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Abstract

We characterise the set of equilibria in a deterministic group contest with the weakest-link impact function, continuous efforts and a private good prize, complementing the results obtained by Chowdhury et al. (2016). We consider a two-stages two-groups model, where in the first stage the agents simultaneously choose the sharing rule, while in the second stage they choose efforts. Despite the existence of within-group symmetric Nash equilibria in pure strategies in the effort stage, there are combinations of possible sharing rules such that no pure strategy effort equilibria exist, hence for these sharing rules, the continuation payoffs are not defined, so that there exist no subgame perfect Nash equilibria in pure stragies. However, limiting ourselves to the restricted sharing rules case, we are able to state that there are continua of subgame perfect equilibria. In this case, by additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterise the set of subgame perfect equilibria in pure strategies.

Preliminary and incomplete. Comments welcome

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1 Introduction

This paper aims at characterising the set of equilibria in a group contest. The contest features the weakest-link impact function,the deterministic contest success function, continuous efforts, and a private good prize. According to Hirshleifer (1983), this setup is suitable for modeling social phenomena like disasters or conflict situations. In these scenarios, selfish behavior can result in collective disasters for the group, even when dealing with private goods, such as relief supplies, that need to be allocated within the group.

To the best of our knowledge, the first paper to use a weakest-link impact function in a strategic setting was Hirshleifer (1983), where he showed that in this case, underprovision of the public good tends to be considerably moderated compared to the cases of perfect substitutability (summation) and of best-shot impact functions. Cornes and Hartley (2007) build more formally on the same approach, characterising the set of equilibria. Both these papers do not consider a group contest setting. Sheremeta (2010) analyzes a group contest model experimentally considering three possible impact functions - weakest-link, best-shot, and perfect substitutability - combining these impact functions with a Tullock contest success function. The results of the experiment confirm that in weakest-link contests there is almost no free riding and all players expend similar positive efforts. Lee (2012) considers a group contest with a group-specific public good prize, weakest-link and a general non-deterministic contests success function. Within a group contest model, Kolmar and Rommeswinkel (2013) generalises previous works considering a CES impact function, so that the weakest-link is a particular case, and a Tullock contests success function.

Sheremeta (2010), Lee (2012), and Kolmar and Rommeswinkel (2013) share the use of a non-deterministic contest success function, while Chowdhury et al. (2016) and Barbieri and Topolyan (2024) consider a deterministic contest success function, so that they are more directly related to this paper. However, these papers consider (group specific) public goods, so there is no need for a within groups sharing rule. Accordingly, our results with a private prize, complement the results obtained by Chowdhury et al. (2016), since they depict the set of Nash equilibria in both pure and mixed strategies in a "max-min group contest" with a public good prize. The authors themselves envisage the possibility of introducing a private good prize in their theoretical framework, as a future line of research.

We consider a two stages two groups contest model, where in the first stage the agents simultaneously choose the sharing rule, while in the second stage they choose efforts. We find that Chowdhury et al. (2016) result about the existence of only within-group symmetric Nash equilibria in pure strategies is robust to the introduction of a private good prize. However, despite the existence of within-group symmetric Nash equilibria in pure strategies in the effort stage, there are combinations of possible sharing rules such that no pure strategy effort equilibria exist. Hence, for these sharing rules, the continuation payoffs are not defined, so in general it is not possible to characterise the set of subgame perfect Nash equilibria in pure strategies, as the continuation payoffs cannot be pinned down along the domain of the first-period action set. However, turning to the restricted sharing rules case, we can conclude that there are continua of subgame perfect equilibria. In this case, by additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterise the set of subgame perfect equilibria in pure strategies.

The paper proceeds as follows. In Section 2, we present the basic model. In Section 3 we characterise the set of puree strategy Nash equilibria of the second effort stage. In Section 4, we discuss the players' behavior in the first stage, while in Section 5 we characterise the set of subgame perfect equilibria of the entire game. Finally, Section 6 presents the conclusions.

2 A Continuous Effort Max-Min Two Group Contest with a Finite Number of Agents

Consider a simple two-groups model that sums up the main characteristics of group contests under complete information. The model is defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;

2. each group has $n_j \geq 4$ members in each group. The total number of agents is $N = n_1 + n_2$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;

3. the **effort** of member $i \in \{1, ..., n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \mathbb{R}_+$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let define the average exerted effort in group j as

$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in \mathbb{R}_+;$$

4. a private **prize** worth v to be allocated to one of the groups;

5. the **impact function** of group j is given by the weakest-link technology

$$X_j = \min \{x_j(i) \in \mathbb{R}_+, i \in \{1, ..., n_j\}\};$$

6. the **contest success function** is given by the *all-pay auction:*

$$p_j(X_1, X_2) = \begin{cases} 1 & if \quad X_j > X_{-j} \\ \frac{1}{2} & if \quad X_j = X_{-j} \\ 0 & if \quad X_j < X_{-j}; \end{cases}$$

7. the **sharing rule**, such that if group $j \in \{1, 2\}$ wins, then a member $i \in \{1, \ldots, n_j\}$ gets a share of the prize

$$q_{ij}\left(x_{1j}, ..., x_{n_j j}\right) =$$

$$= \begin{cases} \underbrace{(1-\alpha_j)}_{incentivation\ part} \frac{x_{ij}}{\sum_{i=1}^{n_j} x_{ij}} + \underbrace{\alpha_j}_{equalising\ part} \frac{1}{n_j} & if \quad \sum_{i=1}^{n_j} x_{ij} > 0 \\ \frac{1}{n_j} & otherwise \end{cases}$$

where α_j is the share of the prize that the members of the winning team get independently of their effort. There are two possible assumptions on the domain of α_j : the **restricted** case with $\alpha_j \in [0, 1]$, and the **unrestricted** case with $\alpha_j \in \mathbb{R}$.

- **When** $\alpha_j \in [0,1]$**,** the sharing rule amounts to pure within-group redistribution
  - α_j towards 0 means that the main bulk of the prize share is distributed according to each agent's effort, the incentivation part, while there is almost no uniform redistribution, the equalising part;
  - α_j towards 1 means that the main bulk of the prize share is distributed according to a uniform redistribution, while almost no part is distributed as a reward for each agent's effort;
  - more generally α_j from 0 to 1 is able to span the entire set of redistribution/rewarding rules, from no uniform redistribution to no effort rewarding.
- **When** $\alpha_j \in \mathbb{R}$**,** the sharing rule goes beyond pure redistribution by allowing groups to engage in cross-subsidisation:
  - $\alpha_j < 0$: conditioned to winning the prize v, a negative value means that group j collects $-\alpha_j \frac{v}{n_j}$ from each of its members, and then distributes $(1-\alpha_j)\, v$ among members in proportion to relative outlay: it is a penalization for group's non-active members;
  - $\alpha_j > 1$: conditioned to winning the prize v, a value greater than 1 means that group j collects $-(1-\alpha_j)\, v$ from its members according to relative outlay, and then distributes $\alpha_j v$ equally among all its members: it is a premium for group's non-active members.

  Let us stress that the application of the sharing rule is conditioned on winning, hence rewards and punisment are applied only if a group wins. The implicit assumption is that screening between free riders and contributors within a group is performed only if the group get the prize, otherwise is too costly. For example, an internal auditing to share a bonus in a team may be performed only after an order is awarded to a company. Similarly, in a conflict the punishment of free riders might be possible only after winning, while a defeat precludes such a possibility.

8. the individual **costs of effort** $C_{ij}\left(x_j\left(i\right)\right) = x_j\left(i\right)$;
9. the **timing**, there are two stages:
   i. in the first stage, the groups choose the equilibrium sharing rule within each group α_j;
   ii. in the second stage all the members of the groups observe the first stage choices (α_1, α_2) and choose simultaneously and independently their effort $x_j\left(i\right)$ and the prize is allocated to one of the two groups according to the contest success function.

Note that this timing structure, used in most papers that endogenise the sharing rule, implies that the groups can precommit to an equilibrium sharing rule, without any subsequent renegotiation. Let us stress that this sequential structure means that, once established, the sharing rule cannot be object of negotiation within or across the groups, there is a commitment to respect the choices of the first stage.

As a consequence of these modelling characteristics, player ij has the **payoff**

$$\pi_{ij}\left(\alpha_j,\alpha_{-j},x_{1j},...,x_{n_jj},x_{1-j},...,x_{n_{-j}-j}\right)=p_jq_{ij}v-x_{ij}=$$

$$=\begin{cases}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)}+\alpha_j\frac{1}{n_j}\right]v-x_j(i) & if \quad \min\{\mathbf{x}_j\}>\min\{\mathbf{x}_{-j}\}\\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)}+\alpha_j\frac{1}{n_j}\right]v-x_j(i) & if \quad \min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}\\ -x_j(i) & if \quad \min\{\mathbf{x}_j\}<\min\{\mathbf{x}_{-j}\}\end{cases}$$

Now, we are able to provide a formal definition of a binary group contest.

Definition 1 *A* Continuous Effort Max-Min Group Contest $CMMGC$ *is a two stages game* $CMMGC=\langle\{1,2\},N,S_j,E_{ij},\pi_{ij}\rangle$ *defined by*

1. *the set of groups* $\{1,2\}$;
2. *the set of players* $N=\{1,...,n_1+n_2\}$;
3. *the set of first-period actions* $S_j=\mathbb{R}$: *for each group* j, *the choice of the share* α_j *in the sharing rule*;
4. *the set of second-period actions* $E_{ij}=\mathbb{R}_+$: *for each player* ij, *the choice of the bid* $x_j(i)$. *Note that because of the sequential structure of the game, each player* ij *second period* ***strategies*** *are maps* $\mathbb{R}^2\mapsto\mathbb{R}_+$;
5. *the payoff functions for each player* $ij\in N$

$$\pi_{ij}(\boldsymbol{\alpha},\mathbf{x})=p_jq_{ij}v-x_j(i)=$$

$$=\begin{cases}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)}+\alpha_j\frac{1}{n_j}\right]v-x_j(i) & if \quad \min\{\mathbf{x}_j\}>\min\{\mathbf{x}_{-j}\}\\ \frac{1}{2}\left[(1-\alpha_j)\frac{x_j(i)}{\sum_j x_j(i)}+\alpha_j\frac{1}{n_j}\right]v-x_j(i) & if \quad \min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}\\ -x_j(i) & if \quad \min\{\mathbf{x}_j\}<\min\{\mathbf{x}_{-j}\}\end{cases}$$

where $\boldsymbol{\alpha}$ *and* $\mathbf{x}$ *are, respectively, the vector of first and second period actions.*

The notation used in this paper is summed up in table 1.

Variable	Meaning
ij or $j(i)$	agent i of group j
$\{1,...,n_j\}$	set of agents in group j
$x_j(i)$ or x_{ji}	effort of agent i in group j
$X_j=\min\{x_j(i)\in\mathbb{R}_+,i\in\{1,...,n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)$	vector of efforts of all agents
$C_{ij}(x_j(i))=x_j(i)$	cost of effort for agent i of group j
$p_j(X_1,X_2)$	probability of group j of winning the contest
$q_{ij}\left(x_{1j},...,x_{n_jj}\right)$	sharing rule for agent i of group j
$\alpha_j\in\mathbb{R}$	equalising part of the sharing rule
α	vector of α_j for $j\in\{1,2\}$
$\pi_{ij}(\boldsymbol{\alpha},\mathbf{x})$	payoff function of agent i of group j
$\gamma_j=\frac{1}{n_j}\sum_{i=1}^{n_j}x_{ij}\in\mathbb{R}_+$	average effort in group j
Table 1	

3 The Set of Second Period Equilibria

Without loss of generality, the second stage equilibria are presented in terms of average effort in each group, i.e. as pairs

$$(\gamma_1, \gamma_2) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

so that geometrically they can be represented in the non-negative quadrant.

Moreover, denote by

$$\gamma'_j = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} \pm e_{ij} \right) \in \mathbb{R} \quad \forall e_{ij} \in \mathbb{R}_{++}$$

the average effort in group j at any individual deviation from x_{ij}.

3.1 Characterization of the Set of Pure Strategy Nash Equilibria in the Second Period.

In this subsection, we characterise the full set of second-period Nash equilibria in pure strategies, to study the interplay within and among groups of the selective incentives induced by the sharing rules, which affect the incentives to free ride, with the strong complementarities, which favour the alignment of effort choice by teammates. Since the payoff functions are not smooth, we approach the problem by direct consideration of all possible strategy profiles, checking whether there is an individual incentive to deviate.

Without loss of generality, let us assume

$$n_2 \geq n_1.$$

3.1.1 Unrestricted Sharing Rules

Suppose $\alpha_i \in \mathbb{R}$**,** so that the sharing rule allows groups to engage in cross-subsidisation.

Proposition 1 *Let $e \in \mathbb{R}_{++}$ be a strictly positive real number. Then, in the CMMGC with unrestricted sharing rules, the set of the second period pure strategy Nash equilibria of the game is characterised as follows:*

1. *when*

$$(\alpha_1, \alpha_2) \in [1, \infty) \times [1, \infty);$$

then there exists a pure strategy equilibrium such that

$$(\gamma_1^*, \gamma_2^*) = (0, 0);$$

2. *when*

$$e \in \left(0, \frac{v}{2n_2}\right] \text{ and } (\alpha_1, \alpha_2) \in \left[1 - \frac{2n_1^2 e}{(n_1 - 1)\, v}, \infty\right) \times \left[1 - \frac{2n_2^2 e}{(n_2 - 1)\, v}, \infty\right)$$

there exists a continuum of pure strategy equilibria such that

$$(\gamma_1^{**}, \gamma_2^{**}) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e};$$

3. when

$$e \in \left(0, \frac{(n_2-1)\,v}{n_2\,(n_2-2)}\right] \quad \text{and} \quad (\alpha_1,\alpha_2) \in \mathbb{R} \times \left\{1-\frac{n_2^2 e}{(n_2-1)\,v}\right\}$$

there exists a continuum of pure strategy equilibria such that

$$\left(\gamma_1^{***},\gamma_2^{***}\right) = (0,e) \quad \text{with } \mathbf{x}_2 = \mathbf{e};$$

4. when

$$e \in \left(0, \frac{(n_1-1)\,v}{n_1\,(n_1-2)}\right] \quad \text{and} \quad (\alpha_1,\alpha_2) \in \left\{1-\frac{n_1^2 e}{(n_1-1)\,v}\right\} \times \mathbb{R}$$

there exists a continuum of pure strategy equilibria such that

$$\left(\gamma_1^{****},\gamma_2^{****}\right) = (e,0) \quad \text{with } \mathbf{x}_1 = \mathbf{e};$$

5. when (α_1,α_2) is not in these regions, there exist no pure strategy equilibria.

Proof. Let consider the different cases.

1. Suppose

$$(\gamma_1,\gamma_2) = (0,0)\ .$$

Then,

$$x_j\,(i) = min\,\{\mathbf{x}_j\} = min\,\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j\,(i) = \sum_{i=1}^{n_{-j}} x_{-j}\,(i) = 0$$

so that

$$\pi_{ij}\,(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[\frac{1}{n_j}\right] v\ .$$

If agent ij deviates to $x_j\,(i) = e' > 0$, then

$$x_j\,(i) = min\,\{\mathbf{x}_j\} = min\,\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j\,(i) = e', \sum_{i=1}^{n_{-j}} x_{-j}\,(i) = 0$$

so that the deviation payoff is

$$\pi_{ij}^D\,(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\,\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right] v - e'\ .$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{v}{2n_j} \geq \frac{1}{2}\left[(1-\alpha_j)\,\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right] v - e' \Leftrightarrow \alpha_j \geq \lim_{e'\to 0} 1 - \frac{2n_j e'}{(n_j-1)\,v} \Leftrightarrow \alpha_j \geq 1\ .$$

2. Suppose

$$(\gamma_1,\gamma_2) = (e,e) \text{ with } e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}\ .$$

Then,

$$x_j(i) = min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\}, \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[\frac{1}{n_j}\right]v - e \ .$$

If agent ij deviates to $x_j(i) = e' = 0$, then

$$x_j(i) = min\{\mathbf{x}_j\} < min\{\mathbf{x}_{-j}\} \, and \sum_{i=1}^{n_j} x_j(i) = (n_j - 1)\gamma_j \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = 0 \leq \frac{v}{2n_j} - e \Leftrightarrow v \geq 2n_j e \ .$$

Note that any deviation $0 < e' < e$ is strictly payoff-dominated by $e' = 0$, so that we can now take into account upward deviations only. If agent ij deviates to $x_j(i) = e' > e$, then

$$x_j(i) = min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\}, \sum_{i=1}^{n_j} x_j(i) = (n_j - 1)\gamma_j + e' \text{ and } \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{(n_j-1)e + e'} + \frac{\alpha_j}{n_j}\right]v - e' < \frac{v}{2n_j} - e \Leftrightarrow$$

$$\Leftrightarrow \alpha_j \geq \lim_{\epsilon \to 0} 1 - \frac{2n_j e}{v} - \frac{2n_j(e+\epsilon)}{(n_j-1)v} \Leftrightarrow \alpha_j \geq 1 - \frac{2n_j e}{v} - \frac{2n_j e}{(n_j-1)v} \ .$$

Therefore,

$$(\gamma_1, \gamma_2) = (e, e) \text{ with } e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}$$

is a Nash equilibrium for any

$$\alpha_j \in \left[1 - \frac{2n_j\gamma_j}{v} - \frac{2n_j\gamma_j}{(n_j-1)v}, \infty\right) \text{ and } v \geq 2n_j\gamma_j \ .$$

3 and 4. Suppose

$$(\gamma_j, \gamma_{-j}) \in (0, e) \quad \forall e \in \mathbb{R}_+ \text{ and } \mathbf{x}_j = \mathbf{e} \ .$$

Then

$$x_j(i) = \min\{\mathbf{x}_j\} = 0 < \min\{\mathbf{x}_{-j}\} = e \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}e \ ,$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = 0.$$

If agent ij deviates to $x_j(i) = e'$, $\forall e' \in \mathbb{R}_{++}$, then

$$\min\{\mathbf{x}_j\} = 0 < \min\{\mathbf{x}_{-j}\} = e \text{ and } \sum_{i=1}^{n_j} x_j(i) = e', \sum_{i=1}^{n_{-j}} x_j(i) = n_{-j}e$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = -e' \ .$$

Hence, for any player ij there is no incentive to deviate. On the other hand,

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{n_{-j}}v - e \ .$$

If agent $i-j$ deviates to any $e' \in \mathbb{R}_{++} \setminus \{e\}$, then

$$min\{\mathbf{x}_j\} = 0 < min\{\mathbf{x}_{-j}\} \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0, \sum_{i=1}^{n_{-j}} x_{-j}(i) = (n_{-j}-1)e + e'$$

so that the deviation payoff is

$$\pi_{i-j}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \left[(1-\alpha_{-j})\frac{e'}{(n_{-j}-1)e + e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \leq \frac{1}{n_{-j}}v - e \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e'}{(n_{-j}-1)v} & \text{if } e' > e \\ \alpha_{-j} \leq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e'}{(n_{-j}-1)v} & \text{if } e' < e \end{cases} \Leftrightarrow \begin{cases} \alpha_{-j} \geq \lim_{\epsilon\to 0} 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}(e+\epsilon)}{(n_{-j}-1)v} \\ \alpha_{-j} \leq \lim_{\epsilon\to 0} 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}(e-\epsilon)}{(n_{-j}-1)v} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \\ \alpha_{-j} \leq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \end{cases} \Leftrightarrow \alpha_{-j} = 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \ .$$

Conversely, if agent $i-j$ deviates to $e' = 0$, then

$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = 0, \sum_{i=1}^{n_{-j}} x_{-j}(i) = (n_{-j}-1)e$$

so that

$$\pi_{i-j}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \leq \frac{1}{n_{-j}}v - e \Leftrightarrow \alpha_{-j} \leq 2\left(1 - \frac{n_{-j}e}{v}\right) \ .$$

Note that

$$2\left(1 - \frac{n_{-j}e}{v}\right) \geq 1 - \frac{n_{-j}e}{v} - \frac{n_{-j}e}{(n_{-j}-1)v} \Leftrightarrow v \geq n_{-j}e - \frac{n_{-j}e}{n_{-j}-1} \ .$$

Hence,

$$(\gamma_j, \gamma_{-j}) \in (0, e) \quad \forall e \in \mathbb{R}_{++} \text{ such that } \mathbf{x}_j = \mathbf{e}$$

is a Nash equilibrium if and only if

$$(\alpha_j, \alpha_{-j}) \in \mathbb{R} \times \left\{1 - \frac{n_{-j}\gamma_{-j}}{v} - \frac{n_{-j}\gamma_{-j}}{(n_{-j}-1)v}\right\} \quad \forall v \geq n_{-j}\gamma_{-j} - \frac{n_{-j}\gamma_{-j}}{n_{-j}-1}.$$

5. Finally, note that similar arguments prove that the following strategy profiles are not second-period Nash equilibria in pure strategies:

1.

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++},\ min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0, 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1\ .^1$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} \leq n_j - 1\ .$$

Then, if $x_j(i) = e \geq min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$,

$$x_j(i) \geq \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e.$$

If agent ij deviates to $e' \geq min\{\mathbf{x}_j\}$ *s.t.* $e' \neq e$, then

$$x_j(i) \geq \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e'\ .$$

In contrast, if agent ij deviates to $e' < min\{\mathbf{x}_j\}$, then

$$x_j(i) = \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = -e'\ .$$

We select the payoff-dominant downward deviation, that is $e' = 0$. Hence, for any player ij there is no incentive to deviate if and only if

(a)

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

[1] Note that $\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}$ stands for the Indicator function taking value 1 when $x_{ij} = min\{\mathbf{x}_j\}$ for any ij.

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{(n_j\gamma_j - e)v} & \text{if } e' > e \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{(n_j\gamma_j - e)v} & \text{if } e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \lim_{\epsilon\to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + \epsilon)}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } e' > e \\ \alpha_j \leq \lim_{\epsilon\to 0} 1 - \frac{2n_j\gamma_j(n_j\gamma_j - max\{\mathbf{x}_j\} + max\{\mathbf{x}_j\} - \epsilon)}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \text{if } e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - min\{\mathbf{x}_j\})v} & \text{if } e' > e \\ \alpha_j \leq 1 - \frac{2(n_j\gamma_j)^2}{(n_j\gamma_j - max\{\mathbf{x}_j\})v} & \text{if } e' < e \end{cases}$$

and

(b)

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq 0 \Leftrightarrow \begin{cases} \alpha_j \geq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j > e \\ v \geq 2n_j\gamma_j & \gamma_j = e \\ \alpha_j \leq \frac{e}{\gamma_j - e}\left(\frac{2n_j\gamma_j}{v} - 1\right) & \gamma_j < e \end{cases}$$

Note that an α_j preventing both upward and downward deviations cannot exist, since the lower and upper bounds at point *a.* do cross, as $max\{\mathbf{x}_j\} > min\{\mathbf{x}_j\}$. Hence,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++},\, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij} = min\{\mathbf{x}_j\}} \leq n_j - 1$$

is not a Nash equilibrium for any $\alpha_j \in \mathbb{R}$.

2.

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = e \in \mathbb{R}_{++} \text{ and } \mathbf{x}_j = \mathbf{e}, \gamma_{-j} \in \mathbb{R}_{++},$$
$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} > 0, 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j} = min\{\mathbf{x}_{-j}\}} \leq n_{-j} - 1\ .$$

The proof follows the corresponding arguments shown at point 1.

3.

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2\ .$$ [2]

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \text{ and } 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2\ .$$

Then,

$$x_j(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

[2] Note that $\mathbb{1}_{x_{ij}=0}$ stands for the Indicator function taking value 1 when $x_{ij} = 0$ for any ij.

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\frac{v}{n_j}\ .$$

If agent ij deviates to $x_j\left(i\right)=e'>0$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j-e+e',\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{e'}+\frac{\alpha_j}{n_j}\right]v-e'\ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j}>\frac{1}{2}\left[\left(1-\alpha_j\right)+\frac{\alpha_j}{n_j}\right]v-1\Leftrightarrow\alpha_j\geq\lim_{e'\to0}1-\frac{2n_je'}{(n_j-1)v}\Leftrightarrow\alpha_j\geq1\ .$$

On the other hand,

$$x_{-j}\left(i\right)\geq\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e\ .$$

If agent $i-j$ deviates to $x_{-j}\left(i\right)=e'\neq e$, then

$$\min\left\{\mathbf{x}_j\right\}=\min\left\{\mathbf{x}_{-j}\right\}=0 \text{ and } \sum_{i=1}^{n_j}x_j\left(i\right)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}\left(i\right)=n_{-j}\gamma_{-j}-e+e'$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}\right)=\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e'}{n_{-j}\gamma_{-j}-e+e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e'\ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\left[\left(1-\alpha_{-j}\right)\frac{e}{n_{-j}\gamma_{-j}}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e\geq\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j-e+e'}+\frac{\alpha_j}{n_j}\right]v-e'\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_{-j}\geq1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e'>e\\ \alpha_{-j}\leq1-\frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j}-e+e'\right)}{(n_{-j}\gamma_{-j}-e)v} & \forall\ e'<e\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\alpha_{-j}\geq\lim_{\epsilon\to0}1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\}+min\{\mathbf{x}_{-j}\}+\epsilon)}{(n_{-j}\gamma_{-j}-min\{\mathbf{x}_{-j}\})v} & \forall\ e'>e\\ \alpha_{-j}\leq\lim_{\epsilon\to0}1-\frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j}-max\{\mathbf{x}_{-j}\}+max\{\mathbf{x}_{-j}\}-\epsilon)}{(n_{-j}\gamma_{-j}-max\{\mathbf{x}_{-j}\})v} & \forall\ e'<e\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v} & \forall\ e' > e \\ \alpha_{-j} \leq 1 - \frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\})v} & \forall\ e' < e \end{cases}$$

Note that the upper and lower bounds obtained above do cross, since $max\{\mathbf{x}_{-j}\} > min\{\mathbf{x}_{-j}\}$. Therefore,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } 2 \leq \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} \leq n_{-j} - 2$$

is not a Nash equilibrium for any $(\alpha_j, \alpha_{-j}) \in \mathbb{R} \times \mathbb{R}$.

4.

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } 2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} \leq n_j - 2\,.$$

The proof follows the corresponding arguments shown at point 3.

5.

$$(\gamma_1, \gamma_2) \quad \text{such that} \quad \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}\, and \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1$$

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j} - 1\,.$$

Then,

$$x_j(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{v}{n_j}\,.$$

If agent ij deviates to any $x_j(i) = e' > 0$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right]v - e'\,.$$

Hence, for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} \geq \frac{1}{2}\left[(1-\alpha_j) + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow \alpha_j \geq \lim_{e'\to 0} 1 - \frac{2n_je'}{(n_j-1)v} \Leftrightarrow \alpha_j \geq 1\,.$$

On the other hand, consider $x_{-j}(i) = 0$, that is

$$x_{-j}(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = e' \in \mathbb{R}_{++}$, then

$$\min\{\mathbf{x}_{-j}\} = \min\{\mathbf{x}_j\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} + e'$$

so that the deviation payoff is

$$\pi^D_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j}+e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \ .$$

Hence, for any player $i-j$ such that $x_{i-j} = 0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \frac{1}{2}\left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j}+e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \Leftrightarrow \alpha_{-j} \geq \lim_{e'\to 0} 1 - \frac{2(n_{-j}\gamma_{-j}+e')}{v} \Leftrightarrow$$

$$\Leftrightarrow \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}}{v} \ .$$

Consider the unique player $i-j$ such that $x_{i-j} = e \in \mathbb{R}_{++}$, that is

$$x_{-j}(i) \geq min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{e}{e} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e \ .$$

It is straightforward to see that deviating to any small $e' > 0$ strictly payoff-dominates any positive effort level $e > e'$ for player $i-j$, so that an equilibrium cannot exist. On the other hand, for completeness sake, if player $i-j$ deviates to $x_{-j}(i) = 0$, then

$$x_{-j}(i) = min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = 0$$

so that the deviation payoff is

$$\pi^D_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{v}{n_{-j}} \ .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_{-j})\frac{e'}{e'}+\frac{\alpha_{-j}}{n_{-j}}\right]v-e' \geq \frac{1}{2}\frac{v}{n_{-j}} \Leftrightarrow \alpha_{-j} \leq \lim_{e'\to 0} 1-\frac{2n_{-j}e'}{(n_{-j}-1)\,v} \Leftrightarrow \alpha_{-j} \leq 1 \ .$$

Therefore,

$$(0,\gamma_{-j}) \text{ such that } \gamma_{-j} \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_{-j}} \mathbb{1}_{x_{i-j}=0} = n_{-j}-1$$

is not a Nash equilibrium for any

$$(\alpha_j,\alpha_{-j}) \in \mathbb{R}\times\mathbb{R} \ .$$

6.

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j-1 \ .$$

Suppose

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j-1 \ .$$

Then, the proof follows the corresponding arguments shown at point v., so that

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = n_j-1$$

is a not a Nash equilibrium for any

$$(\alpha_1,\alpha_2) \in \mathbb{R}\times\mathbb{R} \ .$$

7.

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}\ ,\ min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 \ .$$

Suppose

$$(\gamma_1,\gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0 \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}} = 1 \ .$$

Then, if $x_j(i) = e > min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} > 0$,

$$x_j(i) > \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-e \ .$$

If agent ij deviates to $e' \geq min\left\{\mathbf{x}_j\right\}$ s.t. $e' \neq e$, then

$$x_j(i) \geq \min\left\{\mathbf{x}_j\right\} = \min\left\{\mathbf{x}_{-j}\right\} > 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \ .$$

In contrast, if $x_j(i) = e \geq min\left\{\mathbf{x}_j\right\} = min\left\{\mathbf{x}_1\right\} = min\left\{\mathbf{x}_2\right\} > 0$, and agent ij deviates to $e' < min\left\{\mathbf{x}_j\right\}$, then

$$x_j(i) = \min\left\{\mathbf{x}_j\right\} < \min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = -e' \ .$$

We select the payoff-dominant downward deviation, that is $e' = 0$. On the other hand, if the unique agent ij exerting effort $e = min\left\{\mathbf{x}_j\right\}$ deviates to $e' > e$, then

$$x_j(i) \geq \min\left\{\mathbf{x}_j\right\} > \min\left\{\mathbf{x}_{-j}\right\} \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}\right) = \left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \ .$$

Hence, for any player ij there is no incentive to deviate if and only if

(a)

$$\frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e}{n_j\gamma_j} + \frac{\alpha_j}{n_j}\right]v - e \geq \frac{1}{2}\left[\left(1-\alpha_j\right)\frac{e'}{n_j\gamma_j - e + e'} + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \forall\ e' > e \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - e\right)v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq \lim_{\epsilon\to 0} 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - min\{\mathbf{x}_j\} + min\{\mathbf{x}_j\} + \epsilon\right)}{\left(n_j\gamma_j - min\{\mathbf{x}_j\}\right)v} & \forall\ e' > e \\ \alpha_j \leq \lim_{\epsilon\to 0} 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - max\{\mathbf{x}_j\} + max\{\mathbf{x}_j\} - \epsilon\right)}{\left(n_j\gamma_j - max\{\mathbf{x}_j\}\right)v} & \forall\ e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_j \geq 1 - \frac{2\left(n_j\gamma_j\right)^2}{\left(n_j\gamma_j - min\{\mathbf{x}_j\}\right)v} & \forall\ e' > e \\ \alpha_j \leq 1 - \frac{2n_j\gamma_j\left(n_j\gamma_j - e + e'\right)}{\left(n_j\gamma_j - max\{\mathbf{x}_j\}\right)v} & \forall\ e' < e \end{cases}$$

(b)

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-e\geq 0\Leftrightarrow\begin{cases}\alpha_j\geq\frac{e}{\gamma_j-e}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j>e\\ v\geq 2n_j\gamma_j & \gamma_j=e\\ \alpha_j\leq\frac{e}{\gamma_j-e}\left(\frac{2n_j\gamma_j}{v}-1\right) & \gamma_j<e\end{cases}$$

(c)

$$\frac{1}{2}\left[(1-\alpha_j)\frac{e}{n_j\gamma_j}+\frac{\alpha_j}{n_j}\right]v-e\geq\left[(1-\alpha_j)\frac{e'}{n_j\gamma_j-e+e'}+\frac{\alpha_j}{n_j}\right]v-e'\Leftrightarrow$$

$$\Leftrightarrow\alpha_j\lesseqgtr\frac{(2n_j\gamma_je'-e(n_j\gamma_j-e+e'))v+(e-e')2n_j\gamma_j(n_j\gamma_j-e+e')}{(2n_j\gamma_je'-(\gamma_j+e)(n_j\gamma_j-e+e'))v}$$

Note that the lower and upper bounds at point $a.$ do cross, since $max\{\mathbf{x}_j\}>min\{\mathbf{x}_j\}$. Therefore,

$$(\gamma_1,\gamma_2)\text{ such that }\gamma_j\in\mathbb{R}_{++},min\{\mathbf{x}_1\}=min\{\mathbf{x}_2\}>0\text{ and }\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=min\{\mathbf{x}_j\}}=1$$

is not a Nash equilibrium for any

$$(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R}\ .$$

8.

$$(\gamma_j,\gamma_{-j})\text{ such that }\gamma_j=e\in\mathbb{R}_{++}\text{ and }\mathbf{x}_j=\mathbf{e},\gamma_{-j}\in\mathbb{R}_{++}\ ,$$

$$min\{\mathbf{x}_j\}=min\{\mathbf{x}_{-j}\}>0\text{ and }\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=min\{\mathbf{x}_{-j}\}}=1\ .$$

The proof follows the corresponding arguments shown at point 7.

9.

$$(\gamma_j,\gamma_{-j})\text{ such that }\gamma_j=0\ ,\ \gamma_{-j}\in\mathbb{R}_{++}\text{ and }\sum_{i=1}^{n_{-j}}\mathbb{1}_{x_{i-j}=0}=1\ .$$

Suppose

$$(\gamma_1,\gamma_2)\text{ such that }\gamma_j=0,\gamma_{-j}\in\mathbb{R}_{++},min\{\mathbf{x}_1\}=min\{\mathbf{x}_2\}\text{ and }\mathbb{1}_{x_{i-j}=0}=1\ .$$

Then,

$$x_j(i)=\min\{\mathbf{x}_j\}=\min\{\mathbf{x}_{-j}\}=0\text{ and }\sum_{i=1}^{n_j}x_j(i)=n_j\gamma_j,\sum_{i=1}^{n_{-j}}x_{-j}(i)=n_{-j}\gamma_{-j}$$

so that

$$\pi_{ij}(\gamma_j,\gamma_{-j}|\boldsymbol{\alpha})=\frac{1}{2}\frac{v}{n_j}\ .$$

If agent ij deviates to $x_j(i) = e'$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j - e + e', \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\left[(1-\alpha_j)\frac{e'}{e'} + \frac{\alpha_j}{n_j}\right]v - e' \ .$$

Hence for any player ij there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{v}{n_j} > \frac{1}{2}\left[(1-\alpha_j) + \frac{\alpha_j}{n_j}\right]v - e' \Leftrightarrow \alpha_j \geq \lim_{e'\to 0} 1 - \frac{2n_j}{(n_j-1)v} \Leftrightarrow \alpha_j \geq 1 \ .$$

On the other hand, consider $x_{-j}(i) = 0$, that is

$$x_{-j}(i) = \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\gamma_j, \gamma_{-j}|\boldsymbol{\alpha}) = \frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = e' \in \mathbb{R}_{++}$, then

$$\min\{\mathbf{x}_{-j}\} > \min\{\mathbf{x}_j\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} + e'$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\alpha, \gamma_j, \gamma'_{-j}\right) = \left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j} + e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \ .$$

Hence, for the unique player $i-j$ such that $x_{i-j} = 0$ there is no incentive to deviate if and only if

$$\frac{1}{2}\frac{\alpha_{-j}}{n_{-j}}v \geq \left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j} + e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \frac{2n_{-j}e'\left(v - n_{-j}\gamma_{-j} - e'\right)}{(2n_{-j}e' - n_{-j}\gamma_{-j} - e')v} & \text{if } 2n_{-j}e' - n_{-j}\gamma_{-j} - e' > 0 \\ \alpha_{-j} \leq \frac{2n_{-j}e'\left(v - n_{-j}\gamma_{-j} - e'\right)}{(2n_{-j}e' - n_{-j}\gamma_{-j} - e')v} & \text{if } 2n_{-j}e' - n_{-j}\gamma_{-j} - e' < 0 \end{cases}$$

Consider any player $i-j$ such that $x_{i-j} > 0$, that is

$$x_{-j}(i) > \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j}$$

so that

$$\pi_{i-j}(\alpha, \gamma_j, \gamma_{-j}) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{e}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right] - e \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = e' \in \mathbb{R}_+$, then

$$\min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} = 0 \text{ and } \sum_{i=1}^{n_j} x_j(i) = n_j\gamma_j, \sum_{i=1}^{n_{-j}} x_{-j}(i) = n_{-j}\gamma_{-j} - e + e' ,$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\alpha, \gamma_j, \gamma'_{-j}\right) = \frac{1}{2}\left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j} - e + e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' .$$

Hence, for any player $i-j$ there is no incentive to deviate if and only if

$$\frac{1}{2}\left[(1-\alpha_{-j})\frac{e}{n_{-j}\gamma_{-j}} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e \geq \frac{1}{2}\left[(1-\alpha_{-j})\frac{e'}{n_{-j}\gamma_{-j} - e + e'} + \frac{\alpha_{-j}}{n_{-j}}\right]v - e' \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - e + e'\right)}{(n_{-j}\gamma_{-j} - e)v} & \forall\, e' > e \\ \alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - e + e'\right)}{(n_{-j}\gamma_{-j} - e)v} & \forall\, e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq \lim_{\epsilon\to 0} 1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\} + min\{\mathbf{x}_{-j}\} + \epsilon)}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v} & \forall\, e' > e \\ \alpha_{-j} \leq \lim_{\epsilon\to 0} 1 - \frac{2n_{-j}\gamma_{-j}(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\} + max\{\mathbf{x}_{-j}\} - \epsilon)}{(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\})v} & \forall\, e' < e \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \alpha_{-j} \geq 1 - \frac{2(n_{-j}\gamma_{-j})^2}{(n_{-j}\gamma_{-j} - min\{\mathbf{x}_{-j}\})v} & \forall\, e' > e \\ \alpha_{-j} \leq 1 - \frac{2n_{-j}\gamma_{-j}\left(n_{-j}\gamma_{-j} - e + e'\right)}{(n_{-j}\gamma_{-j} - max\{\mathbf{x}_{-j}\})v} & \forall\, e' < e \end{cases}$$

Note that the upper and lower bounds above do cross, since $max\{\mathbf{x}_{-j}\} > min\{\mathbf{x}_{-j}\}$. Therefore,

$$(\gamma_j, \gamma_{-j}) \text{ such that } \gamma_j = 0, \gamma_{-j} \in \mathbb{R}_{++}, min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} = 0 \text{ and } \mathbb{1}_{x_{i-j}=0} = 1$$

is not a Nash equilibrium for any

$$(\alpha_j, \alpha_{-j}) \in \mathbb{R} \times \mathbb{R} .$$

10.

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in \mathbb{R}_{++} \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=0} = 1 .$$

The proof follows the corresponding arguments shown at point 9.

■

Figure 1 represents graphically the regions of the unrestricted sharing rules with the consequent possible effort equilibria.

From proposition 1 and its proof, it follows immediately the following result.

Corollary 1 *In the CMMGC, there are no second-period within-group asymmetric Nash equilibria in pure strategies.*

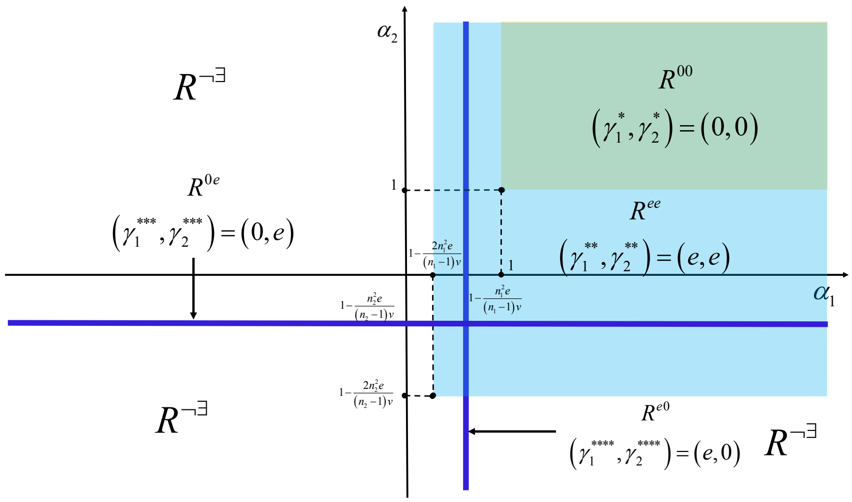


Figure 1: Sharing rules' regions and effort equilibria.

3.1.2 Restricted Sharing Rules

If we consider the restriction of sharing rules to $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, then we get the following result.

Proposition 2 *In the CMMGC with restricted sharing rules, the set of the second period pure strategy Nash equilibria of the game is characterised as follows:*

1. *When*
$$(\alpha_1, \alpha_2) \in \{1\} \times \{1\};$$
then there exists a pure strategy equilibrium such that
$$(\gamma_1^r, \gamma_2^r) = (0,0);$$

2. *If*
$$n_1 \geq 4 \text{ and } n_1 \in \{n_2 - 1, n_2\}$$
then, when
$$e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right] \text{ and } (\alpha_1, \alpha_2) \in [0,1] \times [0,1],$$
there exists a continuum of pure strategy equilibria such that
$$(\gamma_1^{rr}, \gamma_2^{rr}) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e};$$

3. *when*
$$e \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right] \text{ and } (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)\,v}\right\}$$
there exists a continuum of pure strategy equilibria such that
$$(\gamma_1^{rrr}, \gamma_2^{rrr}) = (0,e) \quad \text{such that } \mathbf{x}_2 = \mathbf{e};$$

4. *when*

$$e \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \text{ and } (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)\,v}\right\} \times [0,1]$$

there exists a continuum of pure strategy equilibria such that

$$(\gamma_1^{rrrr}, \gamma_2^{rrrr}) = (e, 0) \quad \text{such that } \mathbf{x}_1 = \mathbf{e}.$$

Proof.

1. Immediate;

2. In the restricted case, the equilibrium

$$(\gamma_j, \gamma_{-j}) = (e, e) \quad \text{such that } \mathbf{x}_{-j} = \mathbf{e}$$

requires the following conditions

(a)

$$\begin{cases} 1 - \frac{2n_1^2 e}{(n_1-1)v} \leq 0 \\ 1 - \frac{2n_2^2 e}{(n_2-1)v} \leq 0 \end{cases} \Leftrightarrow \begin{cases} e \geq \frac{(n_1-1)}{2n_1^2} v \\ e \geq \frac{(n_2-1)}{2n_2^2} v \end{cases} \Leftrightarrow e \geq \frac{(n_1-1)}{2n_1^2} v^3$$

$$e \leq \frac{v}{2n_2}.$$

(b) Hence

$$e \in \left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right]$$

and

$$\left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right] \neq \varnothing \Leftrightarrow \frac{1}{n_2} \geq \frac{(n_1-1)}{n_1^2} \Leftrightarrow n_1^2 - n_2 n_1 + n_2 \geq 0 \Leftrightarrow$$

$$n_1 \leq \frac{n_2 - \sqrt{n_2^2 - 4n_2}}{2} \quad \vee \quad n_1 \geq \frac{n_2 + \sqrt{n_2^2 - 4n_2}}{2}.$$

Thus

$$\left[\frac{(n_1-1)}{2n_1^2} v, \frac{v}{2n_2}\right] \neq \varnothing \Leftrightarrow n_1 \in \left[\frac{n_2 + \sqrt{n_2^2 - 4n_2}}{2}, n_2\right] \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} n_1 = n_2 \\ n_1 = n_2 - 1 \\ n_1 = 2, n_2 = 4. \end{cases}$$

2 and 3. In the restricted case, the equilibrium

$$(\gamma_j, \gamma_{-j}) = (0, e) \quad \text{such that } \mathbf{x}_{-j} = \mathbf{e}$$

requires the following conditions

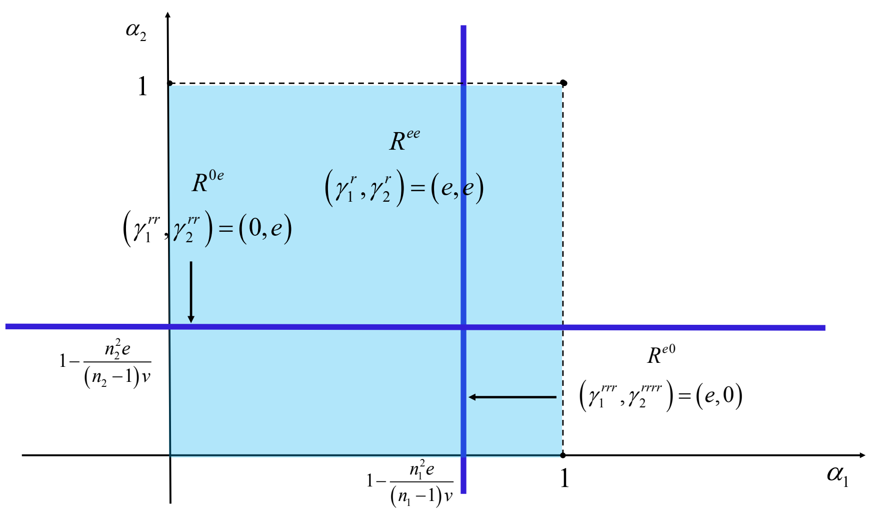
i.
$$\left\{1-\frac{n_{-j}^2 e}{(n_{-j}-1)\,v}\right\} \in [0,1] \Leftrightarrow \begin{cases} 1-\frac{n_{-j}^2 e}{(n_{-j}-1)v} \geq 0 \\ 1-\frac{n_{-j}^2 e}{(n_{-j}-1)v} \leq 1 \end{cases} \Leftrightarrow$$
$$\Leftrightarrow \frac{n_{-j}^2 e}{(n_{-j}-1)\,v} \leq 1 \Leftrightarrow e \leq \frac{(n_{-j}-1)\,v}{n_{-j}^2}$$

ii.
$$v \geq \frac{n_{-j}\,(n_{-j}-2)}{n_{-j}-1} e \Leftrightarrow e \leq \frac{(n_{-j}-1)\,v}{n_{-j}\,(n_{-j}-2)}.$$

However, since
$$\frac{(n_{-j}-1)\,v}{n_{-j}^2} \leq \frac{(n_{-j}-1)\,v}{n_{-j}\,(n_{-j}-2)},$$
only the first inequality is binding.

■

Remark 1 *Note that when* $n_1 \geq 4$ *and* $n_1 \in \{n_2-1, n_2\}$ *, all four type of effort equilibria exist. In figure 2, we represent the region of restricted sharing rules with the consequent effort equilibria.*



4 The Set of First-Period Equilibria

4.1 Unrestricted Sharing Rules

If $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, i.e. if the sharing rules are unrestricted, then it is immediate that for some values of (α_1, α_2) it is not possible to express the continuation payoffs, since for some regions of $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, there exists no second-period equilibrium, so that it is not possible to define a Nash equilibrium in pure strategies in the first period. Therefore, in this case, we might conclude with the following result.

Proposition 3 *In the CMMGC with unrestricted sharing rules, there are no subgame perfect Nash equilibria in pure strategies.*

4.2 Restricted Sharing Rules

If $(\alpha_1, \alpha_2) \in [0,1] \times [0,1]$, i.e. if the sharing rules are restricted, when

$$n_1 \geq 4 \text{ and } n_1 \in \{n_2 - 1, n_2\}$$

then for all values of (α_1, α_2) it is possible to express the continuation payoffs, so that it is possible to try to calculate the subgame perfect equilibria of the entire game.

Hypothesis 1 *Assume that*

$$(\alpha_1, \alpha_2) \in [0,1] \times [0,1] \;\; \textit{and} \;\; n_1 \geq 4 \;\; \textit{and} \;\; n_1 \in \{n_2 - 1, n_2\}.$$

To derive the equilibrium sharing rules, we consider their optimal choice in each group by an utilitarian ruler with payoff function

$$\pi_j^C(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e) = \sum_{i=1}^{n_j} \pi_{ij}^C(\alpha_j, \alpha_{-j} | \gamma_1^e, \gamma_2^e).$$

In the first period, the players may have different expectations on the second period equilibria, for instance:

1. suppose that the groups expect the effort stage equilibrium

$$\forall (\alpha_1, \alpha_2) \in [0,1] \times [0,1], \quad (\gamma_1^I, \gamma_2^I) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e}$$

where

$$e \in \left[\frac{(n_1 - 1) v}{2n_1^2}, \frac{v}{2n_2}\right)$$

so that the players' continuation payoffs are

$$\pi_{ij}(\alpha_j, \alpha_{-j} | \gamma_1^I, \gamma_2^I) = \frac{1}{2}\left[(1 - \alpha_j)\frac{e}{n_j e} + \alpha_j \frac{1}{n_j}\right] v - e = \frac{1}{2n_j} v - e .$$

2. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{II},\gamma_2^{II}\right)=\begin{cases}(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & if\quad (\alpha_1,\alpha_2)\in[0,1)\times[0,1)\\ (0,0) & if\quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\}\end{cases}$$

where

$$e\in\left[\frac{(n_1-1)\,v}{2n_1^2},\frac{v}{2n_2}\right)$$

so that the players' continuation payoffs are

$$\pi_{ij}\left(\alpha_j,\alpha_{-j}|\gamma_1^{II},\gamma_2^{II}\right)=\begin{cases}\frac{1}{2n_j}v-e & if\quad (\alpha_1,\alpha_2)\in[0,1)\times[0,1)\\ \frac{1}{2n_j}v & if\quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\}\end{cases}$$

3. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{III},\gamma_2^{III}\right)=\begin{cases}(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise\\ (0,e) & if\quad (\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\}\end{cases}$$

where

$$e\in\begin{cases}\left[\frac{(n_1-1)v}{2n_1^2},\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{III},\gamma_2^{III}\right)=(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e}\\ \left(0,\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{III},\gamma_2^{III}\right)=(0,e)\end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2|\gamma_1^{III},\gamma_2^{III}\right)=\begin{cases}\frac{1}{2n_1}v-e & otherwise\\ 0 & if\quad (\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\}\end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{III},\gamma_2^{III}\right)=\begin{cases}\frac{1}{2n_2}v-e & otherwise\\ \frac{1}{n_2}v-e & if\quad (\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\}\end{cases}$$

4. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{IV},\gamma_2^{IV}\right)=\begin{cases}(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e} & otherwise\\ (e,0) & if\quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1]\end{cases}$$

where

$$e\in\begin{cases}\left[\frac{(n_1-1)v}{2n_1^2},\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{IV},\gamma_2^{IV}\right)=(e,e)\ \text{ with } \mathbf{x}_j=\mathbf{e}\\ \left(0,\frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{IV},\gamma_2^{IV}\right)=(e,0)\end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2|\gamma_1^{IV},\gamma_2^{IV}\right)=\begin{cases}\frac{1}{2n_1}v-e & otherwise\\ \frac{1}{n_1}v-e & if\quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1]\end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{IV},\gamma_2^{IV}\right)=\begin{cases}\frac{1}{2n_2}v-e & otherwise\\ 0 & if\quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1]\end{cases}$$

5. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^V, \gamma_2^V\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (0,e) & if \quad (\alpha_1, \alpha_2) \in [0,1) \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^V, \gamma_2^V\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^V, \gamma_2^V\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^V, \gamma_2^V\right) = \begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ 0 & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^V, \gamma_2^V\right) = \begin{cases} \frac{1}{2n_2}v - e & otherwise \\ \frac{1}{2n_2}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ \frac{1}{n_2}v - e & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

6. Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e,0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1) \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VI}, \gamma_2^{VI}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VI}, \gamma_2^{VI}\right) = (e,0) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ \frac{1}{n_1}v - e & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{VI}, \gamma_2^{VI}\right) = \begin{cases} \frac{1}{2n_2}v - e & otherwise \\ \frac{1}{2n_2}v & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ 0 & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \end{cases}$$

7.a Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e,0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ (0,e) & if \quad (\alpha_1, \alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (e,0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIa}, \gamma_2^{VIIa}\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} \frac{1}{2n_1} v - e & otherwise \\ \frac{1}{n_1} v - e & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ 0 & if \quad (\alpha_1, \alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1, \alpha_2 | \gamma_1^{VIIa}, \gamma_2^{VIIa}\right) =$$

$$\begin{cases} \frac{1}{2n_2} v - e & otherwise \\ 0 & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0,1] \\ \frac{1}{n_2} v - e & if \quad (\alpha_1, \alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

7.b Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e,0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ (0,e) & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (e,0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIb}, \gamma_2^{VIIb}\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2|\gamma_1^{VIIb},\gamma_2^{VIIb}\right)=$$

$$\begin{cases} \frac{1}{2n_1}v-e & otherwise \\ \frac{1}{n_1}v-e & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times\left\{\left[0,1-\frac{n_2^2e}{(n_2-1)v}\right)\cup\left(1-\frac{n_2^2e}{(n_2-1)v},1\right]\right\} \\ 0 & if \quad (\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{VIIb},\gamma_2^{VIIb}\right)=$$

$$\begin{cases} \frac{1}{2n_2}v-e & otherwise \\ 0 & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times\left\{\left[0,1-\frac{n_2^2e}{(n_2-1)v}\right)\cup\left(1-\frac{n_2^2e}{(n_2-1)v},1\right]\right\} \\ \frac{1}{n_2}v-e & if \quad (\alpha_1,\alpha_2)\in[0,1]\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

8.a Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=$$

$$\begin{cases} (e,e)\ \text{with}\ \mathbf{x}_j=\mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ (e,0) & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1) \\ (0,e) & \left\{\left[0,1-\frac{n_1^2e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2e}{(n_1-1)v},1\right)\right\}\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e\in\begin{cases} \left[\frac{(n_1-1)v}{2n_1^2},\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=(e,e)\ \text{with}\ \mathbf{x}_j=\mathbf{e} \\ \left(0,\frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=(e,0) \\ \left(0,\frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=(0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2|\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=$$

$$\begin{cases} \frac{1}{2n_1}v-e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ \frac{1}{n_1}v-e & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1) \\ 0 & if \quad (\alpha_1,\alpha_2)\in\left\{\left[0,1-\frac{n_1^2e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2e}{(n_1-1)v},1\right)\right\}\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2|\gamma_1^{VIIIa},\gamma_2^{VIIIa}\right)=$$

$$\begin{cases} \frac{1}{2n_2}v-e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1,\alpha_2)\in\{1\}\times\{1\} \\ 0 & if \quad (\alpha_1,\alpha_2)\in\left\{1-\frac{n_1^2e}{(n_1-1)v}\right\}\times[0,1) \\ \frac{1}{n_2}v-e & if \quad (\alpha_1,\alpha_2)\in\left\{\left[0,1-\frac{n_1^2e}{(n_1-1)v}\right)\cup\left(1-\frac{n_1^2e}{(n_1-1)v},1\right)\right\}\times\left\{1-\frac{n_2^2e}{(n_2-1)v}\right\} \end{cases}$$

8.b Suppose that the groups expect the effort stage equilibrium

$$\left(\gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) =$$

$$\begin{cases} (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0,0) & if \quad (\alpha_1,\alpha_2) \in \{1\} \times \{1\} \\ (e,0) & if \quad (\alpha_1,\alpha_2) \in \left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1-\frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1-\frac{n_2^2 e}{(n_2-1)v}, 1\right)\right\} \\ (0,e) & if \quad (\alpha_1,\alpha_2) \in [0,1) \times \left\{1-\frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

where

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) = (e,e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } \left(\gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) = (e,0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } \left(\gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) = (0,e) \end{cases}$$

so that the players' continuation payoffs are

$$\pi_{i1}\left(\alpha_1,\alpha_2 | \gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) =$$

$$\begin{cases} \frac{1}{2n_1}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1,\alpha_2) \in \{1\} \times \{1\} \\ \frac{1}{n_1}v - e & if \quad (\alpha_1,\alpha_2) \in \left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1-\frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1-\frac{n_2^2 e}{(n_2-1)v}, 1\right)\right\} \\ 0 & if \quad (\alpha_1,\alpha_2) \in [0,1) \times \left\{1-\frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

and

$$\pi_{i2}\left(\alpha_1,\alpha_2 | \gamma_1^{VIIIb}, \gamma_2^{VIIIb}\right) =$$

$$\begin{cases} \frac{1}{2n_2}v - e & otherwise \\ \frac{1}{2n_1}v & if \quad (\alpha_1,\alpha_2) \in \{1\} \times \{1\} \\ 0 & if \quad (\alpha_1,\alpha_2) \in \left\{1-\frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1-\frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1-\frac{n_2^2 e}{(n_2-1)v}, 1\right)\right\} \\ \frac{1}{n_2}v - e & if \quad (\alpha_1,\alpha_2) \in [0,1) \times \left\{1-\frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

Notwithstanding the presence of continua of equilibria in the second period, the set of possible continuation payoffs orderings is what matters from the perspective of the first period. However, within each class of equilibria the continuation payoffs can be ordered. For instance, consider the (e,e) equilibrium which is sustained by any $(\alpha_1,\alpha_2) \in [0,1] \times [0,1]$. In this class of equilibria, the effort level e can take up to sixteen different values in the sixteen regions defined by the thresholds equilibrium conditions, as shown in figure 2. The same argument applies to equilibria $(e_l,0),(e_m,0),(e_h,0)$ and $(0,e_l),(0,e_m),(0,e_h)$, where the effort level can take up to four different values in the four regions. Therefore, we can compute the cardinality of classes of continuation payoffs matrices as:

- for $n_1 \in \{n_2-1, n_2\}$, $(16{+}1)\times(16{+}1{+}3{*}4)\times(16{+}1)\times(16{+}1{+}1)\times(16{+}1)\times(16{+}1{+}3{*}4)\times(16{+}1)\times(16{+}1)\times(16{+}1{+}3{*}4)\times(16{+}1{+}3{*}4{*}2)\times(16{+}1{+}3{*}4)\times(16{+}1{+}3{*}4)\times(16{+}1)\times(16{+}1{+}3{*}4)\times(16{+}1)\times(16{+}1) = 3.0622e{+}21 = 3062200000000000000000$.

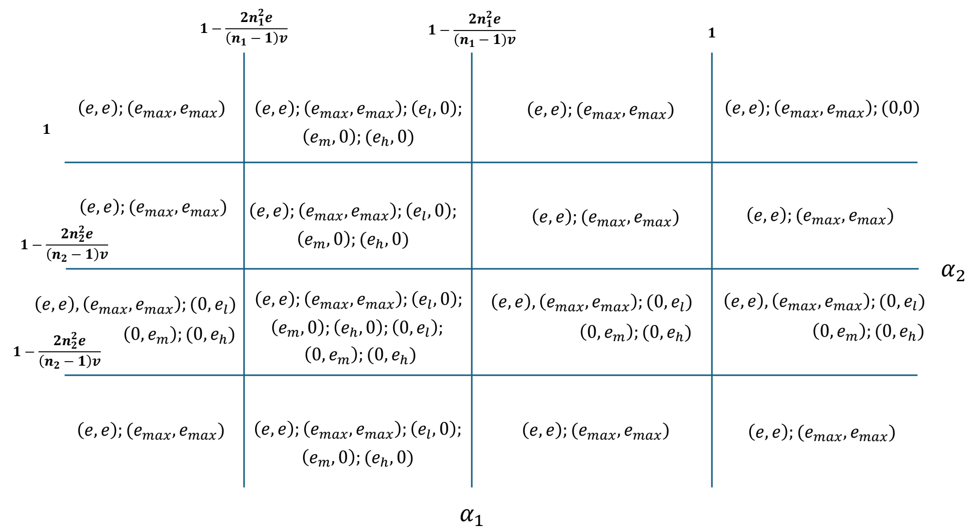


Figure 2: Second-period pure Nash equilibria in $\alpha_1 \times \alpha_2$ space in the restricted case with $n_1 \in \{n_2 - 1, n_2\}$.

Therefore, for computational reasons, we limit ourselves to the case where there is just one effort level over the sixteen regions determined by the equilibrium threshold conditions for each class of equilibria.

With these restrictions the cardinality of continuation payoffs matrices equals:

- for $n_1 \in \{n_2 - 1, n_2\}$, $2\times5\times2\times3\times2\times5\times2\times2\times5\times8\times5\times5\times2\times5\times2\times2 = 96000000$.

Accordingly, we employ a simple algorithm in MATLAB reported in the Appendix to compute the set of first-stage equilibria, and thus the set of subgame perfect Nash equilibria. Nonetheless, given that the between-groups and within-groups symmetric equilibrium (e, e) is sustained by any $(\alpha_1, \alpha_2) \in [0, 1] \times [0, 1]$, even in the absence of the aforementioned restriction it is straightforward to derive the following proposition .

Proposition 4 *In the CMMGC with restricted sharing rules, there exist at least a continuation-payoffs matrix such that the equilibrium sharing rules are:*

$$(\alpha_1^*, \alpha_2^*) = [0, 1] \times [0, 1] \ .$$

4.2.1 The Preference Ordering of Groups' Utilitarian Rulers

Let denote by

$$\succ_j$$

the preference ordering of the utilitarian ruler of group j, as induced by the above payoff function.

Moreover, let denote

- (e_{max}, e_{max}) for (e, e) such that $e = \frac{v}{2n_2}$;
- $(e_l, 0)$ any $(e, 0)$ such that $e \in \left(0, \frac{v}{2n_1}\right)$;
- $(e_m, 0)$ for $(e, 0)$ such that $e = \frac{v}{2n_1}$;
- $(e_h, 0)$ any $(e, 0)$ such that $e \in \left(\frac{v}{2n_1}, \frac{(n_1-1)v}{n_1^2}\right]$;
- $(0, e_l)$ any $(0, e)$ such that $e \in \left(0, \frac{v}{2n_2}\right)$;
- $(0, e_m)$ for $(0, e)$ such that $e = \frac{v}{2n_2}$;
- $(0, e_h)$ any $(0, e)$ such that $e \in \left(\frac{v}{2n_2}, \frac{(n_2-1)v}{n_2^2}\right]$.

Then,

Lemma 1 *The preference relations for the utilitarian rulers are such that:*

- *if* $n_1 = n_2 - 1$,

  $(e_l, 0) \succ_1 (e_m, 0) \sim_1 (0, 0) \succ_1 (e_h, 0) \succ_1 (e, e) \succ_1 (e_{max}, e_{max}) \succ_1 (0, e_l) \sim_1 (0, e_m) \sim_1 (0, e_h)$,

  $(0, e_l) \succ_2 (0, e_m) \sim_2 (0, 0) \succ_2 (0, e_h) \succ_2 (e, e) \succ_2 (e_{max}, e_{max}) \sim_2 (e_l, 0) \sim_2 (e_m, 0) \sim_2 (e_h, 0)$;

- *if* $n_1 = n_2$, *the preference relations for the utilitarian rulers are:*

  $(e_l, 0) \succ_1 (e_m, 0) \sim_1 (0, 0) \succ_1 (e_h, 0) \succ_1 (e, e) \succ_1 (e_{max}, e_{max}) \sim_1 (0, e_l) \sim_1 (0, e_m) \sim_1 (0, e_h)$,

  $(0, e_l) \succ_2 (0, e_m) \sim_2 (0, 0) \succ_2 (0, e_h) \succ_2 (e, e) \succ_2 (e_{max}, e_{max}) \sim_2 (e_l, 0) \sim_2 (e_m, 0) \sim_2 (e_h, 0)$.

Proof. It is immediate to derive the following preference relations for the utilitarian rulers over the different second-period equilibria: ■

- $$(e', 0) \succ_1 (0, e'') \;\forall\; e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \quad \text{and} \quad e'' \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$
  $$(0, e'') \succ_2 (e', 0) \;\forall\; e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \quad \text{and} \quad e'' \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right];$$

Proof.

- if $n_1 = n_2 - 1$

$$(e, e) \succ_1 (0, e'') \;\forall\; e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right] \text{ and } e'' \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$

$$(e, e) \succ_2 (e', 0) \;\forall\; e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right],$$

$$(e, e) \sim_2 (e', 0) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\; e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right];$$

- if $n_1 = n_2$

$$(e,e) \succ_1 (0,e^{''})\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right],$$

$$(e,e) \sim_1 (0,e^{''}) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right];$$

$$(e,e) \succ_2 (e',0)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right) \text{ and } e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right],$$

$$(e,e) \sim_2 (e',0) \text{ if } e = \frac{v}{2n_2} \text{ and } \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right];$$

- $$(0,0) \succ_1 (e,e)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right],$$

$$(0,0) \succ_2 (e,e)\ \forall\ e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right];$$

- $$(e',0) \succ_1 (e,e)\ \forall\ e' \in \left(0, \frac{(n_1-1)\,v}{n_1^2}\right] \text{ and } e \in \left[\frac{(n_1-1)\,v}{2n_1^2}, \frac{v}{2n_2}\right],$$

$$(0,e^{''}) \succ_2 (e,e)\ \forall\ e^{''} \in \left(0, \frac{(n_2-1)\,v}{n_2^2}\right] \text{ and } e \in \left[\frac{(n_2-1)\,v}{2n_2^2}, \frac{v}{2n_2}\right];$$

- $$(e',0) \succ_1 (0,0)\ \forall\ e' \in \left(0, \frac{v}{2n_1}\right)\ ,$$

$$(e',0) \sim_1 (0,0) \text{ if } e' = \frac{v}{2n_1}\ ,$$

$$(0,0) \succ_1 (e',0)\ \forall\ e' \in \left(\frac{v}{2n_1}, \frac{(n_1-1)\,v}{n_1^2}\right]\ ,$$

$$(0,e^{''}) \succ_2 (0,0)\ \forall\ e^{''} \in \left(0, \frac{v}{2n_2}\right),$$

$$(0,e^{''}) \sim_2 (0,0) \text{ if } e^{''} = \frac{v}{2n_2}\ ,$$

$$(0,0) \succ_2 (e^{''},0)\ \forall\ e^{''} \in \left(\frac{v}{2n_2}, \frac{(n_2-1)\,v}{n_2^2}\right]\ .$$

■

Thus, we may conclude with the following result.

Corollary 2 *According to the utilitarian rulers preferences, the Pareto efficient effort equilibria are*

$$(e_l,0), \quad (0,e_l) \quad \textit{and} \quad (0,0).$$

5 The Set of Subgame Perfect Equilibria

From the results illustrated in the previous section, it is immediate to derive the following proposition.

Proposition 5 *In the CMMGC with restricted sharing rules, there are continua of subgame perfect Nash equilibria in pure strategies.*

As an illustration, we provide a few examples of subgame perfect equilibria in pure strategies:

1. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and } e \in \left[\frac{(n_1 - 1)\, v}{2n_1^2}, \frac{v}{2n_2}\right),$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times [0,1]$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, e)\, \forall\, (\alpha_1, \alpha_2) \in [0,1] \times [0,1]$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times [0,1] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, e)$$
as equilibrium outcomes;

2. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and } e \in \left[\frac{(n_1 - 1)\, v}{2n_1^2}, \frac{v}{2n_2}\right),$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \{1\} \times \{1\}$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & if \quad (\alpha_1, \alpha_2) \in [0,1) \times [0,1) \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \end{cases}$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \{1, 1\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, 0)$$
as equilibrium outcomes;

3. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$
$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\}$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0,1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, e)$$
as equilibrium outcomes;

4. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$
$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right] & \text{if } (\gamma_1, \gamma_2) = (e, 0) \end{cases},$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1) v}\right\} \times [0, 1]$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1] \end{cases}$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1) v}\right\} \times [0, 1] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, 0)$$
as equilibrium outcomes;

5. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$
$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \{1\} \times \{1\}$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in [0, 1) \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \{1\} \times \{1\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, 0)$$
as equilibrium outcomes;

6. if
$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$
$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \end{cases},$$
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \{1\} \times \{1\}$$
$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = \begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1) \end{cases}$$
with
$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) \in \{1\} \times \{1\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, 0)$$
as equilibrium outcomes;

7.a if

$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right] & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1)\, v}\right\} \times [0, 1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1] \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right]\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1)\, v}\right\} \times [0, 1] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, 0)$$

as equilibrium outcomes;

7.b if

$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\}$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & otherwise \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right]\right\} \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1)\, v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, e)$$

as equilibrium outcomes;

8.a if

$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1) v}\right\} \times [0, 1]$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & \textit{otherwise} \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times [0, 1) \\ (0, e) & \left\{\left[0, 1 - \frac{n_1^2 e}{(n_1-1)v}\right) \cup \left(1 - \frac{n_1^2 e}{(n_1-1)v}, 1\right)\right\} \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = \left\{1 - \frac{n_1^2 e}{(n_1 - 1) v}\right\} \times [0, 1] \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (e, 0)$$

as equilibrium outcomes;

8.b if

$$n_1 > 2, n_1 \in \{n_2 - 1, n_2\} \text{ and}$$

$$e \in \begin{cases} \left[\frac{(n_1-1)v}{2n_1^2}, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} \\ \left(0, \frac{v}{2n_1}\right) & \text{if } (\gamma_1, \gamma_2) = (e, 0) \\ \left(0, \frac{v}{2n_2}\right) & \text{if } (\gamma_1, \gamma_2) = (0, e) \end{cases},$$

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1) v}\right\}$$

$$\left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) =$$

$$\begin{cases} (e, e) \text{ with } \mathbf{x}_j = \mathbf{e} & \textit{otherwise} \\ (0, 0) & if \quad (\alpha_1, \alpha_2) \in \{1\} \times \{1\} \\ (e, 0) & if \quad (\alpha_1, \alpha_2) \in \left\{1 - \frac{n_1^2 e}{(n_1-1)v}\right\} \times \left\{\left[0, 1 - \frac{n_2^2 e}{(n_2-1)v}\right) \cup \left(1 - \frac{n_2^2 e}{(n_2-1)v}, 1\right)\right\} \\ (0, e) & if \quad (\alpha_1, \alpha_2) \in [0, 1) \times \left\{1 - \frac{n_2^2 e}{(n_2-1)v}\right\} \end{cases}$$

with

$$\left(\alpha_1^{SGP}, \alpha_2^{SGP}\right) = [0, 1] \times \left\{1 - \frac{n_2^2 e}{(n_2 - 1) v}\right\} \text{ and } \left(\gamma_1^{SGP}, \gamma_2^{SGP}\right) = (0, e)$$

as equilibrium outcomes.

6 Conclusion

We complemented the results of Chowdhury et al. (2016) by charactersing the set of equilibria in a deterministic group contest with the weakest-link impact function, continuous efforts and a private good prize. We find that the non-existence of within-group asymmetric Nash equilibria in pure strategies in the effort stage is robust to the introduction of a private good prize. Nevertheless, it is not possible to determine the set of subgame perfect Nash equilibria in pure strategies in our group contest, since the continuation payoffs cannot be specified over the full set of first-period actions for the two groups of agents. However, turning to the restricted sharing rules case, we are able to conclude that there are continua of subgame perfect equilibria. In this case, by additional restrictions on the effort levels of each class of effort equilibria, we are able to computationally characterise the set of subgame perfect equilibria in pure strategies.

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7 Appendix

In this subsection we present the codes written in MATLAB to retrieve the set of first-period equilibria and the set of subgame perfect Nash equilibria in pure strategies for the leading case $n_1 = n_2 - 1$. The lines of code for the case $n_1 = n_2$ are available online at https://sites.google.com/view/andreasorrentino .

```
%%% ALL-PAY, WEAKEST-LINK, CONTINUOUS EFFORTS, GROUP CONTEST

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%% RESTRICTED SHARING RULE CASE %%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%% n_1 = n_2 - 1 CASE %%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Caveat: we will use the term "continuation payoffs matrices" for matrices containing the pos

%% Let's create the matrices needed to get all possible payoffs combinations
Q1 = strings(1,2) ;
Q2 = strings(1,5) ;
Q3 = strings(1,2) ;
Q4 = strings(1,2) ;
Q5 = strings(1,2) ;
Q6 = strings(1,5) ;
Q7 = strings(1,2) ;
Q8 = strings(1,2) ;
Q9 = strings(1,5) ;
Q10 = strings(1,8) ;
Q11 = strings(1,5) ;
Q12 = strings(1,1) ;
Q13 = strings(1,2) ;
Q14 = strings(1,1) ;
Q15 = strings(1,1) ;
Q16 = strings(1,1) ;

% Let's create some auxiliary matrices needed to get subsets of all
% possible continuation payoffs matrices

Q12_s = strings(1,5) ;
Q14_s = strings(1,5) ;
Q15_s = strings(1,2) ;
Q16_s = strings(1,2) ;

%% Let's fill the auxiliary matrices above with the corresponding payoffs
% Caveat: if an error occurs, run these auxiliary matrices
% and the corresponding cartesian product on a different .m file
% and then retrieve the output by load("result_s.mat","result_s")
```

```
Q12_s(1,1) = '(e,e)' ;
Q12_s(1,2) = '(e_max,e_max)' ;
Q12_s(1,3) = '(0,e_l)' ;
Q12_s(1,4) = '(0,e_m)' ;
Q12_s(1,5) = '(0,e_h)' ;

Q14_s(1,1) = '(e,e)' ;
Q14_s(1,2) = '(e_max,e_max)' ;
Q14_s(1,3) = '(e_l,0)' ;
Q14_s(1,4) = '(e_m,0)' ;
Q14_s(1,5) = '(e_h,0)' ;

Q15_s(1,1) = '(e,e)' ;
Q15_s(1,2) = '(e_max,e_max)' ;

Q16_s(1,1) = '(e,e)' ;
Q16_s(1,2) = '(e_max,e_max)' ;

[grid12_s, grid14_s, grid15_s, grid16_s] =
ndgrid(Q12_s, Q14_s, Q15_s, Q16_s); % grid structure to perform the cartesian product
result_s =[grid12_s(:), grid14_s(:),
grid15_s(:), grid16_s(:)]; % cartesian product to obtain all possible payoff profiles


%% Let's fill the remaining matrices with the corresponding payoffs

Q1(1,1) = '(e,e)' ;
Q1(1,2) = '(e_max,e_max)' ;

Q2(1,1) = '(e,e)' ;
Q2(1,2) = '(e_max,e_max)' ;
Q2(1,3) = '(e_l,0)' ;
Q2(1,4) = '(e_m,0)' ;
Q2(1,5) = '(e_h,0)' ;

Q3(1,1) = '(e,e)' ;
Q3(1,2) = '(e_max,e_max)' ;

Q4(1,1) = '(e,e)' ;
Q4(1,2) = '(e_max,e_max)' ;
Q4(1,3) = '(0,0)' ;

Q5(1,1) = '(e,e)' ;
Q5(1,2) = '(e_max,e_max)' ;

Q6(1,1) = '(e,e)' ;
Q6(1,2) = '(e_max,e_max)' ;
```

```
Q6(1,3) = '(e_l,0)' ;
Q6(1,4) = '(e_m,0)' ;
Q6(1,5) = '(e_h,0)' ;

Q7(1,1) = '(e,e)' ;
Q7(1,2) = '(e_max,e_max)' ;

Q8(1,1) = '(e,e)' ;
Q8(1,2) = '(e_max,e_max)' ;

Q9(1,1) = '(e,e)' ;
Q9(1,2) = '(e_max,e_max)' ;
Q9(1,3) = '(0,e_l)' ;
Q9(1,4) = '(0,e_m)' ;
Q9(1,5) = '(0,e_h)' ;

Q10(1,1) = '(e,e)' ;
Q10(1,2) = '(e_max,e_max)' ;
Q10(1,3) = '(e_l,0)' ;
Q10(1,4) = '(e_m,0)' ;
Q10(1,5) = '(e_h,0)' ;
Q10(1,6) = '(0,e_l)' ;
Q10(1,7) = '(0,e_m)' ;
Q10(1,8) = '(0,e_h)' ;

Q11(1,1) = '(e,e)' ;
Q11(1,2) = '(e_max,e_max)' ;
Q11(1,3) = '(0,e_l)' ;
Q11(1,4) = '(0,e_m)' ;
Q11(1,5) = '(0,e_h)' ;

Q13(1,1) = '(e,e)' ;
Q13(1,2) = '(e_max,e_max)' ;

%% For each of the 100 subsets of continuation payoffs-matrices
for l=1:length(result_s)

% Q12(1,1) = '(e,e)' ;
% Q12(1,2) = '(e_max,e_max)' ;
% Q12(1,3) = '(0,e_l)' ;
% Q12(1,4) = '(0,e_m)' ;
Q12(1,1) = result_s(l,1);

% Q14(1,1) = '(e,e)' ;
% Q14(1,2) = '(e_max,e_max)' ;
Q14(1,1) = result_s(l,2);
% Q14(1,4) = '(e_m,0)' ;
```

```
% Q14(1,5) = '(e_h,0)' ;

Q15(1,1) = result_s(1,3);
% Q15(1,2) = '(e_max,e_max)' ;

Q16(1,1) = result_s(1,4);
% Q16(1,2) = '(e_max,e_max)' ;



[grid1, grid2 , grid3, grid4, grid5, grid6, grid7, grid8, grid9, grid10,
grid11, grid12, grid13, grid14, grid15, grid16] = ndgrid(Q1, Q2, Q3,
Q4, Q5, Q6, Q7, Q8, Q9, Q10, Q11, Q12, Q13, Q14, Q15,
Q16); % grid structure to perform the cartesian product
result =[grid1(:), grid2(:), grid3(:), grid4(:),
grid5(:), grid6(:), grid7(:), grid8(:), grid9(:),
grid10(:), grid11(:), grid12(:), grid13(:), grid14(:),
grid15(:), grid16(:)]; % cartesian product to obtain all possible payoff profiles


%%Let's create the length(result) payoff matrices
%%For cycle
pmatrix = strings(length(result)*4,4);
counter = 1;

for i= 1:length(result)
    pmatrix(counter,1:4) = result(i,1:4);
    pmatrix(counter+1,1:4) = result(i,5:8);
    pmatrix(counter+2, 1:4) = result(i,9:12);
    pmatrix(counter+3,1:4) = result(i, 13:16);
    counter = counter + 4;
end

disp(pmatrix);


%% Let's find the set of first-stage equilibria and the set of subgame perfect equilibria.

br1_opt = strings(length(result)*4,4); %% matrix to collect best responses from group 1
br2_opt = strings(length(result)*4,4); %% matrix to collect best responses from group 2
alphaeq_opt = strings(length(result)*4,4); %% matrix to collect first-stage equilibria
sgeq_opt = strings(length(result)*4,8); %% matrix to collect SGP equilibria

%% THE SET OF BEST-RESPONSES FOR GROUP 1

for i=1:length(result)*4
    for j=1:4
```

```
        if pmatrix(i,j) == '(e_l,0)'
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(e_m,0)' | pmatrix(i,j) == '(0,0)' &&
        all(pmatrix(i,1:4) ~= '(e_l,0)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(e_h,0)' && all(pmatrix(i,1:4) ~= '(e_l,0)') &&
               all(pmatrix(i,1:4) ~= '(e_m,0)') && all(pmatrix(i,1:4) ~= '(0,0)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(e,e)' && all(pmatrix(i,1:4) ~= '(e_l,0)') &&
              all(pmatrix(i,1:4) ~= '(e_m,0)') && all(pmatrix(i,1:4) ~= '(0,0)') &&
              all(pmatrix(i,1:4) ~= '(e_h,0)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(e_max, e_max)' && all(pmatrix(i,1:4) ~= '(e_l,0)') &&
              all(pmatrix(i,1:4) ~= '(e_m,0)') && all(pmatrix(i,1:4) ~= '(0,0)') &&
              all(pmatrix(i,1:4) ~= '(e_h,0)') && all(pmatrix(i,1:4) ~= '(e,e)')
           br1_opt(i,j) = pmatrix(i,j);
        elseif pmatrix(i,j) == '(0,e_l)' | pmatrix(i,j) == '(0,e_m)' |
              pmatrix(i,j) == '(0,e_h)' && all(pmatrix(i,1:4) ~= '(e_l,0)') &&
              all(pmatrix(i,1:4) ~= '(e_m,0)') && all(pmatrix(i,1:4) ~= '(0,0)') &&
              all(pmatrix(i,1:4) ~= '(e_h,0)') && all(pmatrix(i,1:4) ~= '(e,e)') &&
              all(pmatrix(i,1:4) ~= '(e_max,e_max)')
           br1_opt(i,j) = pmatrix(i,j);
        end
    end
end

%% THE SET OF BEST-RESPONSES FOR GROUP 2

for j=1:4
    for i= 1:4:length(result)*4-3
        for q=0:3
            if pmatrix(i+q,j) == '(0,e_l)'
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(0,e_m)' | pmatrix(i+q,j) == '(0,0)' &&
            all(pmatrix(i:i+3,j) ~= '(0,e_l)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(0,e_h)' && all(pmatrix(i:i+3,j) ~= '(0,e)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,e_max)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,0)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(e,e)' && all(pmatrix(i:i+3,j) ~= '(0,e)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,e_max)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,0)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,e_h)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            elseif pmatrix(i+q,j) == '(e_max,e_max)' | pmatrix(i+q,j) == '(e_l,0)' |
                   pmatrix(i+q,j) == '(e_m,0)' | pmatrix(i+q,j) == '(e_h,0)' &&
```

```
                   all(pmatrix(i:i+3,j) ~= '(0,e)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,e_max)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,0)') &&
                   all(pmatrix(i:i+3,j) ~= '(0,e_h)') &&
                   all(pmatrix(i:i+3,j) ~= '(e,e)')
               br2_opt(i+q,j) = pmatrix(i+q,j);
            end
        end
    end
end

%% THE SET OF FIRST-PERIOD EQUILIBRIA AND SUBGAME PERFECT EQUILIBRIA

for i= 1:4:length(result)*4-3
    eq_1 = 0;
    for k =i:i+3
        for j = 1:4
            if br1_opt(k,j) == br2_opt(k,j) && br1_opt(k,j) ~= ""
               eq_1 = eq_1 + 1;
               alphaeq_opt(k,j) = br1_opt(k,j); %br1_opt without loss of generality
            end
        end
    end
    if eq_1 > 1 || eq_1 == 1
       sgeq_opt(i:i+3 ,1:4) = alphaeq_opt(i:i+3,1:4);
       sgeq_opt(i:i+3 ,5:8) = pmatrix(i:i+3,1:4);
    end
end



%%% LET'S SPLIT THE MATRICES AND REPRESENT THEM IN A MORE INTUITIVE WAY


sgeq = strings(length(result)*4,8);

for i=1:length(result)*4
    for j=1:8
        if sgeq_opt(i,j) == ""
           sgeq(i,j) = '.';
        elseif sgeq_opt(i,j) ~= ""
            sgeq(i,j) = sgeq_opt(i,j);
        end
    end
end
```

```
for i=1:4:length(result)*4-3
    for j=1:4
        if sgeq(i,1) ~= '.'
           sgeq(i,1) = ' a_1 <  1 - n_1^(2)*e/((n_1 - 1)v) and a_2 = 1 ' ;
        end
        if sgeq(i,2) ~= '.'
           sgeq(i,2) = ' a_1 = 1 - n_1^(2)*e/((n_1 - 1)v) and a_2 = 1 ' ;
        end
        if sgeq(i,3) ~= '.'
           sgeq(i,3) = ' 1 - n_1^(2)*e/((n_1 -1)v) < a_1 < 1 and a_2 = 1 ' ;
        end
        if sgeq(i,4) ~= '.'
            sgeq(i,4) = ' a_1 = 1 and a_2 = 1 ' ;
        end
        if sgeq(i+1,1) ~= '.'
           sgeq(i+1,1) = ' a_1 <  1 - n_1^(2)*e/((n_1 - 1)v) and
           1 - n_2^(2)*e/((n_2 - 1)v) < a_2 < 1 ' ;
        end
        if sgeq(i+1,2) ~= '.'
           sgeq(i+1,2) = ' a_1 = 1 - n_1^(2)*e/((n_1 - 1)v) and
           1 - n_2^(2)*e/((n_2 - 1)v) < a_2 < 1 ' ;
        end
        if sgeq(i+1,3) ~= '.'
           sgeq(i+1,3) = ' 1 - n_1^(2)*e/((n_1 - 1)v) < a_1 < 1 and
           1 - n_2^(2)*e/((n_2 - 1)v) < a_2 < 1 ' ;
        end
        if sgeq(i+1,4) ~= '.'
           sgeq(i+1,4) = ' a_1 = 1 and 1 - n_2^(2)*e/((n_2 - 1)v) < a_2 < 1 ' ;
        end
        if sgeq(i+2,1) ~= '.'
           sgeq(i+2,1) = ' a_1 < 1 - n_1^(2)*e/((n_1 - 1)v) and
           a_2 = 1 - n_2^(2)*e/((n_2 - 1)v) ' ;
        end
        if sgeq(i+2,2) ~= '.'
           sgeq(i+2,2) = ' a_1 = 1 - n_1^(2)*e/((n_1 - 1)v) and
           a_2 = 1 - n_2^(2)*e/((n_2 - 1)v) ';
        end
        if sgeq(i+2,3) ~= '.'
           sgeq(i+2,3) = ' 1 - n_1^(2)*e/((n_1 - 1)v) < a_1 < 1 and
           a_2 = 1 - n_2^(2)*e/((n_2 - 1)v) ';
        end
        if sgeq(i+2,4) ~= '.'
           sgeq(i+2,4) = ' a_1 = 1 and a_2 = 1 - n_2^(2)*e/((n_2 - 1)v ' ;
        end
        if sgeq(i+3,1) ~= '.'
           sgeq(i+3,1) = ' a_1 < 1 - n_1^(2)*e/((n_1 - 1)v) and
           a_2 < 1 - n_2^(2)*e/((n_2 - 1)v ' ;
```

```
        end
        if sgeq(i+3,2) ~= '.'
           sgeq(i+3,2) =  ' a_1 = 1 - n_1^(2)*e/((n_1 - 1)v) and
           a_2 < 1 - n_2^(2)*e/((n_2 - 1)v ' ;
        end
        if sgeq(i+3,3) ~= '.'
           sgeq(i+3,3) =  ' 1 - n_1^(2)*e/((n_1 - 1)v) < a_1 < 1 and
           a_2 < 1 - n_2^(2)*e/((n_2 - 1)v ' ;
        end
        if sgeq(i+3,4) ~= '.'
           sgeq(i+3,4) =  ' a_1 = 1 and a_2 < 1 - n_2^(2)*e/((n_2 - 1)v ' ;
        end
    end
end

% Let's clean the matrix used to get SGP equilibria
sgeq_opt = sgeq_opt(sgeq_opt(:,5) ~= '',:);
% Let's clean the matrix of SGP equilibria
sgeq = sgeq(sgeq(:,5) ~= '.',:);


disp(['There are ' num2str(length(sgeq)/4) '
continuation-payoffs matrices containing equilibria.'])


% Example 1
ex_1 = strings(4,4);
ex_1(1:4,1:4) = ["(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)"];

row_ex_1=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_1(1:4,1:4)
       row_ex_1 = i;
    end
end

% Example 2
ex_2 = strings(4,4);
ex_2(1:4,1:4) = ["(e,e)", "(e,e)", "(e,e)", "(0,0)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)"];

row_ex_2=0;
```

```
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_2(1:4,1:4)
       row_ex_2 = i;
    end
end


% Example 3
ex_3 = strings(4,4);
ex_3(1:4,1:4) = ["(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(0,e_1)", "(0,e_1)", "(0,e_1)", "(0,e_1)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)"];

row_ex_3=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_3(1:4,1:4)
       row_ex_3 = i;
    end
end

% Example 4
ex_4 = strings(4,4);
ex_4(1:4,1:4) = ["(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_4=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_4(1:4,1:4)
       row_ex_4 = i;
    end
end

% Example 5
ex_5 = strings(4,4);
ex_5(1:4,1:4) = ["(e,e)", "(e,e)", "(e,e)", "(0,0)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)";
                "(0,e_1)", "(0,e_1)", "(0,e_1)", "(e,e)";
                "(e,e)", "(e,e)", "(e,e)", "(e,e)"];

row_ex_5=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_5(1:4,1:4)
       row_ex_5 = i;
    end
```

```
end

% Example 6
ex_6 = strings(4,4);
ex_6(1:4,1:4) = ["(e,e)", "(e,e)", "(e,e)", "(0,0)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_6=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_6(1:4,1:4)
       row_ex_6 = i;
    end
end

% Example 7a
ex_7a = strings(4,4);
ex_7a(1:4,1:4) = ["(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                 "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                 "(0,e_1)", "(e_1,0)", "(0,e_1)", "(0,e_1)";
                 "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_7a=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_7a(1:4,1:4)
       row_ex_7a = i;
    end
end

% Example 7b
ex_7b = strings(4,4);
ex_7b(1:4,1:4) = ["(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                 "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                 "(0,e_1)", "(0,e_1)", "(0,e_1)", "(0,e_1)";
                 "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_7b=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_7b(1:4,1:4)
       row_ex_7b = i;
    end
end

% Example 8a
ex_8a = strings(4,4);
ex_8a(1:4,1:4) = ["(e,e)", "(e_1,0)", "(e,e)", "(0,0)";
```

```
                  "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                  "(0,e_1)", "(e_1,0)", "(0,e_1)", "(0,e_1)";
                  "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_8a=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_8a(1:4,1:4)
       row_ex_8a = i;
    end
end

% Example 8a
ex_8b = strings(4,4);
ex_8b(1:4,1:4) = ["(e,e)", "(e_1,0)", "(e,e)", "(0,0)";
                  "(e,e)", "(e_1,0)", "(e,e)", "(e,e)";
                  "(0,e_1)", "(0,e_1)", "(0,e_1)", "(0,e_1)";
                  "(e,e)", "(e_1,0)", "(e,e)", "(e,e)"];

row_ex_8b=0;
for i=1:4:length(sgeq)-3
    if sgeq(i:i+3,5:8) == ex_8b(1:4,1:4)
       row_ex_8b = i;
    end
end

save("sgeq_"+l+"_v_cont_effort_restricted_sharing_rule.mat","sgeq")

clearvars -except Q1 Q2 Q3 Q4 Q5 Q6 Q7 Q8 Q9 Q10 Q11 Q12 Q13 Q14 Q15 Q16 result_s;

end
```