frac{1}{2}\alpha_j \frac{1}{n_j} v \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 2$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j} | \boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j + 1} + \alpha_j \frac{1}{n_j}\right] v - 1;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j \frac{1}{n_j} v \geq \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j + 1} + \alpha_j \frac{1}{n_j}\right] v - 1 \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{\gamma_j + 1} v \leq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \leq \frac{2(n_j\gamma_j + 1)}{v} \Leftrightarrow \alpha_j \geq 1 - \frac{2(n_j\gamma_j + 1)}{v} \ .$$

Thus,

$$(\gamma_1, \gamma_2) \ \ such \ \ that \ \ \gamma_j \in (0,1) \quad \text{with} \quad \sum_{i=1}^{n_j} x_{ij} = 1$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[1 - \frac{2(n_j\gamma_j + 1)}{v}, 1 - \frac{2n_j\gamma_j}{(1-\gamma_j) v}\right].$$

9. Finally, suppose

$$(\gamma_1, \gamma_2) \quad \text{such that } \gamma_j \in (0,1) \quad \text{with} \quad \sum_{i=1}^{n_j} x_{ij} = n_j - 1 \ .$$

Then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 1 \ .$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j \frac{1}{n_j}\right]v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j - 2$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^-, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[\alpha_j \frac{1}{n_j}\right]v;$$

hence, agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_j)\frac{1}{n_j\gamma_j} + \alpha_j\frac{1}{n_j}\right]v - 1 \geq \frac{1}{2}\left[\alpha_j\frac{1}{n_j}\right]v \Leftrightarrow \frac{1}{2}(1-\alpha_j)\frac{1}{n_j\gamma_j}v \geq 1 \Leftrightarrow$$

$$\Leftrightarrow (1-\alpha_j) \geq \frac{2n_j\gamma_j}{v} \Leftrightarrow \alpha_j \leq 1 - \frac{2n_j\gamma_j}{v} \ .$$

Suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\alpha_j \frac{1}{n_j}v \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$\min\{\mathbf{x}_j\} = 1 > \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{n_j}v - 1;$$

agent ij has no incentive to deviate if and only if

$$\frac{1}{2}\alpha_j\frac{1}{n_j}v \geq \frac{1}{n_j}v - 1 \Leftrightarrow \alpha_j \geq \frac{2n_j}{v}\left(\frac{1}{n_j}v - 1\right) \Leftrightarrow \alpha_j \geq 2\left(1 - \frac{n_j}{v}\right).$$

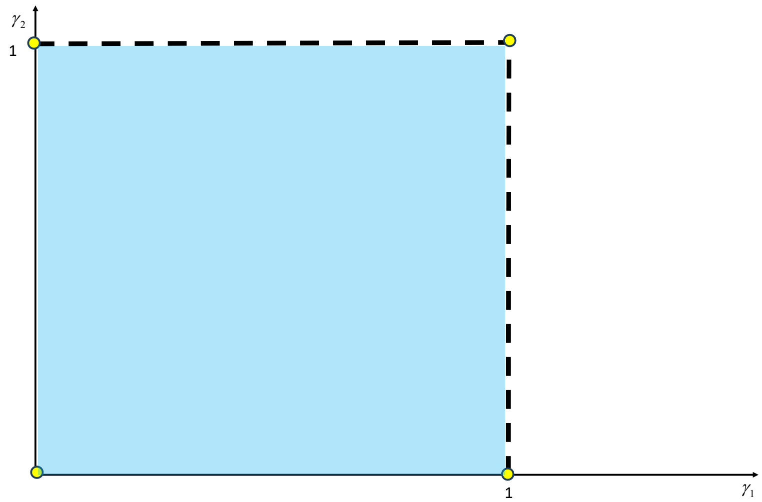

Figure 1: Geometric representation of the set of second-period pure strategy Nash equilibria.

Note that $2\left(1-\frac{n_j}{v}\right)v \leq 1-\frac{2n_j\gamma_j}{v}$ if and only if $v \leq 2$. Thus,

$$(\gamma_1, \gamma_2) \ \ such \ \ that \ \ \gamma_j \in (0,1) \,\text{with} \sum_{i=1}^{n_j} x_{ij} = n_j - 1$$

is a Nash equilibrium if and only if

$$\alpha_j \in \left[2\left(1-\frac{n_j}{v}\right), 1-\frac{2n_j\gamma_j}{v}\right] \ \ \forall\ v \leq 2 \ .$$

Note that strategy profiles $(\gamma_j, \gamma_{-j}) \in (0,1) \times \{1\}$ with $\sum_{i=1}^{n_j} x_{ij} \in \{1, \ldots, n_j - 1\}$ can be easily shown not be pure strategy Nash equilibria in effort stage.

■

Geometrically, if we represent in the space

$$(\gamma_1, \gamma_2) \in [0,1] \times [0,1]$$

the set of second-stage pure strategy Nash equilibria, depending on groups sizes, we cover almost all points corresponding to rational numbers in the unitary square, apart from the two segments

$$(\gamma_1, \gamma_2) \in \{1\} \times (0,1) \quad \text{and} \quad (\gamma_1, \gamma_2) \in (0,1) \times \{1\} \,,$$

as represented in figure 1.

For each of these classes of equilibria, we can compute the continuation payoffs, however the possible combinations of continuation payoffs associated to the possible values of the sharing rules would involve a huge amount of possible combinations: even if it would be computationally feasible to characterize the set of subgame perfect Nash equilibria in pure strategies, it would involve hundreds of billions of iterations as shown in figure 6, so that we decided to limit ourself to the case of within-group symmetric (WGS) second stage equilibria.

3.2 The Set of Pure Strategy Within-Group Symmetric Nash Equilibria in the Second Period.

The following corollary follows immediately from proposition 1.

Corollary 1 *In the BMMGC, the set of the second period WGS pure strategy Nash equilibria of the game is*

1. *if* $v \geq 2\max\{n_1, n_2\}$, *then*

$$(\gamma_1^*, \gamma_2^*) = (1,1) \;\; \textit{for any} \;\; (\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R};$$

2. *if* $v > 0$, *then*

$$(\gamma_1^{**}, \gamma_2^{**}) = (0,0) \;\; \textit{for any} \;\; (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1-1)\,v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2-1)\,v}, \infty\right)$$

3. *if* $v > 0$, *then*

$$\left(\gamma_j^{***}, \gamma_{-j}^{***}\right) = (1,0) \;\; \textit{for any} \;\; (\alpha_j, \alpha_{-j}) \in \left(-\infty, 2\left(1 - \frac{n_j}{v}\right)\right] \times \mathbb{R}.$$

The following result is useful to derive the subgame perfect equilibria of the game.

Corollary 2 *When*

$$v < \frac{2n_j\,(n_j-2)}{n_j-1}$$

for some $j \in \{1,2\}$, *then there exist a region of sharing rules* (α_1, α_2) *such that there exists no second stage pure strategy WGS equilibrium.*

Proof. From the previous result, it is immediate that when

$$1 - \frac{2n_j}{(n_j-1)\,v} > 2\left(1 - \frac{n_j}{v}\right) \quad \text{and} \quad v < 2 \cdot max\{n_1, n_2\}$$

there exists a region of sharing rules (α_1, α_2) such that there exists no second stage pure strategy WGS equilibrium. Note that

$$1 - \frac{2n_j}{(n_j-1)\,v} > 2\left(1 - \frac{n_j}{v}\right) \Leftrightarrow 1 - \frac{2n_j}{(n_j-1)\,v} > 2 - \frac{2n_j}{v} \Leftrightarrow$$

$$\Leftrightarrow \frac{2n_j}{v} - \frac{2n_j}{(n_j-1)\,v} > 1 \Leftrightarrow 2n_j^2 - 4n_j > (n_j-1)\,v \Leftrightarrow$$

$$\Leftrightarrow v < \frac{2n_j\,(n_j-2)}{n_j-1}.$$

Since

$$\frac{2n_j\,(n_j-2)}{n_j-1} < 2n_j$$

then

$$1 - \frac{2n_j}{(n_j-1)\,v} > 2\left(1 - \frac{n_j}{v}\right) \Rightarrow v < 2 \cdot \max\{n_1, n_2\}.$$

■

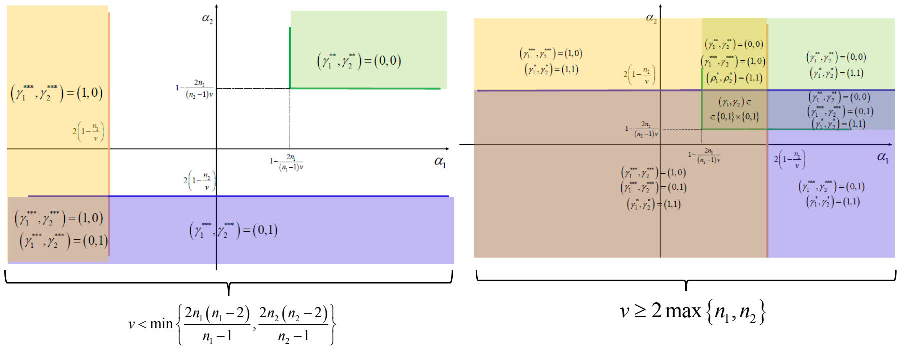

Figure 2: Non existence and multiple WGS equilibria.

Remark 1 *This result means that there are only two possible case for the second stage, either multiple pure strategy WGS equilibria when*

$$v \geq \max\left\{\frac{2n_1(n_1-2)}{n_1-1}, \frac{2n_2(n_2-2)}{n_2-1}\right\},$$

or possible values of the sharing rules such that there exists no pure strategy WGS equilibrium. The following figure represents two possible (extreme) situations.

4 The Equilibrium Choice of the Sharing Rules in the First Period

To derive the optimal sharing rules, we consider their optimal choice in each group by an utilitarian ruler with payoff function

$$\pi_j^C(\alpha_j, \alpha_{-j}|\gamma_1^e, \gamma_2^e) = \sum_{i=1}^{n_j} \pi_{ij}^C(\alpha_j, \alpha_{-j}|\gamma_1^e, \gamma_2^e).$$

Consider the agents' and groups' continuation payoff associated to the different second period equilibria:

1. when

$$(\gamma_1^*, \gamma_2^*) = (1,1), \quad \textit{for any} \quad v \geq 2 \cdot max\{n_1, n_2\} \;\; \textit{and} \;\; (\alpha_1, \alpha_2) \in \mathbb{R}\times\mathbb{R}$$

then

$$\pi_{ij}^C(\alpha_j, \alpha_{-j}|\gamma_1^*, \gamma_2^*) = \frac{1}{2}\frac{1}{n_j}v - 1 \text{ and } \pi_j^C(\alpha_j, \alpha_{-j}|\gamma_1^*, \gamma_2^*) = \frac{v}{2} - n_j\ .$$

2. when

$$(\gamma_1^{**}, \gamma_2^{**}) = (0,0)\,, \quad for \;\; any \;\; \alpha_j \geq 1 - \frac{2n_j}{(n_j - 1)\,v} \;\; with \;\; j \in \{1,2\} \;.$$

then

$$\pi_{ij}^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{**}, \gamma_2^{**}\right) = \frac{1}{2}\frac{1}{n_j} v \;\; \text{and} \;\; \pi_j^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{**}, \gamma_2^{**}\right) = \frac{v}{2} \;.$$

3. when

$$(\gamma_1^{***}, \gamma_2^{***}) = (0,1)\,, \quad for \;\; any \;\; \alpha_j \in \mathbb{R} \;\; \text{and} \;\; any \;\; \alpha_{-j} \leq 2\left(1 - \frac{n_{-j}}{v}\right)$$

then

$$\pi_{ij}^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = 0 \;, \;\; \pi_{i-j}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = \frac{1}{n_{-j}} v - 1$$

$$\text{and} \;\; \pi_j^{C}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = 0 \;, \;\; \pi_{-j}\left(\alpha_j, \alpha_{-j} | \gamma_1^{***}, \gamma_2^{***}\right) = v - n_{-j} \;.$$

Note that in order to find the set of first-period equilibria the continuation payoffs for both groups have to be specified at each element of the Cartesian product of α_1 and α_2 sustaining the set of second-period equilibria for both groups, so that we obtain:

1. if $v \geq 2 \cdot max\,\{n_1, n_2\}$, there are 7776 continuation-payoffs matrices;
2. if $v < 2 \cdot max\,\{n_1, n_2\}$ and $1 - \frac{2n_j}{(n_j-1)v} < 2\left(1 - \frac{n_j}{v}\right)$ with $j \in \{1,2\}$, there are 96 continuation-payoffs matrices;
3. if $v < 2 \cdot max\,\{n_1, n_2\}$ and $1 - \frac{2n_j}{(n_j-1)v} < 2\left(1 - \frac{n_j}{v}\right)$ and $1 - \frac{2n_{-j}}{(n_{-j}-1)v} = 2\left(1 - \frac{n_{-j}}{v}\right)$ with $j \in \{1,2\}$, there are 96 continuation-payoffs matrices;
4. if $v < 2 \cdot max\,\{n_1, n_2\}$ and $1 - \frac{2n_j}{(n_j-1)v} = 2\left(1 - \frac{n_j}{v}\right)$ for any $j \in \{1,2\}$, there are 96 continuation-payoffs matrices.

On the other hand, if $1 - \frac{2n_j}{(n_j-1)v} > 2\left(1 - \frac{n_j}{v}\right) \Leftrightarrow v < \frac{2n_j(n_j-2)}{n_j-1}$, the continuation payoffs cannot be pinned down for some $(\alpha_1, \alpha_2) \in \mathbb{R} \times \mathbb{R}$, so that there exists no optimal sharing rule and thus no pure strategy subgame perfect equilibria.

As reported in figures 5 and 6, the cardinality of the set of continuation-payoffs matrices is obtained by taking the Cartesian product of the number of second-stage equilibria in each interval, determined by the equilibrium thresholds, over the (α_1, α_2) space. Given the magnitude of the cardinality of the set of continuation-payoffs matrices, even for the WGS case, we employ a simple recursive algorithm, written in MATLAB and reported in the Appendix, to compute the optimal sharing rules. Thus, we might conclude with the following result.

Proposition 2 *In the BMMGC, in the first period, there is a continuum of optimal sharing rules such that:*

- *if* $v \geq 2 \cdot max\,\{n_1, n_2\}$
$$(\alpha_1^*, \alpha_2^*) \in \mathbb{R} \times \mathbb{R};$$

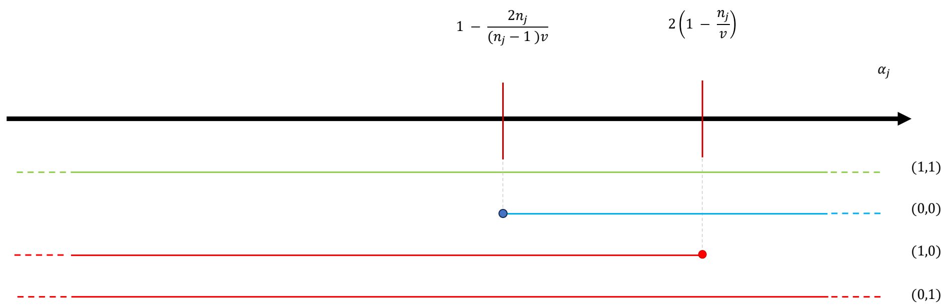

Figure 3: Intervals of α_j sustaining within-group symmetric second-period equilibria $\forall v \geq 2 \cdot max\{n_j, n_{-j}\}$

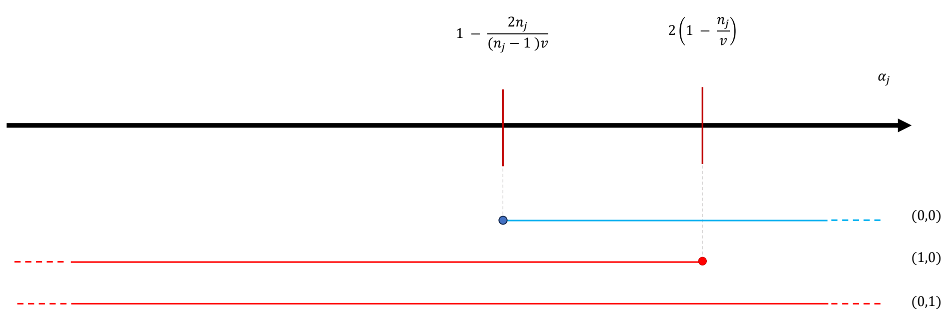

Figure 4: Intervals of α_j sustaining within-group symmetric second-period equilibria $\forall v \geq 2 \cdot max\{n_j, n_{-j}\}$

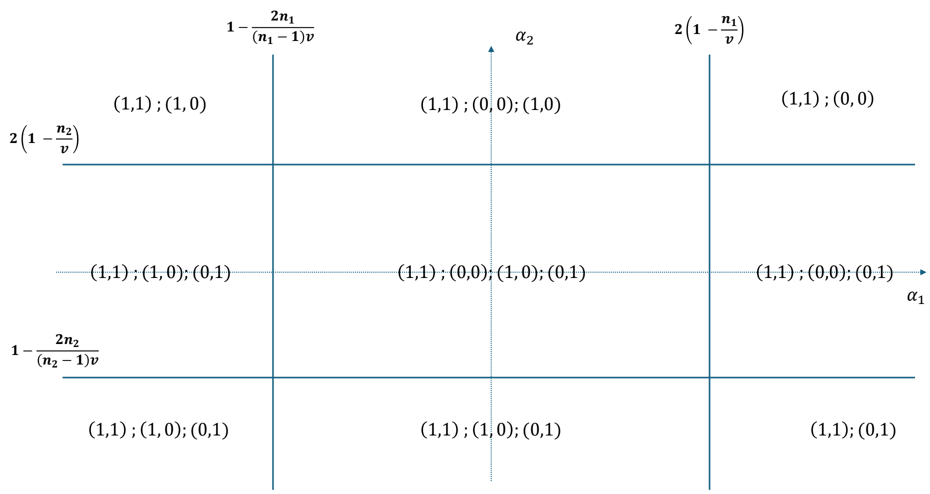

Figure 5: Second-period WGS pure Nash equilibria in $\alpha_1 \times \alpha_2$ space $\forall v \geq 2 \cdot max\{n_j, n_{-j}\}$. Note that the number of continuation-payoffs matrices is $2 \times 3 \times 2 \times 3 \times 4 \times 3 \times 3 \times 3 \times 2 = 7776$.

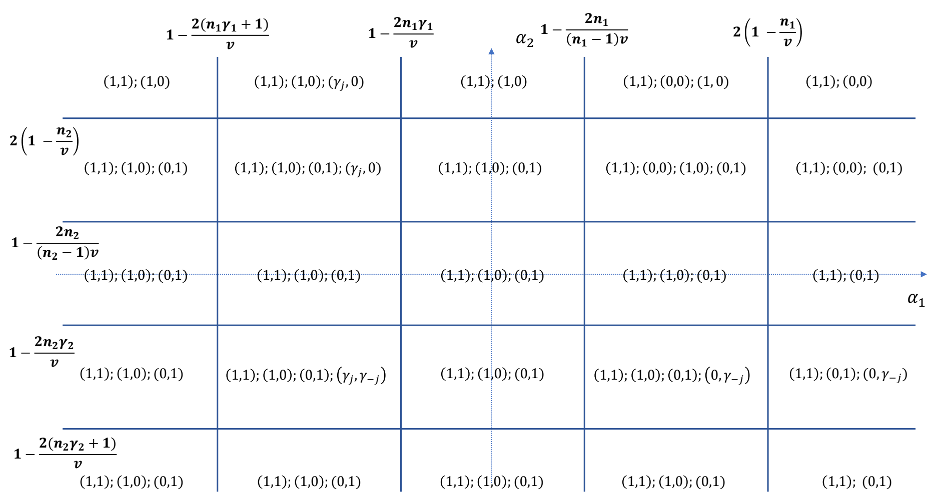

Figure 6: Second-period pure Nash equilibria in $\alpha_1 \times \alpha_2$ space $\forall v \geq 2 \cdot max\{n_j, n_{-j}\}$. Note that for just one asymmetric equilibrium (γ_j, γ_{-j}), the number of continuation-payoffs matrices is $2 \times 3 \times 2 \times 3 \times 2 \times 3 \times 4 \times 3 \times 4 \times 3 \times 3 \times 3 \times 3 \times 3 \times 2 \times 3 \times 4 \times 3 \times 4 \times 3 \times 3 \times 3 \times 3 \times 3 \times 2 = 352,640,000,000$.

- *if* $\frac{2\cdot max\{n_1,n_2\}\cdot(max\{n_1,n_2\}-2)}{max\{n_1,n_2\}-1} \leq v < 2\cdot max\{n_1,n_2\}$

$$(\alpha_1^*,\alpha_2^*)\subset\mathbb{R}\times\mathbb{R};$$

- *otherwise*

$$(\alpha_1^*,\alpha_2^*)\equiv\emptyset\times\emptyset\ .$$

Remark 2 *Proposition 2 means that the set of optimal sharing rules computed over all continuation-payoffs matrices coincides with* $\mathbb{R}\times\mathbb{R}$ *for a sufficiently high prize, that is when* $v\geq 2\cdot max\{n_1,n_2\}$*, the condition sustaining the most competitive second-period WGS pure Nash equilibrium* $(\gamma_j,\gamma_{-j})=(1,1)$*.*

5 The Set of Subgame Perfect Equilibria

From the previous results, it is computationally straightforward to derive the following one.

Proposition 3 *In the BMMGC, there is a continuum of within-group symmetric subgame perfect Nash equilibria in pure strategies.*

Remark 3 *Even though the characterization of within-group asymmetric subgame perfect Nash equilibria is computationally demanding, as discussed in the previous section, it is straightforward to state that in the BMMGC there is a continuum of within-group asymmetric subgame perfect Nash equilibria in pure strategies as well.*

Remark 4 *This results means that in BMMGC there is indeterminacy, in the sense that in equilibrium anything is possible.*

As an illustration, we provide a few examples of pure strategy within-group symmetric Subgame Perfect equilibria:

1. if $v>2\max\{n_1,n_2\}$,

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)\in\mathbb{R}\times\mathbb{R}$$
$$\left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(1,1)\ \ \forall(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R}$$

with

$$(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R}\ \text{ and }\ \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(1,1)$$

as equilibrium outcomes;

2. if $v>2n_1$ and $n_2<v<2n_2$ such that $1-\frac{2n_2}{(n_2-1)v}<2\left(1-\frac{n_2}{v}\right)$,

$$\left(\alpha_1^{SGP},\alpha_2^{SGP}\right)\in\left(-\infty,1-\frac{2n_1}{(n_1-1)\,v}\right)\times\mathbb{R}$$

$$\begin{cases}\left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(1,0) & \text{if } (\alpha_1,\alpha_2)\in\left(-\infty,1-\frac{2n_1}{(n_1-1)v}\right)\times\mathbb{R}\\ \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(1,0) & \text{if } (\alpha_1,\alpha_2)\in\left[1-\frac{2n_1}{(n_1-1)v},2\left(1-\frac{n_1}{v}\right)\right]\times\left(-\infty,1-\frac{2n_2}{(n_2-1)v}\right)\\ \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(0,0) & \text{if } (\alpha_1,\alpha_2)\in\left[1-\frac{2n_1}{(n_1-1)v},\infty\right)\times\left[1-\frac{2n_2}{(n_2-1)v},\infty\right)\\ \left(\gamma_1^{SGP},\gamma_2^{SGP}\right)=(0,1) & \text{if } (\alpha_1,\alpha_2)\in\left[2\left(1-\frac{n_1}{v}\right),\infty\right)\times\left[-\infty,1-\frac{2n_2}{(n_2-1)v}\right)\end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 1 - \frac{2n_1}{(n_1-1)\,v}\right) \times \mathbb{R} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0)$$

as equilibrium outcomes;

3. if $n_1 < v < 2n_1$ and $n_2 < v < 2n_2$ such that $1 - \frac{2n_1}{(n_1-1)v} = 2\left(1 - \frac{n_1}{v}\right)$ and $1 - \frac{2n_2}{(n_2-1)v} < 2\left(1 - \frac{n_2}{v}\right)$,

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in$$

$$\left[1 - \frac{2n_1}{(n_1-1)\,v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right)$$

$$\begin{cases} \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) & \text{if } (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1-1)v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } (\alpha_1, \alpha_2) \in \left(-\infty, 1 - \frac{2n_1}{(n_1-1)v}\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1) & \text{if } (\alpha_1, \alpha_2) \in \mathbb{R} \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right] \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left[1 - \frac{2n_1}{(n_1-1)\,v}, \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0)$$

as equilibrium outcomes;

4. if $n_1 < v < 2n_1$ and $n_2 < v < 2n_2$ such that $1 - \frac{2n_j}{(n_j-1)v} = 2\left(1 - \frac{n_j}{v}\right) \forall j \in \{1,2\}$,

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right] \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right)$$

$$\begin{cases} \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1) & \text{if } (\alpha_1, \alpha_2) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right] \times \mathbb{R} \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,0) & \text{if } (\alpha_1, \alpha_2) \in \left[2\left(1 - \frac{n_1}{v}\right), \infty\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \\ \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (1,0) & \text{if } (\alpha_1, \alpha_2) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right) \times \left(2\left(1 - \frac{n_2}{v}\right), \infty\right) \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \left(-\infty, 2\left(1 - \frac{n_1}{v}\right)\right] \times \left(-\infty, 2\left(1 - \frac{n_2}{v}\right)\right) \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0,1)$$

as equilibrium outcomes.

Remark 5 *The examples above can be rewritten making intervals for (α_1^*, α_2^*) open or closed, depending on which second-period equilibrium is assumed to be played at the internal thresholds $1 - \frac{2n_j}{(n_j-1)v}, 2\left(1 - \frac{n_j}{v}\right)$, $\forall j = 1, 2$.*

6 Extension to K Effort Levels

In this section we try to address some potential questions arising from the results obtained under Proposition 1, 2 and 3, which are in close relation with the assumptions under our binary group contest. In particular, we expand the set of second-stage actions from the binary case to subset of the natural numbers with cardinality at least equal to three.

Definition 2 *A* K-Actions Max-Min Group Contest $KMMGC$ *is a two stages game* $KMMGC = \langle \{1,2\}, N, S_j, K_{ij}, \pi_{ij} \rangle$ *defined by*

1. *the set of groups* $\{1,2\}$;
2. *the set of players* $N = \{1,...,n_1+n_2\}$;
3. *the set of first period actions* $S_j = \mathbb{R}$: *for each group* j, *the choice of the share* α_j *in the sharing rule*;
4. *the set of second period actions* $K_{ij} = \{0,\ldots,K\}$, *with* $K_{ij} \subset \mathbb{N}$ *and* $K \geq 2$: *for each player* ij, *the choice of the bid/effort* $x_j(i)$;
5. *the payoff functions for each player* $ij \in N$

$$\pi_{ij}(\boldsymbol{\alpha}, \mathbf{x}) = p_j q_{ij} v - x_j(i) =$$

$$= \begin{cases} \left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)} + \alpha_j \frac{1}{n_j}\right] v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}$$

where $\boldsymbol{\alpha}$ *and* $\mathbf{x}$ *are, respectively, the vector of first and second period actions.*

Proposition 4 *In the* $KMMGC$, *for any* $\alpha_j \in \mathbb{R}$, *the set of the second-period pure strategy Nash equilibria of the game is characterized as follows:*

1. *if* $v > 0$, *then*

$$(\gamma_j, \gamma_{-j}) = (0, k) \;\; with\; k \in \{1,\ldots,K\} \;\; and\; \mathbf{x}_{-j} = \mathbf{k}$$ [2]

for any

$$(\alpha_j, \alpha_{-j}) \in \begin{cases} \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right)\right] & if\; v < n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \\ \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{(n_{-j}-1)v}\right] & if\; v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \end{cases}$$

2. *if* $v > 0$, *then*

$$(\gamma_1, \gamma_2) = (0, 0)$$

for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1}{(n_1-1)v}, \infty\right) \times \left[1 - \frac{2n_2}{(n_2-1)v}, \infty\right);$$

3. *if* $v \geq 2 \cdot max\{n_1\gamma_1, n_2\gamma_2\}$, *then*

$$(\gamma_1, \gamma_2) = (k, k) \;\; with\; k \in \{1,\ldots,K\} \;\; and\; \mathbf{x}_j = \mathbf{k}$$

for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1\gamma_1}{v} - \frac{2n_1(\gamma_1+1)}{(n_1-1)v}, \infty\right) \times \left[1 - \frac{2n_2\gamma_2}{v} - \frac{2n_2(\gamma_2+1)}{(n_2-1)v}, \infty\right);$$

[2] $\mathbf{x}_{-j} = \mathbf{k}$ with $k \in \{1,\ldots,K\}$ means that all agents in group $-j$ exert the same level of effort k.

4. *if* $v > 2 \cdot max\{n_1\gamma_1, n_2\gamma_2\}$, *then*

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (1, K),\ min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0, 2 \le \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \le n_j - 1$$ [3]

for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1\gamma_1(n_j\gamma_1 + 1)}{(n_1\gamma_1 - min\{\mathbf{x}_1\})v}, 1 - \frac{2n_1\gamma_1(n_1\gamma_1 - 1)}{(n_1\gamma_1 - min\{\mathbf{x}_1\} - 1)v}\right] \times \left[1 - \frac{2n_2\gamma_2(n_2\gamma_2 + 1)}{(n_2\gamma_2 - min\{\mathbf{x}_2\})v}, 1 - \frac{2n_2\gamma_2(n_2\gamma_2 - 1)}{(n_2\gamma_2 - min\{\mathbf{x}_2\} - 1)v}\right],$$
$$max\{\mathbf{x}_1\} = min\{\mathbf{x}_1\} + 1 \text{ and } max\{\mathbf{x}_2\} = min\{\mathbf{x}_2\} + 1;$$

5. *if* $v > 2 \cdot max\{n_1\gamma_1, n_2\gamma_2\}$, *then*

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = k \in \{1, \ldots, K-1\} \text{ and } \mathbf{x}_j = \mathbf{k}, \gamma_{-j} \in (1, K),$$
$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} > 0, 2 \le \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=min\{\mathbf{x}_{-j}\}} \le n_{-j} - 1$$

for any

$$(\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j(\gamma_j + 1)}{(n_j - 1)v}, \infty\right) \times \left[1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} + 1)}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v}, 1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} - 1)}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\} - 1)v}\right],$$
$$max\{\mathbf{x}_{-j}\} = min\{\mathbf{x}_{-j}\} + 1;$$

6. *if* $v > 0$, *then*

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, 1), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } 2 \le \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \le n_{-j} - 2$$ [4]

for any

$$(\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2n_j}{(n_j - 1)v}, \infty\right) \times \left[1 - \frac{2(n_{-j}\gamma_{-j} + 1)}{v}, 1 - \frac{2n_{-j}\gamma_{-j}}{v}\right];$$

7. *if* $v > 0$, *then*

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, 1) \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1$$

for any

$$(\alpha_j, \alpha_{-j}) \in \left[1 - \frac{2n_j}{(n_j - 1)v}, \infty\right] \times \left[1 - \frac{2(n_{-j}\gamma_{-j} + 1)}{v}, 1 - \frac{2n_{-j}}{(n_{-j} - 1)v}\right];$$

[3] Note that $\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}$ stands for the Indicator function taking value 1 when $x_{ij} = min\{\mathbf{x}_j\}$ for any ij.

[4] Note that $\mathbb{1}_{x_{ij}=0}$ stands for the Indicator function taking value 1 when $x_{ij} = 0$ for any ij.

8. *if* $v>0$, *then*

$$(\gamma_j,\gamma_{-j})\ \textit{such that}\ \gamma_j\in(0,1)\,,min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}=0\ \textit{and}\ 2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}\leq n_j-2$$

for any

$$(\alpha_1,\alpha_2)\in\left[1-\frac{2(n_1\gamma_1+1)}{v},1-\frac{2n_1\gamma_1}{v}\right]\times\left[1-\frac{2(n_2\gamma_2+1)}{v},1-\frac{2n_2\gamma_2}{v}\right];$$

9. *if* $v>0$, *then*

$$(\gamma_1,\gamma_2)\ \textit{such that}\ \gamma_j\in(0,1)\ \textit{and}\ \sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}=n_j-1$$

for any

$$(\alpha_1,\alpha_2)\in\left[1-\frac{2\left(n_1\gamma_1+1\right)}{v},1-\frac{2n_1}{\left(n_1-1\right)v}\right]\times\left[1-\frac{2\left(n_2\gamma_2+1\right)}{v},1-\frac{2n_2}{\left(n_2-1\right)v}\right].$$

Proof.

1. Suppose
$$(\gamma_j,\gamma_{-j})=(0,k)\ \text{ with } k\in\{1,\ldots,K\}\ \text{ and } \mathbf{x}_{-j}=\mathbf{k}.$$

 Then

$$x_j\left(i\right)=\min\left\{\mathbf{x}_j\right\}=0<\min\left\{\mathbf{x}_{-j}\right\}=k\ \text{and}\ \sum_{i=1}^{n_j}x_j\left(i\right)=0,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}k\ ,$$

 so that
$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=0\ .$$
 If agent ij deviates to $x_j\left(i\right)=k$, $\forall k\in\{1,\ldots,K\}$, then

$$\min\left\{\mathbf{x}_j\right\}=0<\min\left\{\mathbf{x}_{-j}\right\}=k\ \text{and}\ \sum_{i=1}^{n_j}x_j\left(i\right)=k,\sum_{i=1}^{n_{-j}}x_j\left(i\right)=n_{-j}k$$

 so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=-k\ .$$ [5]

 Hence, for any player ij there is no incentive to deviate. On the other hand,

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{n_{-j}}v-k\ .$$

[5] We will denote $\gamma_j^{'}=\frac{1}{n_j}\left(\sum_{i=1}^{n_j}x_{ij}-k+k'\right)\in[0,K]$ the average group effort at any deviation, both upwards and downwards.

If agent $i-j$ deviates to any $k' \in \{1,\ldots,K\}-\{k\}$, then

$$min\{\mathbf{x}_j\} = 0 < min\{\mathbf{x}_{-j}\} \in \{1,\ldots,K\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0, \sum_{i=1}^{n_{-j}} x_{-j}(i) = (n_{-j}-1)k + k'$$

so that the deviation payoff is

$$\pi_{i-j}^D(\gamma_j, \gamma'_{-j}|\boldsymbol{\alpha}) = \left[(1-\alpha_{-j})\frac{k'}{(n_{-j}-1)k+k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \le \frac{1}{n_{-j}}v - k \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \ge 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}k'}{(n_{-j}-1)v} & \text{if } k' > k \\ \alpha_{-j} \le 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}k'}{(n_{-j}-1)v} & \text{if } k' < k \end{cases}$$

Since the two conditions above have to be valid for any $k' > k$ and any $k' < k \in \{1,\ldots,K\}-\{k\}$, respectively, we get

$$\alpha_{-j} \in \left[1 - \frac{n_{-j}k}{v} - \frac{n_{-j}(k+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}k}{v} - \frac{n_{-j}(k-1)}{(n_{-j}-1)v}\right] .$$

Conversely, if agent $i-j$ deviates to $k' = 0$, then

$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0, \sum_{i=1}^{n_{-j}} x_{-j}(i) = (n_{-j}-1)k$$

so that

$$\pi_{i-j}^D(\gamma_j, \gamma'_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \le \frac{1}{n_{-j}}v - k \Leftrightarrow \alpha_{-j} \le 2\left(1 - \frac{n_{-j}k}{v}\right) .$$

Note that

$$2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right) \ge 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{(n_{-j}-1)v} \Leftrightarrow v \ge n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1} .$$

Hence,

$$(\gamma_j, \gamma_{-j}) = (0, k) \text{ with } k \in \{1,\ldots,K\} \text{ and } \mathbf{x}_{-j} = \mathbf{k}$$

is a Nash equilibrium if and only if

$$(\alpha_j, \alpha_{-j}) \in \begin{cases} \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 2\left(1 - \frac{n_{-j}\gamma_{-j}}{v}\right)\right] & \text{if } v < n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \\ \mathbb{R} \times \left[1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}+1)}{(n_{-j}-1)v}, 1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}(\gamma_{-j}-1)}{(n_{-j}-1)v}\right] & \text{if } v \ge n_{-j}\gamma_{-j} - \frac{n_{-j}(\gamma_{-j}-1)}{n_{-j}-1}; \end{cases}$$

2. Suppose

$$(\gamma_1, \gamma_2) = (0, 0)$$

Then,

$$x_j(i) = min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\frac{1}{n_j}\right]v\ .$$

If agent ij deviates to $x_j\left(i\right)=k'=1$,[6] then

$$x_j\left(i\right)=min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\}\ \text{ and }\ \sum_{i=1}^{n_j}x_j\left(i\right)=k',\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=0$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{1}+\frac{\alpha_j}{n_j}\right]v-1\ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{v}{2n_j}\geq\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{1}+\frac{\alpha_j}{n_j}\right]v-1\Leftrightarrow\alpha_j\geq1-\frac{2n_j}{\left(n_j-1\right)v}\ .$$

3. Suppose

$$\left(\gamma_1,\gamma_2\right)=\left(k,k\right)\ \text{ with } k\in\left\{1,\ldots,K\right\}\ \text{ and }\ \mathbf{x}_j=\mathbf{k}\ .$$

Then,

$$x_j\left(i\right)=min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\},\sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j\ \text{ and }\ \sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\frac{1}{n_j}\right]v-k\ .$$

If agent ij deviates to $x_j\left(i\right)=k'=0$, then

$$x_j\left(i\right)=min\left\{\mathbf{x}_j\right\}<min\left\{\mathbf{x}_{-j}\right\}and\sum_{i=1}^{n_j}x_j\left(i\right)=\left(n_j-1\right)\gamma_j\ \text{ and }\ \sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=0\leq\frac{v}{2n_j}-k\Leftrightarrow v\geq2n_jk\ .$$

Note that any deviation $0<k'<k$ is strictly payoff-dominated by $k'=0$, so that we can now take into account upward deviations only. If agent ij deviates to $x_j\left(i\right)=k'>k$, then

$$x_j\left(i\right)=min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\},\sum_{i=1}^{n_j}x_j\left(i\right)=\left(n_j-1\right)\gamma_j+k'\ \text{ and }\ \sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

[6] Note that any deviation to $k'>1$ would be strictly payoff-dominated by $k'=1$.

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma'_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k'}{\left(n_j-1\right)k+k'}+\frac{\alpha_j}{n_j}\right]v-k'\leq\frac{v}{2n_j}-k\Leftrightarrow$$

$$\Leftrightarrow\alpha_j\geq 1-\frac{2n_j\gamma_j}{v}-\frac{2n_j\left(\gamma_j+1\right)}{\left(n_j-1\right)v}\ .$$

Therefore,

$$\left(\gamma_1,\gamma_2\right)=\left(k,k\right)\ \text{with}\ k\in\{1,\ldots,K\}\ \text{and}\ \mathbf{x}_j=\mathbf{k}$$

is a Nash equilibrium for any

$$\alpha_j\in\left[1-\frac{2n_j\gamma_j}{v}-\frac{2n_j\left(\gamma_j+1\right)}{(n_j-1)v},\infty\right)\ \text{and}\ v\geq 2n_j\gamma_j\ .$$

4. Suppose

$$\left(\gamma_1,\gamma_2\right)\ \text{such that}\ \gamma_j\in\left(1,K\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}>0\ \text{and}\ 2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}\leq n_j-1\ .$$

Then, if $x_j\left(i\right)=k\in\left\{min\left\{\mathbf{x}_j\right\},\ldots,K\right\}\geq min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}>0$,

$$x_j\left(i\right)\geq\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}>0\ \text{and}\ \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-k\ .$$

If agent ij deviates to $k'\in\left\{min\left\{\mathbf{x}_j\right\},\ldots,K\right\}-\{k\}$, then

$$x_j\left(i\right)\geq\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}>0\ \text{and}\ \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j-k+k',\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma'_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{k'}{n_j\gamma_j-k+k'}+\frac{\alpha_j}{n_j}\right]v-k'\ .$$

In contrast, if agent ij deviates to $k'\in\left\{0,\ldots,min\left\{\mathbf{x}_j\right\}-1\right\}$, then

$$x_j\left(i\right)=\min\left\{\mathbf{x}_j\right\}<\min\left\{\mathbf{x}_{-j}\right\}\ \text{and}\ \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j-k+k',\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma'_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=-k'\ .$$

We select the payoff-dominant downward deviation, that is $k'=0$. Hence, for any player ij there is no incentive to deviate if and only if a.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-k\geq\frac{1}{2}\left[(1-\alpha_j)\frac{k'}{n_j\gamma_j-k+k'}+\frac{\alpha_j}{n_j}\right]v-k'\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_j\geq 1-\frac{2n_j\gamma_j\left(n_j\gamma_j-k+k'\right)}{(n_j\gamma_j-k)v} & \text{if } k'>k\\ \alpha_j\leq 1-\frac{2n_j\gamma_j\left(n_j\gamma_j-k+k'\right)}{(n_j\gamma_j-k)v} & \text{if } k'<k\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_j\geq 1-\frac{2n_j\gamma_j(n_j\gamma_j-min\{\mathbf{x}_j\}+min\{\mathbf{x}_j\}+1)}{(n_j\gamma_j-min\{\mathbf{x}_j\})v} & \text{if } k'>k\\ \alpha_j\leq 1-\frac{2n_j\gamma_j(n_j\gamma_j-max\{\mathbf{x}_j\}+max\{\mathbf{x}_j\}-1)}{(n_j\gamma_j-max\{\mathbf{x}_j\})v} & \text{if } k'<k\end{cases}$$

and

b.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-k\geq 0\Leftrightarrow\begin{cases}\alpha_j\geq\frac{k}{\gamma_j-k}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j>k\\ v\geq 2n_j\gamma_j & \gamma_j=k\\ \alpha_j\leq\frac{k}{\gamma_j-k}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j<k\end{cases}$$

Note that an α_j preventing upward and downward deviations can exist if and only if the lower and upper bounds at point *a.* do not cross, that is for $max\{\mathbf{x}_j\}=min\{\mathbf{x}_j\}+1$. Therefore, combining these two sets of conditions above we get that for any player ij there is no incentive to deviate if and only if

$$\alpha_j\in\left[1-\frac{2n_j\gamma_j\,(n_j\gamma_j+1)}{(n_j\gamma_j-min\{\mathbf{x}_j\})\,v},1-\frac{2n_j\gamma_j\,(n_j\gamma_j-1)}{(n_j\gamma_j-min\{\mathbf{x}_j\}-1)\,v}\right]\text{ and } v>2n_j\gamma_j\,.$$

Hence,

$$(\gamma_1,\gamma_2)\text{ such that }\gamma_j\in(1,K)\,,\,min\{\mathbf{x}_1\}=min\{\mathbf{x}_2\}>0\text{ and }2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}\leq n_j-1$$

is a Nash equilibrium for any

$$\alpha_j\in\left[1-\frac{2n_j\gamma_j\,(n_j\gamma_j+1)}{(n_j\gamma_j-min\{\mathbf{x}_j\})\,v},1-\frac{2n_j\gamma_j\,(n_j\gamma_j-1)}{(n_j\gamma_j-min\{\mathbf{x}_j\}-1)\,v}\right],$$

$$max\{\mathbf{x}_j\}=min\{\mathbf{x}_j\}+1\text{ and }v>2n_j\gamma_j\;.$$

5. Suppose

$$(\gamma_j,\gamma_{-j})\text{ such that }\gamma_j=k\in\{1,\ldots,K-1\}\text{ and }\mathbf{x}_j=\mathbf{k},\gamma_{-j}\in(1,K)\,,$$

$$min\{\mathbf{x}_j\}=min\{x_{-j}\}>0,2\leq\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}\leq n_j-1\;.$$

Then, the proof is a direct application of what shown at points 3. and 4., so that

$$(\gamma_j,\gamma_{-j})\text{ such that }\gamma_j=k\in\{1,\ldots,K-1\}\text{ and }\mathbf{x}_j=\mathbf{k},\gamma_{-j}\in(1,K)\,,$$

$$min\left\{\mathbf{x}_j\right\} = min\left\{x_{-j}\right\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1$$

is a Nash equilibrium for any

$$(\alpha_j, \alpha_{-j}) \in$$
$$\left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j\left(\gamma_j + 1\right)}{(n_j - 1)v}, \infty\right) \times$$
$$\left[1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} + 1\right)}{\left(n_{-j}\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\}\right)v}, 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - min\left\{\mathbf{x}_{-j}\right\} - 1\right)v}\right],$$
$$max\left\{\mathbf{x}_{-j}\right\} = min\left\{\mathbf{x}_{-j}\right\} + 1 \text{ and } v > 2 \cdot max\left\{n_j\gamma_j, n_{-j,\gamma_{-j}}\right\} .$$

6. Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in (0, K), min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} \text{ and } 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2 .$$

Then,

$$x_j\left(i\right) = \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{v}{n_j} .$$

If agent ij deviates to $x_j\left(i\right) = 1$[7], then

$$\min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j - k + 1, \sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi^D_{ij}\left(\gamma'_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1 - \alpha_j\right)\frac{1}{1} + \frac{\alpha_j}{n_j}\right]v - 1 .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[\left(1 - \alpha_j\right) + \frac{\alpha_j}{n_j}\right]v - 1 \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j}{(n_j - 1)v} .$$

On the other hand,

$$x_{-j}\left(i\right) \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j\left(i\right) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}\left(i\right) = n_{-j}\gamma_{-j}$$

[7] Any deviation $k' > 1$ would not deliver a greater payoff than the one attained at $k' = 1$.

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k\ .$$

If agent $i-j$ deviates to $x_{-j}\left(i\right)=k'\in\{0,\ldots,K\}-\{k\}$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}-k+k'$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}'|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k'}{n_{-j}\gamma_{-j}-k+k'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k'\ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k\geq\frac{1}{2}\left[\left(1-\alpha_{j}\right)\frac{k'}{n_{j}\gamma_{j}-k+k'}+\frac{\alpha_{j}}{n_{j}}\right]v-k'\Leftrightarrow$$

$$\Leftrightarrow\alpha_{-j}\geq1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-k+k'\right)}{\left(n_{-j}\gamma_{-j}-k\right)v}\qquad\forall k\in\{0,\ldots,K\}\text{ and }k'\in\{k+1,\ldots,K\}$$

$$\alpha_{-j}\leq1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-k+k'\right)}{\left(n_{-j}\gamma_{-j}-k\right)v}\qquad\forall k\in\{0,\ldots,K\}\text{ and }k'\in\{0,\ldots,k-1\}$$

Note that the bounds above are decreasing in both k and k', so that for the lower bound we set $k=0$ and $k'=1$, whereas for the upper bound we set $k=K$ and $k'=K-1$. However, the bounds do not cross if and only if $K=1$, so that $max\left\{\mathbf{x}_{-j}\right\}=1$. Therefore,

$$\left(\gamma_j,\gamma_{-j}\right)\text{ such that }\gamma_j=0,\gamma_{-j}\in\left(0,1\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}=0\text{ and }2\leq\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}\leq n_{-j}-2$$

is a Nash equilibrium for any

$$\left(\alpha_j,\alpha_{-j}\right)\in\left[1-\frac{2n_j}{(n_j-1)v},\infty\right)\times\left[1-\frac{2(n_{-j}\gamma_{-j}+1)}{v},1-\frac{2n_{-j}\gamma_{-j}}{v}\right]$$

7. Suppose

$$\left(\gamma_1,\gamma_2\right)\text{ such that }\gamma_j=0,\gamma_{-j}\in\left(0,K\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}\text{ and }\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}=n_{-j}-1\ .$$

Then,

$$x_j\left(i\right)=\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_j}\ .$$

If agent ij deviates to $x_j(i) = 1$,[8] then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - k + 1, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_j)\frac{1}{1} + \frac{\alpha_j}{n_j}\right]v - 1 \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[(1-\alpha_j) + \frac{\alpha_j}{n_j}\right]v - 1 \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j}{(n_j-1)v}$$

On the other hand, consider $x_{-j}(i) = 0$, that is

$$x_{-j}(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = k' \in \{1, \ldots, K\}$, then

$$\min\{\mathbf{x}_{-j}\} = \min\{\mathbf{x}_j\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} + k'$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}'|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{k'}{n_{-j}\gamma_{-j} + k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \ .$$

Hence, for any player $i-j$ such that $x_{i-j} = 0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \frac{1}{2}\left[(1-\alpha_{-j})\frac{k'}{n_{-j}\gamma_{-j} + k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \Leftrightarrow \alpha_{-j} \geq 1 - \frac{2\left(n_{-j}\gamma_{-j} + k'\right)}{v}$$

Given that the condition must hold for all $k' \in \{1, \ldots, K\}$, it must be such that

$$\alpha_{-j} \geq 1 - \frac{2\left(n_{-j}\gamma_{-j} + 1\right)}{v} \ .$$

Consider the unique player $i-j$ such that $x_{i-j} = k \in \{1, \ldots, K\}$, that is

$$x_{-j}(i) > min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

[8] Any deviation $k' > 1$ would not deliver a greater payoff than the one attained at $k' = 1$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{k}{k}+\frac{\alpha_{-j}}{n_{-j}}\right]v-k$$

It is straightforward to see that the deviation $k'=1$ strictly payoff-dominates any positive effort level k for player $i-j$. On the other hand, if player $i-j$ deviates to $x_{-j}\left(i\right)=0$, then

$$x_{-j}\left(i\right)=min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_j\left(i\right)=0,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=0$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}'|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_{-j}}\ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{1}{1}+\frac{\alpha_{-j}}{n_{-j}}\right]v-1\geq\frac{1}{2}\frac{v}{n_{-j}}\Leftrightarrow\alpha_{-j}\leq1-\frac{2n_{-j}}{\left(n_{-j}-1\right)v}$$

Therefore,

$$\left(0,\gamma_{-j}\right)\ \text{such that}\ \gamma_{-j}\in\left(0,1\right)\ \text{and}\ \sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}=n_{-j}-1$$

is a Nash equilibrium if and only if

$$\left(\alpha_j,\alpha_{-j}\right)\in\left[1-\frac{2n_j}{\left(n_j-1\right)v},\infty\right)\times\left[1-\frac{2\left(n_{-j}\gamma_{-j}+1\right)}{v},1-\frac{2n_{-j}}{\left(n_{-j}-1\right)v}\right]\ .$$

8. Suppose

$$\left(\gamma_j,\gamma_{-j}\right)\ \text{such that}\ \gamma_j\in\left(0,1\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}=0\ \text{and}\ 2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}\leq n_j-2\ .$$

Then, the proof follows the corresponding arguments shown at point 6., so that

$$\left(\gamma_j,\gamma_{-j}\right)\ \text{such that}\ \gamma_j\in\left(0,1\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}=0\ \text{and}\ 2\leq\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}\leq n_j-2$$

is a Nash equilibrium for any

$$\left(\alpha_1,\alpha_2\right)\in\left[1-\frac{2(n_1\gamma_1+1)}{v},1-\frac{2n_1\gamma_1}{v}\right]\times\left[1-\frac{2(n_2\gamma_2+1)}{v},1-\frac{2n_2\gamma_2}{v}\right]\ .$$

9. Suppose

$$\left(\gamma_1,\gamma_2\right)\ \text{such that}\ \gamma_j\in\left(0,K\right)\ \text{and}\ \sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=0}=n_j-1\ .$$

Then, the proof follows the corresponding arguments shown at point 7., so that

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j - 1$$

is a Nash equilibrium for any

$$(\alpha_1, \alpha_2) \in \left[1 - \frac{2(n_1\gamma_1 + 1)}{v}, 1 - \frac{2n_1}{(n_1 - 1)v}\right] \times \left[1 - \frac{2(n_2\gamma_2 + 1)}{v}, 1 - \frac{2n_2}{(n_2 - 1)v}\right] .$$

Finally, note that similar arguments prove that the following strategy profiles are not second-period Nash equilibria in pure strategies:

i. (γ_1, γ_2) such that $\gamma_j \in (1, K)$, $min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1$.

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (1, K), min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 .$$

Then, if $x_j(i) = k \in \{min\{\mathbf{x}_j\} + 1, \ldots, K\} > min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$,

$$x_j(i) > \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1 - \alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k .$$

If agent ij deviates to $k' \in \{min\{\mathbf{x}_j\}, \ldots, K\} - \{k\}$, then

$$x_j(i) \geq \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1 - \alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' .$$

In contrast, if $x_j(i) = k \in \{min\{\mathbf{x}_j\}, \ldots, K\} \geq min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$, and agent ij deviates to $k' \in \{0, \ldots, min\{\mathbf{x}_j\} - 1\}$, then

$$x_j(i) = \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}) = -k' .$$

We select the payoff-dominant downward deviation, that is $k' = 0$. On the other hand, if the unique agent ij exerting effort $k = min\{\mathbf{x}_j\}$ deviates to $k' \in \{min\{\mathbf{x}_j + 1\}, \ldots, K\}$, then

$$x_j(i) \geq \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - k + k', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j', \gamma_{-j}|\boldsymbol{\alpha}\right) = \left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' .$$

Hence, for any player ij there is no incentive to deviate if and only if

a.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq \frac{1}{2}\left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - k + k'\right)}{(n_j\gamma_j - k)v} & \text{if } k' > k \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - k + k'\right)}{(n_j\gamma_j - k)v} & \text{if } k' < k \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + 1)}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } k' > k \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} - 1 + min\{\mathbf{x}_j\})}{(n_j\gamma_j - min\{\mathbf{x}_j\} - 1)v} & \text{if } k' < k \end{cases}$$

b.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq 0 \Leftrightarrow \begin{cases} \alpha_j \geq \frac{k}{\gamma_j - k}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j > k \\ v \geq 2n_j\gamma_j & \gamma_j = k \\ \alpha_j \leq \frac{k}{\gamma_j - k}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j < k \end{cases}$$

c.

$$\frac{1}{2}\left[(1-\alpha_j)\frac{k}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - k \geq \left[(1-\alpha_j)\frac{k'}{n_j\gamma_j - k + k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \lesseqgtr \frac{(2n_j\gamma_j k' - k(n_j\gamma_j - k + k'))v + (k - k')2n_j\gamma_j(n_j\gamma_j - k + k')}{(2n_j\gamma_j k' - (\gamma_j + k)(n_j\gamma_j - k + k'))v}$$

Note that the bounds at point *a.* are decreasing in both k and k', so that for the lower bound we set $k = min\{\mathbf{x}_j\}$ and $k' = min\{\mathbf{x}_j\} + 1$, whereas for the upper bound we set $k = \mathbf{x}_j$ and $k' = \mathbf{x}_j - 1$. However, the bounds do not cross if and only if $max\{\mathbf{x}_j\} = min\{\mathbf{x}_j\} + 1$. Consequently, we can substitute these values in the condition at point *c.* which holds for the unique player ij exerting effort $k = min\{\mathbf{x}_j\}$ and, for $v > 2n_j\gamma_j$ as resulting from conditions a. and b., we obtain

$$\alpha_j \geq \frac{(2n_j\gamma_j(min\{\mathbf{x}_j\} + 1) - min\{\mathbf{x}_j\}(n_j\gamma_j + 1))v - 2n_j\gamma_j(n_j\gamma_j + 1)}{(2n_j\gamma_j(min\{\mathbf{x}_j\} + 1) - (\gamma_j + min\{\mathbf{x}_j\})(n_j\gamma_j + 1))v} > 1$$

However, it follows

$$\left[1-\frac{2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(n_j\gamma_j-min\left\{\mathbf{x}_j\right\}\right)v},1-\frac{2n_j\gamma_j\left(n_j\gamma_j-1\right)}{\left(n_j\gamma_j-min\left\{\mathbf{x}_j\right\}-1\right)v}\right]\cap$$

$$\cap\left[\frac{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right)-min\left\{\mathbf{x}_j\right\}\left(n_j\gamma_j+1\right)\right)v-2n_j\gamma_j\left(n_j\gamma_j+1\right)}{\left(2n_j\gamma_j\left(min\left\{\mathbf{x}_j\right\}+1\right)-\left(\gamma_j+min\left\{\mathbf{x}_j\right\}\right)\left(n_j\gamma_j+1\right)\right)v},\infty\right)=\emptyset.$$

Therefore,

$$\left(\gamma_1,\gamma_2\right)\text{ such that }\gamma_j\in\left(1,K\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}>0\text{ and }\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=min\left\{\mathbf{x}_j\right\}}=1$$

is not a Nash equilibrium for any

$$\left(\alpha_1,\alpha_2\right)\in\mathbb{R}\times\mathbb{R}$$

ii. $\left(\gamma_j,\gamma_{-j}\right)$ such that $\gamma_j=k\in\left\{1,\ldots,K-1\right\}$ and $\mathbf{x}_j=\mathbf{k},\gamma_{-j}\in\left(1,K\right)$, $min\left\{\mathbf{x}_j\right\}=min\left\{\mathbf{x}_{-j}\right\}>0$ and $\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=min\left\{\mathbf{x}_{-j}\right\}}=1$.
Then, the proof directly follows from what shown at point i.

iii. $\left(\gamma_j,\gamma_{-j}\right)$ such that $\gamma_j=0$ and $\gamma_{-j}\in\left(0,K\right)$ and $\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}=1$.
Suppose

$$\left(\gamma_1,\gamma_2\right)\text{ such that }\gamma_j=0,\gamma_{-j}\in\left(0,K\right),min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}\text{ and }\mathbb{1}_{x_{i-j}=0}=1\ .$$

Then,

$$x_j\left(i\right)=\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_j}\ .$$

If agent ij deviates to $x_j\left(i\right)=1$,[9] then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0\text{ and }\sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j-k+1,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j',\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{1}{1}+\frac{\alpha_j}{n_j}\right]v-1\ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j}\geq\frac{1}{2}\left[\left(1-\alpha_j\right)+\frac{\alpha_j}{n_j}\right]v-1\Leftrightarrow\alpha_j\geq1-\frac{2n_j}{(n_j-1)v}$$

[9] Any deviation $k'>1$ would not deliver a greater payoff than the one attained at $k'=1$

On the other hand, consider $x_{-j}(i) = 0$, that is

$$x_{-j}(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = k \in \{1, \ldots, K\}$, then

$$\min\{\mathbf{x}_{-j}\} > \min\{\mathbf{x}_j\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} + k$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\alpha, \gamma_j, \gamma'_{-j}\right)\left(\gamma_j, \gamma'_{-j}|\boldsymbol{\alpha}\right) = \left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}+k} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k \ .$$

Hence for the unique player $i-j$ such that $x_{i-j} = 0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}+k} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \frac{2n_{-j}k(v-n_{-j}\gamma_{-j}-k)}{(2n_{-j}k-n_{-j}\gamma_{-j}-k)v} & \text{if } 2n_{-j}k - n_{-j}\gamma_{-j} - k > 0 \\ \alpha_{-j} \leq \frac{2n_{-j}k(v-n_{-j}\gamma_{-j}-k)}{(2n_{-j}k-n_{-j}\gamma_{-j}-k)v} & \text{if } 2n_{-j}k - n_{-j}\gamma_{-j} - k < 0 \end{cases}$$

Consider any player $i-j$ such that $x_{i-j} > 0$, that is

$$x_{-j}(i) > \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right] - k \ .$$

If agent ij deviates to $x_j(i) = k' \in \{0, \ldots, K\}$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} - k + k'$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma'_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{k'}{n_{-j}\gamma_{-j}-k+k'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - k' \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_{-j})\frac{k}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right] - k \geq \frac{1}{2}\left[(1-\alpha_j)\frac{k'}{n_j\gamma_j-k+k'} + \frac{\alpha_j}{n_j}\right]v - k' \Leftrightarrow$$

$$\Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right)v} \qquad \forall k \in \{1, \ldots, K\} \text{ and } k' \in \{k+1, \ldots, K\}$$

$$\alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - k + k'\right)}{\left(n_{-j}\gamma_{-j} - k\right)v} \qquad \forall k \in \{1, \ldots, K\} \text{ and } k' \in \{0, \ldots, k-1\}$$

Note that the bounds above are decreasing in both k and k', so that for the lower bound we set $k = 1$ and $k' = 2$, whereas for the upper bound we set $k = K$ and $k' = K - 1$. However, the bounds do not cross if and only if $K = 2$, so that $max\,\{\mathbf{x}_{-j}\} = 2$. Nonetheless, in the case $2n_j k - n_j\gamma_j - k > 0$, which always holds for at least $k = max\,\{\mathbf{x}_j\}$, we have

$$\left[\frac{2n_{-j}k\left(v - n_{-j}\gamma_{-j} - k\right)}{\left(2n_{-j}k - n_{-j}\gamma_{-j} - k\right)v}, \infty\right) \cap \left[1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} + 1\right)}{\left(n_{-j}\gamma_{-j} - 1\right)v}, 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - 1\right)}{\left(n_{-j}\gamma_{-j} - 2\right)v}\right] = \emptyset\,.$$

Therefore,

(γ_j, γ_{-j}) such that $\gamma_j = 0, \gamma_{-j} \in (0, K)\,, min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} = 0$ and $\mathbb{1}_{x_{i-j}=0} = 1$

is not a Nash equilibrium for any

$$(\alpha_j, \alpha_{-j}) \in \mathbb{R} \times \mathbb{R}\,.$$

iv. (γ_1, γ_2) such that $\gamma_j \in (0, K)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = 1$. The proof directly follows from what shown at point iii. .

■

Remark 6 *Regarding the first stage, it is straightforward to highlight that it is not possible to express the continuation payoffs for any possible interval on* $\boldsymbol{\alpha}$*, since some intervals do not sustain any second-period equilibrium, so that there are no Nash equilibria in pure strategies in the first stage. The last point remains valid despite the presence of within-group asymmetric Nash equilibria in pure strategies in the second period.*

Consider the following example which should clarify the argument above.

Example 1 *Suppose* $K_{ij} = \{0, 1, 2\}$ *and* $v \geq 2 \cdot max\,\{n_1, n_2\}$*, and focus on within-group symmetric SGP equilibria. Consider*

$$(\alpha_1, \alpha_2) \in \mathbb{R} \times \left(-\infty, 1 - \frac{2n_2}{v} - \frac{3n_2}{\left(n_2 - 1\right)v}\right):$$

no second-period equilibrium is sustained by these intervals on α*, so that we cannot specify second-period strategy profiles for any interval on* $\boldsymbol{\alpha} \in \mathbb{R} \times \mathbb{R}$*, and thus the groups' continuation payoffs.*

Therefore, we can conclude with the following proposition.

Proposition 5 *In the KMMGC, there are no subgame perfect Nash equilibria in pure strategies.*

Therefore, the discrete actions case does not represent a generalization of the binary actions case. The reason for this result is that in $KMMGC$ the sharing rule parameters sustaining second-period equilibria must ensure that both upward and downward deviations are not profitable, for them to be equilibria, not just upward or downward ones, as in the binary actions setup. This difference translates into more restrictive conditions over the $\boldsymbol{\alpha}$ parameters, so that there are intervals over $\mathbb{R} \times \mathbb{R}$ not sustaining second-period equilibria.

7 Conclusions

In this paper, we innovate on the existing literature by characterizing the entire set of second-period pure strategy Nash equilibria in binary max-min group contest with a private good prize. Moreover, we computationally characterize the set of within-group symmetric subgame perfect pure Nash equilibria. Depending on the size of the private good prize with respect to groups' size, we find conditions such that both the set of first-period equilibria in pure strategies and the set of within-group symmetric subgame perfect Nash equilibria in pure strategies have the cardinality of the continuum, i.e. there is indeterminacy. Our results show the interplay between the complementarities induced by the weakest-link impact function and the selective incentives set by the endogenous sharing rule. Then, we check the consequences of expanding the set of second-period actions from the binary case to any subset of the natural numbers with cardinality at least equal to three. In this case, we are able to show that subgame perfect equilibria in pure strategies do not exist. Finally, regarding one of the main contributions to the all-pay auction under complete information literature brought by the work of Chowdhury et al. (2016), that is the existence of within-group symmetric pure strategies Nash equilibria in a deterministic group contest with the weakest-link impact function, we do confirm the robustness of this result in case of a private good prize, as it was conceived by the authors themselves.

Finally, we stress that our theoretical framework has many applications in settings where actions are binary and within-group complementarities are a salient feature of competition between groups for rival and excludable goods, as discussed in the introduction.

References

Baik, K. H., Kim, I.-G., and Na, S. (2001). Bidding for a group-specific public-good prize. *Journal of Public Economics*, 82(3):415–429.

Baik, K. H. and Lee, D. (2012). Do rent-seeking groups announce their sharing rules? *Economic Inquiry*, 50(2):348–363.

Baik, K. H. and Lee, S. (2001). Strategic groups and rent dissipation. *Economic Inquiry*, 39(4):672–684.

Barbieri, S., Kovenock, D., Malueg, D. A., and Topolyan, I. (2019). Group contests with private information and the "weakest link". *Games and Economic Behavior*, 118:382–411.

Barbieri, S. and Malueg, D. A. (2016). Private-information group contests: Best-shot competition. *Games and Economic Behavior*, 98:219–234.

Barbieri, S., Malueg, D. A., and Topolyan, I. (2014). The best-shot all-pay (group) auction with complete information. *Economic Theory*, 57(3):603–640.

Barbieri, S. and Topolyan, I. (2021). Private-information group contests with complementarities. *Journal of Public Economic Theory*, 23(5):772–800.

Baye, M. R., Kovenock, D., and de Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, 8(2):291–305.

Chowdhury, S. M., Lee, D., and Sheremeta, R. M. (2013). Top guns may not fire: Best-shot group contests with group-specific public good prizes. *Journal of Economic Behavior Organization*, 92:94–103.

Chowdhury, S. M., Lee, D., and Topolyan, I. (2016). The max-min group contest: Weakest-link (group) all-pay auction. *Southern Economic Journal*, 83(1):105–125.

Clark, D. J. and Riis, C. (1998). Contest success functions: an extension. *Economic Theory*, 11:201–204.

Gilli, M. and Sorrentino, A. (2024). Characterization of the set of equilibria in max-min group contests with continuous efforts and a private good prize. *Mimeo.*

Gilli, M. and Tedeschi, P. (2022). Europe and transnational terrorism. a strategic analysis of normative spillovers. *European Union Politics.*

Hillman, A. L. and Riley, J. G. (1989). Politically contestable rents and transfers*. *Economics & Politics*, 1(1):17–39.

Hirshleifer, J. (1983). From weakest-link to best-shot: The voluntary provision of public goods. *Public Choice*, 41(3):371–386.

Kolmar, M. and Rommeswinkel, H. (2013). Contests with group-specific public goods and complementarities in efforts. *Journal of Economic Behavior Organization*, 89:9–22.

Lee, D. (2012). Weakest-link contests with group-specific public good prizes. *European Journal of Political Economy*, 28(2):238–248.

Nitzan, S. (1991). Collective rent dissipation. *The Economic Journal*, 101(409):1522–1534.

Skaperdas, S. (1996). Contest success functions. *Economic Theory*, 7(2):283–290.

9 Appendix

9.1 Representation of Second-Period Nash Equilibria in pure strategies

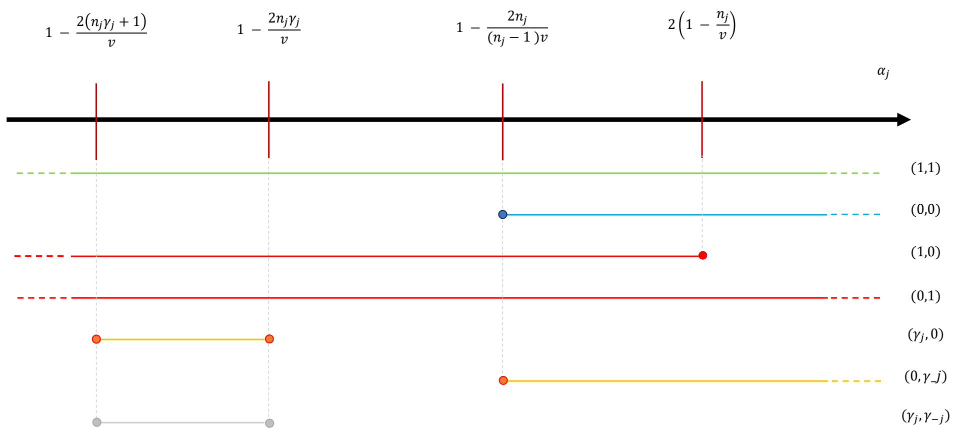

Figure 7: Intervals of α_j sustaining second-period equilibria $\forall v \geq 2 \cdot max\,\{n_j, n_{-j}\}$

9.2 Codes

In this subsection we present the codes written in MATLAB to retrieve the set of first-period equilibria and the set of WGS subgame perfect Nash equilibria in pure strategies for the leading case $v \geq 2\{n_1, n_2\}$. The lines of code for the remaining cases are available online at https://sites.google.com/view/andreasorrentino .

```
%%% ALL-PAY, WEAKEST-LINK, BINARY ACTIONS, GROUP CONTEST

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%% v > 2max{n_1,n_2 } CASE %%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Let's create the matrices needed to get all possible payoffs combinations
Q1 = strings(1,2) ;
Q2 = strings(1,3) ;
Q3 = strings(1,2) ;
Q4 = strings(1,3) ;
Q5 = strings(1,4) ;
Q6 = strings(1,3) ;
Q7 = strings(1,3) ;
Q8 = strings(1,3) ;
Q9 = strings(1,2) ;

%% Let's fill the matrices above with the corresponding payoffs

Q1(1,1) = '(1,1)' ;
Q1(1,2) = '(1,0)' ;

Q2(1,1) = '(1,1)' ;
Q2(1,2) = '(0,0)' ;
Q2(1,3) = '(1,0)' ;

Q3(1,1) = '(1,1)' ;
Q3(1,2) = '(0,0)' ;

Q4(1,1) = '(1,1)' ;
Q4(1,2) = '(1,0)' ;
Q4(1,3) = '(0,1)' ;

Q5(1,1) = '(1,1)' ;
Q5(1,2) = '(0,0)' ;
Q5(1,3) = '(1,0)' ;
Q5(1,4) = '(0,1)' ;

Q6(1,1) = '(1,1)' ;
Q6(1,2) = '(0,0)' ;
```

```
Q6(1,3) = '(0,1)' ;

Q7(1,1) = '(1,1)' ;
Q7(1,2) = '(1,0)' ;
Q7(1,3) = '(0,1)' ;

Q8(1,1) = '(1,1)' ;
Q8(1,2) = '(1,0)' ;
Q8(1,3) = '(0,1)' ;

Q9(1,1) = '(1,1)' ;
Q9(1,2) = '(0,1)' ;


[grid1, grid2 , grid3, grid4, grid5, grid6, grid7, grid8, grid9] = ndgrid(Q1, Q2,
Q3, Q4, Q5, Q6, Q7, Q8, Q9); % grid structure to perform the cartesian product
result = [grid1(:), grid2(:), grid3(:), grid4(:), grid5(:), grid6(:), grid7(:),
grid8(:), grid9(:)]; % cartesian product to obtain all possible payoff profiles

%%Let's create the 7776 payoff matrices
%%For cycle
pmatrix = strings(length(result)*3,3);
counter = 1;

for i= 1:length(result)
    pmatrix(counter,1:3) = result(i,1:3);
    pmatrix(counter+1,1:3) = result(i,4:6);
    pmatrix(counter+2, 1:3) = result(i,7:9);
    counter = counter + 3;
end

disp(pmatrix);


%% Let's find the set of first-period equilibria and the set of subgame perfect equilibria.

% Remember:
% For group 1: (1,0) > (0,0) > (1,1) > (0,1)
% For group 2: (0,1) > (0,0) > (1,1) > (1,0)

br1_opt = strings(length(result)*3,3); %% matrix to collect best responses from group 1
br2_opt = strings(length(result)*3,3); %% matrix to collect best responses from group 2
alphaeq_opt = strings(length(result)*3,3); %% matrix to collect first-stage equilibria
sgeq_opt = strings(length(result)*3,6); %% matrix to collect SGP equilibria

%% THE SET OF BEST-RESPONSES FOR GROUP 1
```

```
for i=1:length(result)*3
    for j=1:3
        if pmatrix(i,j) == '(1,0)'
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(0,0)' && all(pmatrix(i,1:3) ~= '(1,0)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(1,1)' && all(pmatrix(i,1:3) ~= '(1,0)') &&
            all(pmatrix(i,1:3) ~= '(0,0)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(0,1)' && all(pmatrix(i,1:3) ~= '(1,0)') &&
           all(pmatrix(i,1:3) ~= '(0,0)') && all(pmatrix(i,1:3) ~= '(1,1)')
           br1_opt(i,j) = pmatrix(i,j);
        end
    end
end

%% THE SET OF BEST-RESPONSES FOR GROUP 2

for j=1:3
    for i= 1:3:length(result)*3-2
        for q=0:2
            if pmatrix(i+q,j) == '(0,1)'
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(0,0)' && all(pmatrix(i:i+2,j) ~= '(0,1)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,1)' && all(pmatrix(i:i+2,j) ~= '(0,1)') &&
                   all(pmatrix(i:i+2,j) ~= '(0,0)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,0)' && all(pmatrix(i:i+2,j) ~= '(0,1)') &&
                   all(pmatrix(i:i+2,j) ~= '(0,0)') && all(pmatrix(i:i+2,j) ~= '(1,1)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            end
        end
    end
end


%% THE SET OF FIRST-PERIOD EQUILIBRIA AND SUBGAME PERFECT EQUILIBRIA

for i= 1:3:length(result)*3-2
    eq_1 = 0;
    for k =i:i+2
        for j = 1:3
            if br1_opt(k,j) == br2_opt(k,j) && br1_opt(k,j) ~= ""
               eq_1 = eq_1 + 1;
               alphaeq_opt(k,j) = br1_opt(k,j); %br1_opt without loss of generality
            end
```

```
        end
    end
    if eq_1 > 1 || eq_1 == 1
       sgeq_opt(i:i+2 ,1:3) = alphaeq_opt(i:i+2,1:3);
       sgeq_opt(i:i+2 ,4:6) = pmatrix(i:i+2,1:3);
    end
end



%%% LET'S SPLIT THE MATRICES AND REPRESENT THEM IN A MORE INTUITIVE WAY


sgeq = strings(length(result)*3,6);

for i=1:length(result)*3
    for j=1:6
        if sgeq_opt(i,j) == ""
           sgeq(i,j) = '.';
        elseif sgeq_opt(i,j) ~= ""
            sgeq(i,j) = sgeq_opt(i,j);
        end
    end
end

%% CAVEAT: here we assume closed intervals [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)].
% It is possible to consider open intervals, so that we would have:
% (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% Or again:
% a) (-\infty, 1 - 2n_j/((n_j-1)v)) ; [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% b) (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)];
% [2(1-n_j/v), \infty)


for i=1:3:length(result)*3-2
    if sgeq(i,1) ~= '.'
       sgeq(i,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq(i,2) ~= '.'
       sgeq(i,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq(i,3) ~= '.'
       sgeq(i,3) = ' a_1 > 2(1 - n1/v) and a_2 > 2(1 - n2/v) ';
    end
```

```
    if sgeq(i+1,1) ~= '.'
       sgeq(i+1,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq(i+1,2) ~= '.'
       sgeq(i+1,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq(i+1,3) ~= '.'
       sgeq(i+1,3) = ' a_1 > 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq(i+2,1) ~= '.'
       sgeq(i+2,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v) ';
    end
    if sgeq(i+2,2) ~= '.'
       sgeq(i+2,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
    if sgeq(i+2,3) ~= '.'
       sgeq(i+2,3) = ' a_1 > 2(1 - n_1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
end

sgeq_opt = sgeq_opt(sgeq_opt(:,4) ~= '',:); % cleaning the matrix
% used to get SGP equilibria
sgeq = sgeq(sgeq(:,4) ~= '.',:); % cleaning the matrix of SGP equilibria

disp(['There are ' num2str(length(sgeq)/3) ' continuation-payoffs matrices
containing equilibria.'])


%% Check in order to verify whether the set of first-stage equilibria is RxR or not
Check = zeros(3,3);
% Are there first-row elements not being an equilibrium
% in any continuation payoff matrix?
Check(1,1:3) = all(sgeq(1:3:length(sgeq) -2,1:3)==".");
% Are there second-row elements not being an equilibrium
% in any continuation payoff matrix?
Check(2,1:3) = all(sgeq(2:3:length(sgeq) -1,1:3)==".");
% Are there third-row elements not being an equilibrium
% in any continuation payoff matrix?
Check(3,1:3) = all(sgeq(3:3:length(sgeq),1:3)==".");
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%% v = 2max{n_1,n_2} = 2n_1 CASE %%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Let's find the set of first-stage equilibria and the set of subgame perfect equilibria.

% Remember:
% For group 1: (1,0) == (0,0) > (1,1) == (0,1)
% For group 2: (0,1) > (0,0) > (1,1) > (1,0)

br1_opt_b = strings(length(result)*3,3); %% matrix to collect best responses from group 1
br2_opt_b = strings(length(result)*3,3); %% matrix to collect best responses from group 2
alphaeq_opt_b = strings(length(result)*3,3); %% matrix to collect first-stage equilibria
sgeq_opt_b = strings(length(result)*3,6); %% matrix to collect SGP equilibria

%% THE SET OF BEST-RESPONSES FOR GROUP 1

for i=1:length(result)*3
    for j=1:3
        if pmatrix(i,j) == '(1,0)' | pmatrix(i,j) == '(0,0)'
           br1_opt_b(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(1,1)' | pmatrix(i,j) == '(0,1)' &&
               all(pmatrix(i,1:3) ~= '(1,0)') && all(pmatrix(i,1:3) ~= '(0,0)')
               br1_opt_b(i,j) = pmatrix(i,j);
        end
    end
end

%% THE SET OF BEST-RESPONSES FOR GROUP 2

for j=1:3
    for i= 1:3:length(result)*3-2
        for q=0:2
            if pmatrix(i+q,j) == '(0,1)'
               br2_opt_b(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(0,0)' &&
                   all(pmatrix(i:i+2,j) ~= '(0,1)')
                   br2_opt_b(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,1)' &&
                   all(pmatrix(i:i+2,j) ~= '(0,1)') && all(pmatrix(i:i+2,j) ~= '(0,0)')
                   br2_opt_b(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,0)' && all(pmatrix(i:i+2,j) ~= '(0,1)') &&
                   all(pmatrix(i:i+2,j) ~= '(0,0)') && all(pmatrix(i:i+2,j) ~= '(1,1)')
                   br2_opt_b(i+q,j) = pmatrix(i+q,j);
            end
        end
```

```
    end
end


%% THE SET OF FIRST-PERIOD EQUILIBRIA AND SUBGAME PERFECT EQUILIBRIA

for i= 1:3:length(result)*3-2
    eq_1 = 0;
    for k =i:i+2
        for j = 1:3
            if br1_opt_b(k,j) == br2_opt_b(k,j) && br1_opt_b(k,j) ~= ""
               eq_1 = eq_1 + 1;
               alphaeq_opt_b(k,j) = br1_opt_b(k,j); %br1_opt without loss of generality
            end
        end
    end
    if eq_1 > 1 || eq_1 == 1
       sgeq_opt_b(i:i+2 ,1:3) = alphaeq_opt_b(i:i+2,1:3);
       sgeq_opt_b(i:i+2 ,4:6) = pmatrix(i:i+2,1:3);
    end
end



%%% LET'S SPLIT THE MATRICES AND REPRESENT THEM IN A MORE INTUITIVE WAY


sgeq_b = strings(length(result)*3,6);

for i=1:length(result)*3
    for j=1:6
        if sgeq_opt_b(i,j) == ""
           sgeq_b(i,j) = '.';
        elseif sgeq_opt_b(i,j) ~= ""
           sgeq_b(i,j) = sgeq_opt_b(i,j);
        end
    end
end

%% CAVEAT: here we assume closed intervals [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)].
% It is possible to consider open intervals, so that we would have:
% (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% Or again:
% a) (-\infty, 1 - 2n_j/((n_j-1)v)) ; [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% b) (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)];
```

```
% [2(1-n_j/v), \infty)


for i=1:3:length(result)*3-2
    if sgeq_b(i,1) ~= '.'
       sgeq_b(i,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_b(i,2) ~= '.'
       sgeq_b(i,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_b(i,3) ~= '.'
       sgeq_b(i,3) = ' a_1 > 2(1 - n1/v) and a_2 > 2(1 - n2/v) ';
    end
    if sgeq_b(i+1,1) ~= '.'
       sgeq_b(i+1,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_b(i+1,2) ~= '.'
       sgeq_b(i+1,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_b(i+1,3) ~= '.'
       sgeq_b(i+1,3) = ' a_1 > 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_b(i+2,1) ~= '.'
       sgeq_b(i+2,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v) ';
    end
    if sgeq_b(i+2,2) ~= '.'
       sgeq_b(i+2,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
    if sgeq_b(i+2,3) ~= '.'
       sgeq_b(i+2,3) = ' a_1 > 2(1 - n_1/v) and a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
end

sgeq_opt_b = sgeq_opt_b(sgeq_opt_b(:,4) ~= '',:); % cleaning the matrix
 % used to get SGP equilibria
sgeq_b = sgeq_b(sgeq_b(:,4) ~= '.',:); % cleaning the matrix of SGP equilibria


disp(['There are ' num2str(length(sgeq_b)/3) ' continuation-payoffs matrices
containing SGP equilibria.'])
```

```
%% Check in order to verify whether the set of first-stage equilibria is RxR or not
Check_b = zeros(3,3);
% Are there first-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_b(1,1:3) = all(sgeq_b(1:3:length(sgeq_b) -2,1:3)==".");
% Are there second-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_b(2,1:3) = all(sgeq_b(2:3:length(sgeq_b) -1,1:3)==".");
% Are there third-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_b(3,1:3) = all(sgeq_b(3:3:length(sgeq_b),1:3)==".");


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%% v = 2max{n_1,n_2} = 2n_2 CASE %%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Let's find the set of first-stage equilibria and the set of subgame perfect equilibria.

% Remember:
% For group 1: (1,0) > (0,0) > (1,1) > (0,1)
% For group 2: (0,1) == (0,0) > (1,1) == (1,0)

br1_opt_c = strings(length(result)*3,3); %% matrix to collect best responses from group 1
br2_opt_c = strings(length(result)*3,3); %% matrix to collect best responses from group 2
alphaeq_opt_c = strings(length(result)*3,3); %% matrix to collect first-stage equilibria
sgeq_opt_c = strings(length(result)*3,6); %% matrix to collect SGP equilibria

%% THE SET OF BEST-RESPONSES FOR GROUP 1

for i=1:length(result)*3
    for j=1:3
        if pmatrix(i,j) == '(1,0)'
           br1_opt_c(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(0,0)' && all(pmatrix(i,1:3) ~= '(1,0)')
               br1_opt_c(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(1,1)' && all(pmatrix(i,1:3) ~= '(1,0)') &&
               all(pmatrix(i,1:3) ~= '(0,0)')
               br1_opt_c(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(0,1)' && all(pmatrix(i,1:3) ~= '(1,0)') &&
               all(pmatrix(i,1:3) ~= '(0,0)') && all(pmatrix(i,1:3) ~= '(1,1)')
               br1_opt_c(i,j) = pmatrix(i,j);
        end
    end
end
```

```
%% THE SET OF BEST-RESPONSES FOR GROUP 2

for j=1:3
    for i= 1:3:length(result)*3-2
        for q=0:2
            if pmatrix(i+q,j) == '(0,1)' | pmatrix(i+q,j) == '(0,0)'
               br2_opt_c(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,1)' | pmatrix(i+q,j) == '(1,0)' &&
                    all(pmatrix(i:i+2,j) ~= '(0,1)') && all(pmatrix(i:i+2,j) ~= '(0,0)')
                    br2_opt_c(i+q,j) = pmatrix(i+q,j);
            end
        end
    end
end


%% THE SET OF FIRST-PERIOD EQUILIBRIA AND SUBGAME PERFECT EQUILIBRIA

for i= 1:3:length(result)*3-2
    eq_1 = 0;
    for k =i:i+2
        for j = 1:3
            if br1_opt_c(k,j) == br2_opt_c(k,j) && br1_opt_c(k,j) ~= ""
               eq_1 = eq_1 + 1;
               alphaeq_opt_c(k,j) = br1_opt_c(k,j); %br1_opt without loss of generality
            end
        end
    end
    if eq_1 > 1 || eq_1 == 1
       sgeq_opt_c(i:i+2 ,1:3) = alphaeq_opt_c(i:i+2,1:3);
       sgeq_opt_c(i:i+2 ,4:6) = pmatrix(i:i+2,1:3);
    end
end



%%% LET'S SPLIT THE MATRICES AND REPRESENT THEM IN A MORE INTUITIVE WAY


sgeq_c = strings(length(result)*3,6);

for i=1:length(result)*3
    for j=1:6
        if sgeq_opt_c(i,j) == ""
           sgeq_c(i,j) = '.';
        elseif sgeq_opt_c(i,j) ~= ""
           sgeq_c(i,j) = sgeq_opt_c(i,j);
```

```
        end
    end
end


%% CAVEAT: here we assume closed intervals [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)].
% It is possible to consider open intervals, so that we would have:
% (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% Or again:
% a) (-\infty, 1 - 2n_j/((n_j-1)v)) ; [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% b) (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)];
% [2(1-n_j/v), \infty)


for i=1:3:length(result)*3-2
    if sgeq_c(i,1) ~= '.'
       sgeq_c(i,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_c(i,2) ~= '.'
       sgeq_c(i,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_c(i,3) ~= '.'
       sgeq_c(i,3) = ' a_1 > 2(1 - n1/v) and a_2 > 2(1 - n2/v) ';
    end
    if sgeq_c(i+1,1) ~= '.'
       sgeq_c(i+1,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_c(i+1,2) ~= '.'
       sgeq_c(i+1,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_c(i+1,3) ~= '.'
       sgeq_c(i+1,3) = ' a_1 > 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_c(i+2,1) ~= '.'
       sgeq_c(i+2,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v) ';
    end
    if sgeq_c(i+2,2) ~= '.'
       sgeq_c(i+2,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
```

```
    if sgeq_c(i+2,3) ~= '.'
       sgeq_c(i+2,3) = ' a_1 > 2(1 - n_1/v) and a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
end

sgeq_opt_c = sgeq_opt_c(sgeq_opt_c(:,4) ~= '',:); % cleaning the matrix
% used to get SGP equilibria
sgeq_c = sgeq_c(sgeq_c(:,4) ~= '.',:); % cleaning the matrix of SGP equilibria


disp(['There are ' num2str(length(sgeq_c)/3) ' continuation-payoffs matrices
containing SGP equilibria.'])


%% Check in order to verify whether the set of first-stage equilibria is RxR or not
Check_c = zeros(3,3);
% Are there first-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_c(1,1:3) = all(sgeq_c(1:3:length(sgeq_c) -2,1:3)==".");
% Are there second-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_c(2,1:3) = all(sgeq_c(2:3:length(sgeq_c) -1,1:3)==".");
% Are there third-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_c(3,1:3) = all(sgeq_c(3:3:length(sgeq_c),1:3)==".");





%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%% v = 2max{n_1,n_2} = 2n_1 = 2n_2 CASE %%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


%% Let's find the set of first-stage equilibria and the set of subgame perfect equilibria.

% Remember:
% For group 1: (1,0) == (0,0) > (1,1) == (0,1)
% For group 2: (0,1) == (0,0) > (1,1) == (1,0)

br1_opt_d = strings(length(result)*3,3); %% matrix to collect best responses from group 1
br2_opt_d = strings(length(result)*3,3); %% matrix to collect best responses from group 2
alphaeq_opt_d = strings(length(result)*3,3); %% matrix to collect first-stage equilibria
sgeq_opt_d = strings(length(result)*3,6); %% matrix to collect SGP equilibria

%% THE SET OF BEST-RESPONSES FOR GROUP 1
```

```
for i=1:length(result)*3
    for j=1:3
        if pmatrix(i,j) == '(1,0)' | pmatrix(i,j) == '(0,0)'
           br1_opt_d(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(1,1)' | pmatrix(i,j) == '(0,1)' &&
               all(pmatrix(i,1:3) ~= '(1,0)') && all(pmatrix(i,1:3) ~= '(0,0)')
               br1_opt_d(i,j) = pmatrix(i,j);
        end
    end
end

%% THE SET OF BEST-RESPONSES FOR GROUP 2

for j=1:3
    for i= 1:3:length(result)*3-2
        for q=0:2
            if pmatrix(i+q,j) == '(0,1)' | pmatrix(i+q,j) == '(0,0)'
               br2_opt_d(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(1,1)' | pmatrix(i+q,j) == '(1,0)' &&
                   all(pmatrix(i:i+2,j) ~= '(0,1)') && all(pmatrix(i:i+2,j) ~= '(0,0)')
                   br2_opt_d(i+q,j) = pmatrix(i+q,j);
            end
        end
    end
end


%% THE SET OF FIRST-PERIOD EQUILIBRIA AND SUBGAME PERFECT EQUILIBRIA

for i= 1:3:length(result)*3-2
    eq_1 = 0;
    for k =i:i+2
        for j = 1:3
            if br1_opt_d(k,j) == br2_opt_d(k,j) && br1_opt_d(k,j) ~= ""
               eq_1 = eq_1 + 1;
               alphaeq_opt_d(k,j) = br1_opt_d(k,j); %br1_opt without loss of generality
            end
        end
    end
    if eq_1 > 1 || eq_1 == 1
       sgeq_opt_d(i:i+2 ,1:3) = alphaeq_opt_d(i:i+2,1:3);
       sgeq_opt_d(i:i+2 ,4:6) = pmatrix(i:i+2,1:3);
    end
end
```

```
%%% LET'S SPLIT THE MATRICES AND REPRESENT THEM IN A MORE INTUITIVE WAY


sgeq_d = strings(length(result)*3,6);

for i=1:length(result)*3
    for j=1:6
        if sgeq_opt_d(i,j) == ""
           sgeq_d(i,j) = '.';
        elseif sgeq_opt_d(i,j) ~= ""
            sgeq_d(i,j) = sgeq_opt_d(i,j);
        end
    end
end


%% CAVEAT: here we assume closed intervals [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)].
% It is possible to consider open intervals, so that we would have:
% (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% Or again:
% a) (-\infty, 1 - 2n_j/((n_j-1)v)) ; [1 - 2n_j/((n_j-1)v) , 2(1-n_j/v));
% [2(1-n_j/v), \infty)
% b) (-\infty, 1 - 2n_j/((n_j-1)v)] ; (1 - 2n_j/((n_j-1)v) , 2(1-n_j/v)];
% [2(1-n_j/v), \infty)


for i=1:3:length(result)*3-2
    if sgeq_d(i,1) ~= '.'
       sgeq_d(i,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_d(i,2) ~= '.'
       sgeq_d(i,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 > 2(1 - n2/v) ' ;
    end
    if sgeq_d(i,3) ~= '.'
       sgeq_d(i,3) = ' a_1 > 2(1 - n1/v) and a_2 > 2(1 - n2/v) ';
    end
    if sgeq_d(i+1,1) ~= '.'
       sgeq_d(i+1,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_d(i+1,2) ~= '.'
       sgeq_d(i+1,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
```

```
    if sgeq_d(i+1,3) ~= '.'
       sgeq_d(i+1,3) = ' a_1 > 2(1 - n1/v) and
       1 - 2n_2/((n_2 - 1)v) <= a_2 <= 2(1 - n2/v) ';
    end
    if sgeq_d(i+2,1) ~= '.'
       sgeq_d(i+2,1) = ' a_1 <  1 - 2n_1/((n_1 - 1)v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v) ';
    end
    if sgeq_d(i+2,2) ~= '.'
       sgeq_d(i+2,2) = ' 1 - 2n_1/((n_1 - 1)v) <= a_1 <= 2(1 - n1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
    if sgeq_d(i+2,3) ~= '.'
       sgeq_d(i+2,3) = ' a_1 > 2(1 - n_1/v) and
       a_2 <  1 - 2n_2/((n_2 - 1)v)';
    end
end

sgeq_opt_d = sgeq_opt_d(sgeq_opt_d(:,4) ~= '',:); % cleaning the matrix
% used to get SGP equilibria
sgeq_d = sgeq_d(sgeq_d(:,4) ~= '.',:); % cleaning the matrix of SGP equilibria


disp(['There are ' num2str(length(sgeq_d)/3) ' continuation-payoffs matrices
containing SGP equilibria.'])


%% Check in order to verify whether the set of first-stage equilibria is RxR or not
Check_d = zeros(3,3);
% Are there first-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_d(1,1:3) = all(sgeq_d(1:3:length(sgeq_d) -2,1:3)==".");
% Are there second-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_d(2,1:3) = all(sgeq_d(2:3:length(sgeq_d) -1,1:3)==".");
% Are there third-row elements not being an equilibrium
% in any continuation payoff matrix?
Check_d(3,1:3) = all(sgeq_d(3:3:length(sgeq_d),1:3)==".");
```