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A Max-Min Two-Group Contest with Binary Actions and Incomplete Information à la Global Games

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Abstract

In this paper we introduce incomplete information à la global games into a max-min two-group contest with binary actions and we characterize the set of equilibria. Depending on whether the complete information assumption is relaxed on the value of the prize or on the cost of providing effort, we obtain different results in terms of equilibrium selection: in the first case, there exist both an equilibrium in (monotonic) switching strategies and an equilibrium robust to incomplete information in the sense of Kajii and Morris [1997], in which no player exerts effort in both groups, whereas in the second one there exists a unique equilibrium in (monotonic) switching-strategies.

Keywords: Group contests, incomplete information, global games.
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1 Introduction

Complete information games with strategic complementarities do often display multiple Nash equilibria. This is the case for group contests, as well. When within-group strong complementarities in group contests are modelled by the weakest-link impact function which equates the minimum of the efforts provided by the teammates, this multiplicity translates into the continuum of pure strategy Nash equilibria result found by Chowdhury, Lee, and Topolyan [2016], who term deterministic group contests with the weakest-link impact function as "max-min group contests". The authors employ a public good prize, whereas Gilli and Sorrentino [2024a] and Gilli and Sorrentino [2024b] analyze the private good prize setting with binary and continuous effort provision choices, respectively, and confirm the strong multiplicity result. These works share the complete information assumption. Departing from the information completeness assumption, Barbieri, Kovenock, Malueg, and Topolyan [2019] confirm the multiplicity result for a deterministic group contest with the weakest-link impact function, a public good prize and incomplete information about the cost of exerting effort. The authors focus on the interplay between dispersed private information about the cost of effort and the weakest-link impact function and characterize the set of Bayes-Nash equilibria in pure strategies. Despite allowing for very general distributions of the cost of effort, with common support being an unnecessary assumption, nevertheless, the uniqueness result they get is limited to nondegenerate equilibria without mass-points at the top, along with a continuum of nondegenerate equilibria with mass-points at the top and degenerate equilibria. On the other hand, global games introduced by Carlsson and van Damme [1993a] relax the complete information assumption in 2×2 games in such a way that a unique equilibrium in switching strategies is selected as the noise vanishes, independently from its distribution, as the result of iterated deletion of (interim) strictly-dominated strategies. Therefore, in this paper we pursue the research agenda of merging these two classes of games by introducing incomplete information à la global games in max-min binary two-group contests. We perturb complete information about the value of the prize contested and about the cost of providing effort, separately. In the first case, we find both a unique equilibrium in (monotonic) switching-strategies and an equilibrium robust to incomplete information à la Kajii and Morris [1997], in which no player exerts effort; in the second one, we obtain a unique equilibrium in (monotonic) switching-strategies. The properties of the payoff-structure of the underline complete information games are the key to understand this difference. Our results are closely related to the generalization of payoffs perturbation of 2×2 games to the n-player case of stag hunt games due to Carlsson and van Damme [1993b], where it is apparent that risk-dominance fails to be the appropriate equilibrium selection criterion, when going beyond the two-player case. However, we will not make claims about limit-uniqueness, independence of both specific distributional assumptions and the number of parameters involved as, in turn, Carlsson and van Damme [1993a] do.

As stressed in Gilli and Sorrentino [2024a] for research groups, international alliances and group strikes, many economic applications can be modelled as competition between groups with agents choosing whether to exert effort or not, rather than selecting an effort level from a continuous support. Military conflict is an additional example that can be brought to the fore, where soldiers can decide whether to abide with orders or not, for instance. Sports, music performances and research activities are some settings presented by Barbieri et al. [2019] as possible applications of their perfectly-discriminating group contest with perfect within-group complementarities. Back to Chowdhury et al. [2016], R&D competition, negative campaigning on multiple dimensions for products or elections, cybersecurity conflict are additional examples presented as being suitable examples of max-min group contests. Moreover, contests with binary decisions have been the object of a wide theoretical and experimental literature, spanning from corporate science, to sabotage activities and contests for status, as reviewed by Sheremeta [2018].

The paper is structured as follows. Section 2 presents two examples which should clarify the parallelism between group contests and the supermodular payoff structure perturbed in the global games à la Carlsson and van Damme [1993a] and how equilibrium selection naturally arises when modelling incomplete information à la global games. In Section 3 the formal model with both complete information and incomplete information is presented under two different specifications. Section 4 delivers the conclusions.

2 An Example[1]

Let us consider a deterministic group contest defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;
2. each group consists of $n_j = 2$ members. The total number of agents is $N = n_1 + n_2 = 4$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;
3. the **choice** of member $i \in \{1, 2\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let $x_j(i) = 1$ be denoted by a and $x_j(i) = 0$ by $\overline{a}$;
4. a club good **prize** worth $v \in \mathbb{R}$ to be allocated to one of the groups: thus, the prize v can be worth negative utils, which means that it can be a bad;
5. the **impact function** of group j is given by the weakest-link technology
$$X_j = \min\{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\};$$
6. the **contest success function** is given by the *all-pay auction:*
$$p_j(X_1, X_2) = \begin{cases} 1 & if \quad X_j > X_{-j} \\ \frac{1}{2} & if \quad X_j = X_{-j} \\ 0 & if \quad X_j < X_{-j}; \end{cases}$$
7. the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$.

As a consequence of these modelling characteristics, player ij has the **payoff**

$$\pi_{ij}(x_{11}, x_{12}, x_{21}, x_{22}) = p_j v - x_{ij} =$$
$$= \begin{cases} v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}.$$

Let players $1, 2$ belong to group 1 and players $3, 4$ to group 2. Consider the following geometric representation of the game, where player 3 "moves horizontally", while player 4 "moves vertically":

[1] This section is a direct application of the example carried out by Carlsson and van Damme [1993a] in their introduction.

3 — a (left tables) / $\overline{a}$ (right tables)

a

1/2	a	$\overline{a}$
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$-1; 0; v-1; v-1$
$\overline{a}$	$0; -1; v-1; v-1$	$0; 0; v-1; v-1$

$\overline{a}$

1/2	a	$\overline{a}$
a	$v-1; v-1; 0; -1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$

a

1/2	a	$\overline{a}$
a	$v-1; v-1; -1; 0$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$

1/2	a	$\overline{a}$
a	$v-1; v-1; 0; 0$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

$\overline{a}$

4

It is straightforward to derive the following properties:

- if $v > 2$, there are four strict Nash equilibria in pure strategies

$$NE \equiv \{(a, a, a, a); (\overline{a}, \overline{a}, a, a); (a, a, \overline{a}, \overline{a}); (\overline{a}; \overline{a}; \overline{a}; \overline{a})\}$$

  and a Nash equilibrium in symmetric strictly-mixed strategies $\sigma_i^*(a) = \frac{2}{v}$ $\forall i \in \{1, 2, 3, 4\}$;

- if $v = 2$, there are four Nash equilibria in pure strategies

$$NE \equiv \{(a, a, a, a); (\overline{a}, \overline{a}, a, a); (a, a, \overline{a}, \overline{a}); (\overline{a}; \overline{a}; \overline{a}; \overline{a})\};$$

- if $v < 2$, the unique Nash equilibrium derived by strict-dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;

- if $v > 2$, $(a, a, \overline{a}, \overline{a})$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 1 and $(\overline{a}, \overline{a}, a, a)$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 2;[2]

- if $v > 4$, (a, a, a, a) and $(a, a, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$ are the risk-dominant equilibrium strategy profiles for group 2. As a matter of fact, let us compare the Nash products of (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$. Then, for group 1:

$$\left(\frac{v}{2} - 1\right)\left(\frac{v}{2} - 1\right) > (0 - (-1))(0 - (-1)) \Leftrightarrow \left(\frac{v}{2} - 1\right)^2 > 1 \Leftrightarrow v > 4,$$

  that is, for $v > 4$, (a, a) is associated with the largest Nash product.
  Moreover, let us compare the Nash products of $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$. Then, for group 1:

$$\left(v - 1 - \frac{v}{2}\right)\left(v - 1 - \frac{v}{2}\right) > \left(\frac{v}{2} - \left(\frac{v}{2} - 1\right)\right)\left(\frac{v}{2} - \left(\frac{v}{2} - 1\right)\right) \Leftrightarrow \left(\frac{v}{2} - 1\right)^2 > 1 \Leftrightarrow v > 4\ ,$$

  that is, for $v > 4$, (a, a) is associated with the largest Nash product.
  The same inequalities hold symmetrically for group 2 as well;

[2] For the formulation of payoff-dominace and risk-dominance concepts see Harsanyi and Selten [1988].

- if $2 < v < 4$, $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2 . Clearly this follows from what shown at the previous point for both groups;
- overall, there is a one-sided dominance region: for $v < 2$, a is a strictly dominated action.

Finally, note that, for $2 < v < 4$, $(a, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium strategy profile for group 1, whereas $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1, and, symmetrically, $(\overline{a}, \overline{a}, a, a)$ is the payoff-dominant equilibrium strategy profile for group 2, whereas $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2. Hence, there is a tension between payoff-dominance and risk-dominance.

Now let us consider a slight variation of the game above and let:

- the individual **costs of effort** $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se, while the prize v is always worth positive utils, so that it is a good.

Then, we have the following representation of the game, where player 3 "moves horizontally" and player 4 "moves vertically":

3: a — 4: a

1/2	a	$\overline{a}$
a	$\frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}-c$	$-c; 0; v-c; v-c$
$\overline{a}$	$0; -c; v-c; v-c$	$0; 0; v-c; v-c$

3: $\overline{a}$ — 4: a

1/2	a	$\overline{a}$
a	$v-c; v-c; 0; -1$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-c$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-c$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-c$

3: a — 4: $\overline{a}$

1/2	a	$\overline{a}$
a	$v-c; v-c; -1; 0$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}-c; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}$

3: $\overline{a}$ — 4: $\overline{a}$

1/2	a	$\overline{a}$
a	$v-c; v-c; 0; 0$	$\frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-c; \frac{v}{2}; \frac{v}{2}$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$

It is straightforward to derive the following properties:

- if $c < 0$, the unique Nash equilibrium derived by strict dominance is (a, a, a, a);
- if $c > \frac{v}{2}$, the unique Nash equilibrium derived by strict dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;
- if $0 \leq c < \frac{v}{2}$, there are four strict Nash equilibria in pure strategies

$$NE = \{(a, a, a, a), (a, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, a), (\overline{a}, \overline{a}, \overline{a}, \overline{a})\}$$

  and an equilibrium in symmetric strictly mixed strategies $\sigma_i^*(a) = \frac{2c}{v} \quad \forall i \in \{1, 2, 3, 4\}$;
- if $c = \frac{v}{2}$, there are four Nash equilibria in pure strategies

$$NE = \{(a, a, a, a), (a, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, a), (\overline{a}, \overline{a}, \overline{a}, \overline{a})\};$$

- if $c < \frac{v}{2}$, $(a, a, \overline{a}, \overline{a})$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 1 and $(\overline{a}, \overline{a}, a, a)$ payoff-dominates $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for group 2;

- if $0 \le c < \frac{v}{4}$, (a, a, a, a) and $(a, a, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$ are the risk-dominant equilibrium strategy profiles for group 2 . As a matter of fact, let us compare the Nash products of (a, a, a, a) and $(\overline{a}, \overline{a}, a, a)$. Then, for group 1:

$$\left(\frac{v}{2} - c\right)\left(\frac{v}{2} - c\right) > (0 - (-c))\,(0 - (-c)) \Leftrightarrow \left(\frac{v}{2} - c\right)^2 > c^2 \Leftrightarrow v > 4c \Leftrightarrow c < \frac{v}{4},$$

  that is, for $c < \frac{v}{4}$, (a, a) is associated with the largest Nash product.
  Moreover, let us compare the Nash products of $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$. Then, for group 1:

$$\left(v - c - \frac{v}{2}\right)\left(v - c - \frac{v}{2}\right) > \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)\left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right) \Leftrightarrow \left(\frac{v}{2} - c\right)^2 > c^2 \Leftrightarrow c < \frac{v}{4}\ ,$$

  that is, for $c < \frac{v}{4}$, (a, a) is associated with the largest Nash product.
  The same inequalities hold symmetrically for group 2 as well;

- if $\frac{v}{4} < c < \frac{v}{2}$, $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1 and, symmetrically, $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2 . Clearly this follows from what shown at the previous point for both groups;

- overall, there are two dominance regions: for $c < 0$, $\overline{a}$ is a strictly dominated action; for $c > \frac{v}{2}$, a is a strictly dominated action.

Finally, note that, for $\frac{v}{4} < c < \frac{v}{2}$, $(a, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium strategy profile for group 1, whereas $(\overline{a}, \overline{a}, a, a)$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 1, and, symmetrically, $(\overline{a}, \overline{a}, a, a)$ is the payoff-dominant equilibrium strategy profile for group 2, whereas $(a, a, \overline{a}, \overline{a})$ and $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for group 2. Hence, there is a tension between payoff-dominance and risk-dominance.

In turn, we would like to draw a possible comparison with the classical example due to Carlsson and van Damme [1993a] about a 2×2 game under complete information, reported in table 1.

	α_2	β_2
α_1	x, x	$x, 0$
β_1	$0, x$	$4, 4$

Table 1: Game $g(x)$ by Carlsson and van Damme [1993a] .

Carlsson and van Damme [1993a] highlight the following properties of this game under complete information:

- if $x > 4$, the unique Nash equilibrium derived by strict dominance is (α_1, α_2);

- if $x < 0$, the unique Nash equilibrium derived by strict dominance is (β_1, β_2);

- if $0 < x < 4$, there are two strict Nash equilibria, that is (α_1, α_2) and (β_1, β_2);
- if $x \in (2, 4)$, (α_1, α_2) is the risk-dominant equilibrium;
- if $x \in (0, 2)$, (β_1, β_2) is the risk-dominant equilibrium;
- overall, there are two dominance regions.

Finally, note that, for $2 < x < 4$, (β_1, β_2) is the payoff-dominant equilibrium, whereas (α_1, α_2) is the risk-dominant equilibrium: there is a tension between payoff-dominance and risk-dominance.

2.1 Incomplete Information about the Prize

Let us consider the case where the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$. Henceforth, we refer to this game as $g(v)$. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$, e.g. $[1, 5]$;
- given the realization v, each player $i \in \{1, 2, 3, 4\}$ idiosyncratically observes the realization of a random variable V_i, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $V_1 - v$, $V_2 - v$, $V_3 - v$ and $V_4 - v$ are independent;
- after these idiosyncratic observations, each player $i \in \{1, 2, 3, 4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game of $g(v)$;
- note that $E(V|v_i) = v_i$, if i observes $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_i \sim U(v_i - \varepsilon, v_i + \varepsilon)$;
- furthermore, for $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around v_i with support $[v_i - 2\varepsilon, v_i + 2\varepsilon]$. Hence, $Prob[V_{-i} < v_i|v_i] = Prob[V_{-i} > v_i|v_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon < \left|\frac{v}{2} - 1\right|$ and suppose player $i \in \{1, 2, 3, 4\}$ observes $v_i < 2$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $v_i < 2$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $v_{-i} < 2$, then player i, observing $v_i = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff from not exerting effort, that is choosing $\overline{a}_i$, will be at least $\frac{3}{4}$, so that a_i can be discarded by iterated dominance for $v_i = 2$, since the conditionally expected payoff from exerting effort equals $\frac{1}{4}$. Let v_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $v_i^* = 4$. Note that $v_i = 4$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $v_i = 4$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}(-1) + \frac{1}{8}(4 - 1) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}(4 - 1) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}(4 - 1) + \frac{1}{8}\left(\frac{4}{2} - 1\right) = \frac{3}{2},$$

while the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8} * 0 + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8}\left(\frac{4}{2}\right) = \frac{3}{2}.$$

The cutoff $v_i^* = 4$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for $v_i \in [v - \varepsilon, v + \varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(-1)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)+\frac{1}{8}(v_i-1)+\frac{1}{8}\left(\frac{v_i}{2}-1\right)=$$

$$=\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right)+\frac{1}{8}\left(\frac{v_i}{2}\right).$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower dominance region and an upper dominance region.

Hence, in $g(v)$ under incomplete information à la global games, for sufficiently small ε, there is a unique equilibrium in (monotonic) cutoff strategies, such that $\forall i \in \{1,2,3,4\}$:

$$x_i^*(v_i) = \begin{cases} 1 & \text{if } v_i > 4 \\ 0 & \text{if } v_i \leq 4 \end{cases}$$

Nonetheless, given the absence of an upward dominance region, the following equilibrium $\forall i \in \{1,2,3,4\}$ exists as in De Mesquita [2011]:

$$x_i^{**}(v_i) = 0 \ \forall v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon].$$

As a matter of fact, at $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$ any deviation is strictly dominated for any $v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, so that $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$ is robust to incomplete information in the sense of Kajii and Morris [1997].

2.2 Incomplete Information about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Henceforth, we refer to this game as $g(c)$. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$;
- given the realization c, each player $i \in \{1,2,3,4\}$ idiosyncratically observes the realization of a random variable C_i, uniform on $[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_1 - c$, $C_2 - c$, $C_3 - c$ and $C_4 - c$ are independent;
- after these idiosyncratic observations, each player $i \in \{1,2,3,4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game $g(c)$;
- note that $E(C|c_i) = c_i$, if i observes $c_i \in [\underline{c}+\varepsilon, \overline{c}-\varepsilon]$ so that $C|c_i \sim U(c_i - \varepsilon, c_i + \varepsilon)$;
- furthermore, for $c_i \in [\underline{c}+\varepsilon, \overline{c}-\varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around c_i with support $[c_i - 2\varepsilon, c_i + 2\varepsilon]$. Hence, $Prob[C_{-i} < c_i|c_i] = Prob[C_{-i} > c_i|c_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon < \left|\frac{2\overline{c}-v}{4}\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $c_i > \frac{v}{2}$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1,2,3,4\}$ whenever she observes $c_i > \frac{v}{2}$. Suppose $i=1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2,3,4\}$ are forced to play $\overline{a}$ whenever they observe $c_{-i} > \frac{v}{2}$, then player i, observing $c_i = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff from not exerting effort, that is choosing $\overline{a}_i$ will be at least $\frac{3}{8}v$, so that a_i can be discarded by iterated dominance for $c_i = \frac{v}{2}$, since the conditionally expected payoff from exerting effort equals $\frac{v}{8}$. Let c_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^* = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$
$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v\ .$$

The cutoff $c_i^* = \frac{v}{4}$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c-\epsilon, c+\epsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)=$$
$$=\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right).$$

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region.

Again, let us assume $\varepsilon < \left|-\frac{c}{2}\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $c_i < 0$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is positive and greater than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1,2,3,4\}$ whenever she observes $c_i < 0$. Iterating this dominance argument, suppose $i=1$ without loss of generality. Then, if players $-i \in \{2,3,4\}$ are forced to play a whenever they observe $c_{-i} < 0$, player i, observing $c_i = 0$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to (a_2, a_3, a_4). Thus, i's conditionally expected payoff from exerting effort, that is choosing a_i will be at least $\frac{5}{8}v$, so that $\overline{a}_i$ can be discarded by iterated dominance for $c_i = 0$, since the conditionally expected payoff from not exerting effort equals $\frac{3}{8}v$. Let c_i^{**} be the smallest observation such that $\overline{a}_i$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^{**} = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$
$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}*0+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v\ .$$

The cutoff $c_i^{**} = \frac{v}{4}$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c - \varepsilon, c + \varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2} - c_i\right)+\frac{1}{8}\left(-c_i\right)+\frac{1}{8}\left(v - c_i\right)+\frac{1}{8}\left(\frac{v}{2} - c_i\right)+\frac{1}{8}\left(v - c_i\right)+\frac{1}{8}\left(\frac{v}{2} - c_i\right)+\frac{1}{8}\left(v - c_i\right)+\frac{1}{8}\left(\frac{v}{2} - c_i\right) =$$

$$= \frac{1}{8} * 0 + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) + \frac{1}{8}\left(\frac{v}{2}\right) .$$

Hence, $c_i^* = c_i^{**}$ and in $g(c)$ under incomplete information à la global games, for sufficiently small ε, there exists a unique equilibrium in switching strategies such that $\forall i \in \{1, 2, 3, 4\}$

$$x_i^*(c_i) = \begin{cases} 1 & \text{if } c_i < \frac{v}{4} \\ 0 & \text{if } c_i \geq \frac{v}{4} \end{cases} .$$

2.3 Observations

Overall, we can state some general points from the example above:

- under complete information, there are multiple Nash equilibria in pure strategies in a max-min two-group four-player contest with binary actions and a public good prize, independently from whether the cost of effort equates effort itself or a parameter belonging to the set of real numbers.

- Focusing on the geometric representation of each two-player group it is apparent the similarity with the symmetric 2x2 game by Carlsson and van Damme [1993a]: in both of them there is a supermodular payoff-structure and the cardinality of the actions set is equal to two, as in classical stag hunt games. Accordingly, symmetric Nash equilibria in pure strategies naturally emerge.

- In both examples we highlight a tension between payoff-dominance and risk-dominance, as in the example due to Carlsson and van Damme [1993a].

- Relaxing complete information à la global games induces the existence of an equilibrium in (monotonic) switching strategies, whose cutoff coincides with the one of the risk-dominance region.

- Equilibrium selection happens even for "a pinch of uncertainty", no matter how small ε is.

- Whether the selection induced delivers uniqueness or not crucially depends on the properties of the payoffs structure under complete information: in particular, the presence of both an upward and a downward dominance region is conducive to a unique equilibrium in (monotonic) switching strategies by deletion of interim-strictly dominated strategies when departing from the complete information assumption in the sense of Carlsson and van Damme [1993a].

- Finally, note that whether the risk-dominant equilibrium in $g(v_i)$ and $g(c_i)$ coincides with the risk-dominant equilibrium in the actual game selected by Nature, i.e. $g(v)$ and $g(c)$ respectively, or not, depends on whether ε is sufficiently small, that is for $\varepsilon < |v - 2|$ and $\varepsilon < \left|c - \frac{v}{4}\right|$, respectively.

Once highlighted the main properties of our example, the general model and the mechanisms guiding to the related results should be more transparent in the next section.

3 The Model

Let us consider a deterministic group contest defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;
2. each group has $n_j \geq 2$ members, where $n_1 \geq n_2$ without loss of generality. The total number of agents is $n_1 + n_2 = N$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;
3. the **choice** of member $i \in \{1, \ldots, n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let $x_j(i) = 1$ be denoted by a and $x_j(i) = 0$ by $\overline{a}$; let us define the average exerted effort in group j, or rather the participation rate in group j as

$$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1] ;$$

4. a club good **prize** worth $v \in \mathbb{R}$ to be allocated to one of the two groups: thus, the prize v can be worth negative utils, which means that it can be a bad;
5. the **impact function** of group j is given by the weakest-link technology

$$X_j = \min \{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\} ;$$

6. the **contest success function** is given by the *all-pay auction:*

$$p_j(X_1, X_2) = \begin{cases} 1 & if \;\; X_j > X_{-j} \\ \frac{1}{2} & if \;\; X_j = X_{-j} \\ 0 & if \;\; X_j < X_{-j}; \end{cases}$$

7. the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$.

As a consequence of these modelling characteristics, player ij has the expected **payoff**

$$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2) = p_j v - x_{ij} =$$

$$= \begin{cases} v - x_j(i) & if \;\; \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}v - x_j(i) & if \;\; \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \;\; \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\}. \end{cases}$$

Now we are able to provide a formal definition of a binary max-min group contest with a public good prize.

Definition 1 *A* Binary Max-Min Group Contest $BMMGC^*$ *is a one-stage game* $BMMGC^* = \langle\{1, 2\}, N, B_{ij}, \pi_{ij}\rangle$ *defined by*

1. *the set of groups* $\{1, 2\}$;
2. *the set of players* $N = \{1, ..., n_1 + n_2\}$;

3. *the set of actions* $B_{ij} = \{0,1\}$: *for each player* ij, *the choice of the effort* $x_j(i)$;

4. *the payoff functions for each player* $ij \in N$

$$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2) = p_j v - x_{ij} =$$

$$= \begin{cases} v - x_j(i) & if \quad \min\{\mathbf{x}_j\} > \min\{\mathbf{x}_{-j}\} \\ \frac{1}{2}v - x_j(i) & if \quad \min\{\mathbf{x}_j\} = \min\{\mathbf{x}_{-j}\} \\ -x_j(i) & if \quad \min\{\mathbf{x}_j\} < \min\{\mathbf{x}_{-j}\} \end{cases}.$$

The notation used in this paper is summed up in table 1.

Variable	Meaning
ij or $j(i)$	agent i of group j
$\{1,...,n_j\}$	set of agents in group j
$x_j(i)$ or x_{ji}	effort of agent i in group j
$X_j = \min\{x_j(i) \in \{0,1\}, i \in \{1,...,n_j\}\}$	impact of effort of all agents in group j
$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$	vector of efforts of all agents
$C_{ij}(x_j(i)) = x_j(i)$	cost of effort for agent i of group j
$p_j(X_1, X_2)$	probability of group j of winning the contest
$\pi_{ij}(\mathbf{x}_1, \mathbf{x}_2)$	payoff function of agent i of group j
$\gamma_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_{ij} \in [0,1]$	share of active agents in group j
Table 1	

Moreover, when $\gamma_j \in (0,1)$, denote by

$$\gamma_j^+ = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} + 1\right) \in [0,1]$$

the share of active agents at a marginal increase and by

$$\gamma_j^- = \frac{1}{n_j}\left(\sum_{i=1}^{n_j} x_{ij} - 1\right) \in [0,1]$$

the share of active agents at a marginal decrease.

To simplify notation and presentation, the NE of the $BMMGC^*$ will be presented in terms of share of active agents, i.e. $(\gamma_1, \gamma_2) \in [0,1] \times [0,1]$.

Proposition 1 *In the* $BMMGC^*$,

- *if* $v > 2$, *there are four strict Nash equilibria in pure strategies*

$$NE \equiv \{(\gamma_1,\gamma_2) = (1,1); (\gamma_1,\gamma_2) = (1,0); (\gamma_1,\gamma_2) = (0,1); (\gamma_1,\gamma_2) = (0,0)\}$$

  and a Nash equilibrium in within-group symmetric strictly-mixed strategies

$$\sigma_{ij}^*(x_{ij} = 1) = \left(\frac{2}{v}\right)^{1/(n_j - 1)} \quad \forall i \in \{1,\ldots,n_j\} \text{ and } j \in \{1,2\};$$

- *if $v = 2$, there are four Nash equilibria in pure strategies*

$$NE \equiv \{(\gamma_1, \gamma_2) = (1,1)\,; (\gamma_1, \gamma_2) = (1,0)\,; (\gamma_1, \gamma_2) = (0,1)\,; (\gamma_1, \gamma_2) = (0,0)\}\,;$$

- *if $v < 2$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0,0) \ .$$

Overall, there is a one-sided dominance region: for $v < 2$, $x_{ij} = 1$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$.

Proof.

- Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\} \ .^3$$

Then,

$$min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\} \ .$$

Suppose $x_j\,(i) = 1$, then

$$\pi_{ij}\,(\gamma_j, \gamma_{-j}) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j\,(i) = 0$, then

$$min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is an incentive to deviate $\forall v \in \mathbb{R}$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \ \forall v \in \mathbb{R} \ .$$

Thus,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\}$$

is not a Nash equilibrium $\forall v \in \mathbb{R}$.

- Suppose

$$(\gamma_1, \gamma_2) = (1,1) \ .$$

Then,

$$min\,\{\mathbf{x}_1\} = min\,\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}\,(\gamma_j, \gamma_{-j}) = \frac{v}{2} - 1 \ .$$

[3] $\mathbb{1}_{x_{ij}=1}$ stands for the Indicator random variable taking value 1 when player ij chooses $x_{ij} = 1$, that is she exerts effort.

If agent ij deviates to $x_j(i) = 0$, then

$$min\{\mathbf{x}_j\} = 0 < min\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = 0 \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} - 1 \geq 0 \Leftrightarrow v \geq 2 \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (1, 1)$$

is a Nash equilibrium in pure strategies $\forall v \geq 2$.

- Suppose

$$(\gamma_j, \gamma_{-j}) = (1, 0) \, \forall j \in \{1, 2\} \ .$$

Then,

$$min\{\mathbf{x}_j\} = 1 > min\{\mathbf{x}_{-j}\} = 0$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = v - 1 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$min\{\mathbf{x}_j\} = 0 = min\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for agent ij there is no incentive to deviate if and only if

$$v - 1 \geq \frac{v}{2} \Leftrightarrow v \geq 2 \ .$$

On the other hand, if agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$min\{\mathbf{x}_j\} = 1 > min\{\mathbf{x}_{-j}\} = 0$$

so that the deviation payoff is

$$\pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+\right) = -1 \ .$$

Hence, for player $i-j$ deviating is a strictly dominated action as

$$\pi_{i-j}(\gamma_j, \gamma_{-j}) = 0 > \pi_{i-j}^D\left(\gamma_j, \gamma_{-j}^+\right) = -1 \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) = (1, 0) \ \ \forall j \in \{1, 2\}$$

is a Nash equilibrium in pure strategies for any $v \geq 2$.

- Suppose

$$(\gamma_1, \gamma_2) = (0,0) \ .$$

Then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}(\gamma_1, \gamma_2) = \frac{v}{2} \ \forall j \in \{1,2\} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

Hence, for player ij deviating is a strictly dominated action as

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} > \pi_{ij}^D\left(\gamma_j^D, \gamma_{-j}\right) = \frac{v}{2} - 1 \ \forall j \in \{1,2\} \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (0,0)$$

is a Nash equilibrium in pure strategies for any $v \in \mathbb{R}$.

- Let $\sigma_{ij}(x_{ij} = 1)$ be the within-group symmetric randomization over pure strategy $x_{ij} = 1$ for player ij, then

$$EU_{ij}(x_{ij} = 1) = EU_{ij}(x_{ij} = 0) \Leftrightarrow$$

$$\Leftrightarrow Prob((n_j - 1)\gamma_j = n_j - 1) \cdot Prob(n_{-j}\gamma_{-j} < n_{-j}) \cdot (v-1) + \Big(Prob((n_j - 1)\gamma_j < n_j - 1) \cdot$$

$$Prob(n_{-j}\gamma_{-j} < n_{-j}) + Prob((n_j - 1)\gamma_j = n_j - 1) \cdot Prob(n_{-j}\gamma_{-j} = n_{-j}) \Big) \left(\frac{v}{2} - 1\right) +$$

$$+ Prob((n_j - 1)\gamma_j < n_j - 1) \cdot Prob(n_{-j}\gamma_{-j} = n_{-j}) \cdot (-1) = Prob(n_{-j}\gamma_{-j} < n_{-j}) \cdot \frac{v}{2} \Leftrightarrow$$

$$\Leftrightarrow (\sigma_{ij}(x_{ij} = 1))^{n_j - 1} \cdot (1 - (\sigma_{i-j}(x_{i-j} = 1))^{n_{-j}}) \cdot (v-1) + \Big(1 - (\sigma_{ij}(x_{ij} = 1))^{n_j - 1} \cdot$$

$$(1 - (\sigma_{i-j}(x_{i-j} = 1))^{n_{-j}}) + (\sigma_{ij}(x_{ij} = 1))^{n_j - 1} \cdot (\sigma_{i-j}(x_{i-j} = 1))^{n_{-j}} \Big) \cdot \left(\frac{v}{2} - 1\right) +$$

$$+ \Big(1 - (\sigma_{ij}(x_{ij} = 1))^{n_j - 1}\Big) \cdot (\sigma_{i-j}(x_{i-j} = 1))^{n_{-j}} \cdot (-1) = (1 - (\sigma_{i-j}(x_{i-j} = 1))^{n_{-j}}) \cdot \frac{v}{2} \Leftrightarrow$$

$$\Leftrightarrow \sigma_{ij}^*(x_{ij} = 1) = \left(\frac{2}{v}\right)^{1/(n_j - 1)} \ \forall i \in \{1, \ldots, n_j\} \text{ and } j \in \{1,2\} \ .$$

Thus,

$$\sigma_{ij}^*(x_{ij} = 1) = \left(\frac{2}{v}\right)^{1/(n_j - 1)} \ \forall i \in \{1, \ldots, n_j\} \text{ and } j \in \{1,2\} \ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall v > 2$.

■

The following result is immediate from proposition 1.

Corollary 1 *In the $BMMGC^*$, there are no within-group asymmetric Nash equilibria in pure strategies.*

Moreover, it is easy to prove the following result.

Proposition 2 *In the $BMMGC^*$,*

- *for $v > 2$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is the payoff-dominant equilibrium for group $j \in \{1,2\}$;*
- *for $v > 4$, $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (1,0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1,2\}$;*
- *for $2 < v < 4$, $(\gamma_j, \gamma_{-j}) = (0,1)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1,2\}$.*

Remark 1 *Note that, for $2 < v < 4$ and any group $j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is the payoff-dominant equilibrium strategy profile for group $j \in \{1,2\}$, whereas $(\gamma_j, \gamma_{-j}) = (0,1)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1,2\}$: there is a tension between payoff-dominance and risk-dominance.*

Proof. Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten [1988], it is straightforward to state that:

- for $v > 2$, $(\gamma_1, \gamma_2) = (1,0)$ payoff-dominates $(\gamma_1, \gamma_2) = (0,0)$ for group 1 and $(\gamma_1, \gamma_2) = (0,1)$ payoff-dominates $(\gamma_1, \gamma_2) = (0,0)$ for group 2, since

$$\pi_{ij}(\gamma_j = 1, \gamma_{-j} = 0) = v - 1 > \pi_{ij}(\gamma_j = 0, \gamma_{-j} = 0) = \frac{v}{2} \Leftrightarrow v > 2 \ .$$

- for $v > 4$, $\gamma_j = 1$ is the risk-dominant equilibrium strategy profile for group $j \in \{1,2\}$. As a matter of fact, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (0,1)$. Then, for group j:

$$\left(\frac{v}{2} - 1\right)^{n_j} > (0 - (-1))^{n_j} \Leftrightarrow \left(\frac{v}{2} - 1\right)^{n_j} > 1 \Leftrightarrow v > 4,$$

  that is, for $v > 4$, $(\gamma_1, \gamma_2) = (1,1)$ is associated with the largest Nash product.
  Moreover, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$. Then, for group j:

$$\left(v - 1 - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - \left(\frac{v}{2} - 1\right)\right)^{n_j} \Leftrightarrow \left(\frac{v}{2} - 1\right)^{n_j} > 1 \Leftrightarrow v > 4 \ ,$$

  that is, for $v > 4$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product;

- for $2 < v < 4$, $(\gamma_j, \gamma_{-j}) = (0,1)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1,2\}$. Clearly this follows from what shown at the previous point for both groups.

■

Now consider a slight variation of the game above and let:

- the individual cost of effort $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se;

- the club good prize $v > 0$, i.e. the prize v is always worth positive utils, so that it is a good.

Henceforth, we term this variation as $BMMGC^{*b}$. Then, it is straightforward to derive the following results.

Proposition 3 *In the $BMMGC^{*b}$,*

- *if $c < 0$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (1, 1) \ ;$$

- *if $c > \frac{v}{2}$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0, 0) \ ;$$

- *if $0 < c < \frac{v}{2}$, there are four strict Nash equilibria in pure strategies*

$$NE = \{(\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0)\}$$

and an equilibrium in within-group symmetric strictly mixed strategies $\sigma_i^* (x_{ij} = 1) = \left(\frac{2c}{v}\right)^{1/(n_j-1)} \ \forall i \in \{1, \ldots, n_j\}$ and $j \in \{1, 2\}$;

- *if $c = 0$, the set of Nash equilibria in pure strategies is*

$$\begin{aligned} NE = & \left\{ (\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0) \right\} \cup \\ & \cup \left\{ (\gamma_1, \gamma_2) \ \text{such that } \gamma_j \in (0, 1) \ \text{and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1, 2\} \right\} \cup \\ & \cup \left\{ (\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \ \text{such that } \gamma_j \in (0, 1) \ \text{and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1, 2\} \right\} \ ; \end{aligned}$$

- *if $c = \frac{v}{2}$, there are four Nash equilibria in pure strategies*

$$NE = \{(\gamma_1, \gamma_2) = (1, 1), (\gamma_1, \gamma_2) = (1, 0), (\gamma_1, \gamma_2) = (0, 1), (\gamma_1, \gamma_2) = (0, 0)\} \ .$$

Overall, there are two dominance regions: for $c < 0$, $x_{ij} = 0$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$: for $c > \frac{v}{2}$, $x_{ij} = 1$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$.

Proof.

- Suppose

$$(\gamma_1, \gamma_2) \ \text{such that } \gamma_j \in (0, 1) \ \text{and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1, 2\} \ .$$

  Then,

$$min \{\mathbf{x}_1\} = min \{\mathbf{x}_2\} \ .$$

  Suppose $x_j (i) = 1$, then

$$\pi_{ij} (\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

On the other hand, suppose $x_j(i) = 0$ and $c \leq 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2}$$

If agent ij deviates to $x_j(i) = 1$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^+, \gamma_{-j}\right) = \begin{cases} v - c & \text{if } n_j\gamma_j = n_j - 1 \\ \frac{v}{2} - c & \text{otherwise} \end{cases} .$$

Hence, for player ij there is no incentive to deviate if and only if

$$c = 0 \text{ and } n_j\gamma_j < n_j - 1 \ .$$

Thus,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1,2\}$$

is a Nash equilibrium if and only if $c = 0$.

- Suppose

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\} \ .$$

Then,

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\} \ .$$

Suppose $x_j(i) = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \text{ and } \pi_{i-j}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in \mathbb{R}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

On the other hand, suppose $x_j(i) = 0$ and $c \leq 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2}$$

If agent ij deviates to $x_j(i) = 1$, then

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j}) = \begin{cases} v - c & \text{if } n_j\gamma_j = n_j - 1 \\ \frac{v}{2} - c & \text{otherwise} \end{cases} .$$

Hence, for player ij there is no incentive to deviate if and only if

$$c = 0 \text{ and } n_j\gamma_j < n_j - 1 \ .$$

If agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$min\{\mathbf{x}_j\} = min\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{i-j}^D(\gamma_j, \gamma_{-j}^+) = \frac{v}{2} - c \ .$$

Hence, for player $i-j$ there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1,2\} \ .$$

is a Nash equilibrium if and only if $c = 0$.

- Suppose

$$(\gamma_1, \gamma_2) = (1,1) \ .$$

Then,

$$min\{\mathbf{x}_1\} = min\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$min\{\mathbf{x}_j\} = 0 < min\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}\right)=0\ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2}-c\geq 0\Leftrightarrow c\leq\frac{v}{2}\ .$$

Thus,

$$(\gamma_1,\gamma_2)=(1,1)$$

is a Nash equilibrium in pure strategies $\forall c\leq\frac{v}{2}$.

- Suppose

$$(\gamma_j,\gamma_{-j})=(1,0)\,\forall j\in\{1,2\}\ .$$

Then,

$$min\left\{\mathbf{x}_j\right\}=1>min\left\{\mathbf{x}_{-j}\right\}=0$$

so that

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=v-c\ .$$

If agent ij deviates to $x_j\left(i\right)=0$, then

$$min\left\{\mathbf{x}_j\right\}=0=min\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-},\gamma_{-j}\right)=\frac{v}{2}\ .$$

Hence, for agent ij there is no incentive to deviate if and only if

$$v-c\geq\frac{v}{2}\Leftrightarrow c\leq\frac{v}{2}\ .$$

On the other hand, if agent $i-j$ deviates to $x_{-j}\left(i\right)=1$, then

$$min\left\{\mathbf{x}_j\right\}=1>min\left\{\mathbf{x}_{-j}\right\}=0$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=-c\ .$$

Hence, for player $i-j$ there is no incentive to deviate if and only if

$$\pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=0\geq\pi_{i-j}^{D}\left(\gamma_j,\gamma_{-j}^{+}\right)=-c\Leftrightarrow c\geq 0\ .$$

Thus,

$$(\gamma_j,\gamma_{-j})=(1,0)\ \ \forall j\in\{1,2\}$$

is a Nash equilibrium in pure strategies for any $0\leq c\leq\frac{v}{2}$.

- Suppose

$$(\gamma_1,\gamma_2)=(0,0)\ .$$

Then

$$min\left\{\mathbf{x}_1\right\}=min\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_1,\gamma_2\right)=\frac{v}{2}\ \forall j\in\{1,2\}\ .$$

If agent ij deviates to $x_j(i)=1$, then

$$min\{\mathbf{x}_1\}=min\{\mathbf{x}_2\}$$

so that

$$\pi_{ij}^D\left(\gamma_j^+,\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}\geq\pi_{ij}^D\left(\gamma_j^D,\gamma_{-j}\right)=\frac{v}{2}-c\Leftrightarrow c\geq 0\ \forall j\in\{1,2\}\ .$$

Thus,

$$(\gamma_1,\gamma_2)=(0,0)$$

is a Nash equilibrium in pure strategies for any $c\geq 0$.

- Let $\sigma_{ij}(x_{ij}=1)$ be the within-group symmetric randomization over pure strategy $x_{ij}=1$ for player ij $\forall j\in\{1,2\}$, then

$$EU_{ij}(x_{ij}=1)=EU(x_{ij}=0)\Leftrightarrow$$

$$\Leftrightarrow Prob\left((n_j-1)\gamma_j=n_j-1\right)\cdot Prob\left(n_{-j}\gamma_{-j}<n_{-j}\right)\cdot(v-c)+\Big(Prob\left((n_j-1)\gamma_j<n_j-1\right)\cdot$$
$$Prob\left(n_{-j}\gamma_{-j}<n_{-j}\right)+Prob\left((n_j-1)\gamma_j=n_j-1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=n_{-j}\right)\Big)\left(\frac{v}{2}-c\right)+$$
$$+Prob\left((n_j-1)\gamma_j<n_j-1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=n_{-j}\right)\cdot(-c)=Prob\left(n_{-j}\gamma_{-j}<n_{-j}\right)\cdot\frac{v}{2}\Leftrightarrow$$
$$\Leftrightarrow\left(\sigma_{ij}(x_{ij}=1)\right)^{n_j-1}\cdot\left(1-\left(\sigma_{i-j}(x_{i-j}=1)\right)^{n_{-j}}\right)\cdot(v-c)+\Big(1-\left(\sigma_{ij}(x_{ij}=1)\right)^{n_j-1}\cdot$$
$$\left(1-\left(\sigma_{i-j}(x_{i-j}=1)\right)^{n_{-j}}\right)+\left(\sigma_{ij}(x_{ij}=1)\right)^{n_j-1}\cdot\left(\sigma_{i-j}(x_{i-j}=1)\right)^{n_{-j}}\Big)\cdot\left(\frac{v}{2}-c\right)+$$
$$+\Big(1-\left(\sigma_{ij}(x_{ij}=1)\right)^{n_j-1}\Big)\cdot\left(\sigma_{i-j}(x_{i-j}=1)\right)^{n_{-j}}\cdot(-c)=\left(1-\left(\sigma_{i-j}(x_{i-j}=1)\right)^{n_{-j}}\right)\cdot\frac{v}{2}\Leftrightarrow$$
$$\Leftrightarrow\sigma_{ij}^*(x_{ij}=1)=\left(\frac{2c}{v}\right)^{1/(n_j-1)}\quad\forall i\in\{1,\ldots,n_j\}\text{ and }j\in\{1,2\};$$

Thus,

$$\sigma_{ij}^*(x_{ij}=1)=\left(\frac{2c}{v}\right)^{1/(n_j-1)}\quad\forall i\in\{1,\ldots,n_j\}\text{ and }j\in\{1,2\}$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall\ 0<c<\frac{v}{2}$.

■

As before, it is easy prove the following result.

Proposition 4 *In the* $BMMGC^{*b}$,

- *for* $0\leq c<\frac{v}{2}$, $(\gamma_j,\gamma_{-j})=(1,0)$ *is the payoff-dominant equilibrium for group* $j\in\{1,2\}$;
- *for* $0\leq c<\frac{v}{4}$, $(\gamma_j,\gamma_{-j})=(1,1)$ *and* $(\gamma_j,\gamma_{-j})=(1,0)$ *are the risk-dominant equilibrium strategy profiles for group* $j\in\{1,2\}$;
- *for* $\frac{v}{4}<c<\frac{v}{2}$, $(\gamma_j,\gamma_{-j})=(0,1)$ *and* $(\gamma_j,\gamma_{-j})=(0,0)$ *are the risk-dominant equilibrium strategy profiles for group* $j\in\{1,2\}$.

Remark 2 *Note that, for* $\frac{v}{4} < c < \frac{v}{2}$ *and any group* $j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (1,0)$ *is the payoff-dominant equilibrium strategy profile for group* $j \in \{1,2\}$, *whereas* $(\gamma_j, \gamma_{-j}) = (0,1)$ *and* $(\gamma_j, \gamma_{-j}) = (0,0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1,2\}$*: there is a tension between payoff-dominance and risk-dominance.*

Proof. Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that in the $BMMGC^{*b}$:

- if $0 < c \leq \frac{v}{2}$, $(\gamma_1, \gamma_2) = (1,0)$ payoff-dominates $(\gamma_1, \gamma_2) = (0,0)$ for group 1 and $(\gamma_1, \gamma_2) = (0,1)$ payoff-dominates $(\gamma_1, \gamma_2) = (0,0)$ for group 2, since

$$\pi_{ij}\left(\gamma_j = 1, \gamma_{-j} = 0\right) = v - c > \pi_{ij}\left(\gamma_j = 0, \gamma_{-j} = 0\right) = \frac{v}{2} \Leftrightarrow c < \frac{v}{2};$$

- if $c = 0$, for group j, $(\gamma_j, \gamma_{-j}) = (1,0)$ payoff-dominates $(\gamma_1, \gamma_2) = (0,0), (\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2, (\gamma_1, \gamma_2)$ such that $\gamma_j \in (0,1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 2 \ \forall j \in \{1,2\}$ since

$$\begin{cases} \pi_{ij}\left(\gamma_j = 1, \gamma_{-j} = 0\right) = v \\ \pi_{ij}\left(\gamma_j = 0, \gamma_{-j} = 0\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j, 0\right) = \frac{v}{2} \\ \pi_{ij}\left(\gamma_j = 0, \gamma_{-j} = 1\right) = 0 \ ; \end{cases}$$

- if $0 < c < \frac{v}{4}$, $\gamma_1 = 1$ is the risk-dominant equilibrium strategy profile for group 1 and, symmetrically,$\gamma_2 = 1$ for group 2 . As a matter of fact, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (0,1) \, \forall j \in \{1,2\}$. Then, for group j:

$$\left(\frac{v}{2} - c\right)^{n_j} > (0 - (-c))^{n_j} \Leftrightarrow \left(\frac{v}{2} - c\right)^{n_j} > c^{n_j} \Leftrightarrow v > 4c \Leftrightarrow c < \frac{v}{4},$$

  that is, for $c < \frac{v}{4}$, $(\gamma_j, \gamma_{-j}) = (1,1)$ is associated with the largest Nash product.
  Moreover, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$. Then, for group j:

$$\left(v - c - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)^{n_j} \Leftrightarrow \left(\frac{v}{2} - c\right)^{n_j} > c^{n_j} \Leftrightarrow c < \frac{v}{4} \ ,$$

  that is, for $c < \frac{v}{4}$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product;

- if $c = 0$, $\gamma_1 = 1$ is the risk-dominant equilibrium strategy profile for group 1 and, symmetrically,$\gamma_2 = 1$ for group 2 . As a matter of fact, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,1)$ and $(\gamma_j, \gamma_{-j}) = (0,1) \, \forall j \in \{1,2\}$. Then, for group j:

$$\left(\frac{v}{2} - c\right)^{n_j} > (0 - (-c))^{n_j} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0^{n_j} \Leftrightarrow v > 0 \ ,$$

  that is, for $c = 0$, $(\gamma_j, \gamma_{-j}) = (1,1)$ is associated with the largest Nash product.
  Moreover, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$. Then, for group 1:

$$\left(v - c - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)^{n_j} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0^{n_j} \Leftrightarrow v > 0 \ ,$$

  that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product.

Furthermore, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}} \leq n_j - 2 \; \forall j \in \{1,2\}$. Then, for group 1:

$$\left(v - c - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - c - \frac{v}{2}\right)^{n_j \gamma_j} \cdot \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)^{n_j(1-\gamma_j)} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0 \Leftrightarrow v > 0 \; ,$$

that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product.

Finally, let us compare the Nash products of $(\gamma_j, \gamma_{-j}) = (1,0)$ and (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}} \leq n_j - 2 \; \forall j \in \{1,2\}$. Then, for group j:

$$\left(v - c - \frac{v}{2}\right)^{n_j} > \left(\frac{v}{2} - c - \frac{v}{2}\right)^{n_j \gamma_j} \cdot \left(\frac{v}{2} - \left(\frac{v}{2} - c\right)\right)^{n_j(1-\gamma_j)} \Leftrightarrow \left(\frac{v}{2}\right)^{n_j} > 0 \Leftrightarrow v > 0 \; ,$$

that is, for $v > 0$, $(\gamma_j, \gamma_{-j}) = (1,0)$ is associated with the largest Nash product;

- if $\frac{v}{4} < c < \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (0,1)$ and $(\gamma_j, \gamma_{-j}) = (0,0)$ are the risk-dominant equilibrium strategy profiles for group $j \in \{1,2\}$. Clearly this follows from what shown at the previous point for both groups.

■

3.1 Incomplete Information à la global games about the Prize

Let us consider the case where the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$, that is the $BMMGC^*$ model. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$, e.g. $[1,5]$;
- given the realization v, each player $ij \in \{1, \ldots, n_j\} \; \forall j \in \{1,2\}$ idiosyncratically observes the realization of a random variable V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$, so that the players' observation errors $V_{ij} - v \; \forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1,2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\} \; \forall j \in \{1,2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, we refer to this game as $g_1(v)$. Then, we are able to obtain the following result.

Proposition 5 *In the $g_1(v)$, there is a unique equilibrium in (monotonic) switching strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1,2\}$:*

$$x_{ij}^*(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > 2^{n_j} \\ 0 & \text{if } v_{ij} \leq 2^{n_j} \end{cases}$$

other than the equilibrium

$$x_{ij}^{**}(v_{ij}) = 0 \; \forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon] \, .$$

Remark 3 *The absence of an upward dominance region is conducive to the existence of the equilibrium $x_{ij}^{**}(v_{ij}) = 0 \; \forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, so that the equilibrium $(\gamma_1, \gamma_2) = (0,0)$ in the $BMMGC^*$ is robust to incomplete information in the sense of Kajii and Morris (1997).*

Remark 4 *The existence of an equilibrium in (monotonic) switching strategies in the* $g_1(v)$ *is ensured as long as* $0 < \varepsilon < \left|\frac{\underline{v}}{2} - 1\right|$. *However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small* ε *is.*

Remark 5 *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e.* $v_{ij} = 2^{n_j}$, *does not coincide with the one of the risk-dominance region, that is* $v_{ij} = 4$ *for any* $j \in \{1,2\}$, *differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and van Damme [1993b] for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when we depart from the 2×2 case.*

Proof. In the $g_1(v)$, note that $E(V|v_{ij}) = v_{ij}$, if i observes $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_{ij} \sim U(v_{ij} - \varepsilon, v_{ij} + \varepsilon)$. Furthermore, for $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with support $[v_{ij} - 2\varepsilon, v_i + 2\varepsilon]$. Hence, $Prob[V_{-ij} < v_{ij}|v_{ij}] = Prob[V_{-ij} > v_{ij}|v_{ij}] = \frac{1}{2}$ $\forall ij \in \{1,\ldots,n_j\}$ and $j \in \{1,2\}$.

Now, suppose player $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ observes $v_{ij} < 2$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ whenever she observes $v_{ij} < 2$. Iterating this dominance argument, if players $-ij \in \{1,\ldots,N-1\}$ are forced to play $x_{-ij} = 0$ whenever they observe $v_{-ij} < 2$, then player ij, observing $v_{ij} = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j - 1 + n_{-j}}$ to $\sum_{-ij=1}^{N-1} x_{-ij} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $1 - 2^{-n_{-j}}$, so that $x_{ij} = 1$ can be discarded by iterated dominance for $v_{ij} = 2$, since the conditionally expected payoff from exerting effort equals $2^{1-n_j} - 2^{-n_{-j}}$. Note that we imposed by assumption that $0 < \varepsilon < \left|\frac{\underline{v}}{2} - 1\right|$, so that $v_{ij} - 2\varepsilon > \underline{v}$ for $v_{ij} = 2$. Let v_{ij}^* be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $v_{ij}^* = 2^{n_j}$. Note that $v_{ij} = 4$ is the threshold for the risk-dominance regions. As a matter of fact, when $v_{ij} = 2^{n_j}$, the conditionally expected payoff from exerting effort equals

$$Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(2^{n_j} - 1\right) + \Big(Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot$$

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big) \left(\frac{2^{n_j}}{2} - 1\right) +$$

$$+ Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot (-1) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(2^{n_j} - 1\right) + \left(\left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) +\right.$$

$$\left. + \left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(\frac{1}{2}\right)^{n_j - 1}\right) \left(\frac{2^{n_j}}{2} - 1\right) + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n-j} \cdot (-1) = 2^{n_j - n_{-j} - 1} \cdot \left(2^{n_{-j}} - 1\right),$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{2^{n_j}}{2} \Leftrightarrow \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{2^{n_j}}{2} = 2^{n_j - n_{-j} - 1} \cdot \left(2^{n_{-j}} - 1\right).$$

The cutoff $v_{ij}^* = 2^{n_j}$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for

$v_{ij} \in [v-\varepsilon, v+\varepsilon]$ solving

$$\left(\frac{1}{2}\right)^{n_j-1} \cdot \left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot (v_{ij}-1) + \left(\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1-\left(\frac{1}{2}\right)^{n_j-1}\right) + \right.$$

$$\left. + \left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{v_{ij}}{2}-1\right) + \left(1-\left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{1}{2}\right)^{n_{-j}} \cdot (-1) = \left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v_{ij}}{2}.$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower and an upper dominance region.

Hence, in $g_1(x)$ under incomplete information à la global games there is a unique equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1,\ldots,n_j\}$ and $\forall j \in \{1,2\}$:

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > 2^{n_j} \\ 0 & \text{if } v_{ij} \le 2^{n_j} \end{cases}$$

Nonetheless, given the absence of an upward dominance region, the following equilibrium $\forall ij \in \{1,\ldots,n_j\}$ and $\forall j \in \{1,2\}$ exists as in De Mesquita [2011]:

$$x^{**}_{ij}(v_{ij}) = 0 \ \forall v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon].$$

Note that at $(\gamma_1,\gamma_2)=(0,0)$ any deviation is strictly dominated for any $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, so that $(\gamma_1,\gamma_2)=(0,0)$ in the $BMMGC^*$ is robust to incomplete information in the sense of Kajii and Morris [1997]. ■

Moreover, we are able to calculate the probability of winning the prize v for both groups at the unique equilibrium in (monotonic) switching strategies, as shown by the following result.

Proposition 6 *In the $g_1(v)$, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium equals:*

- *if $\underline{v}+\varepsilon \le 2^{n_j} \le \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \le 2^{n_{-j}} \le \overline{v}-\varepsilon$,*

$$\begin{aligned} Prob\,(j\ wins\ v) = {} & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \cdot \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right] + \\ & + \frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right] \cdot \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right] + \\ & + \frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j} \cdot \left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}} \quad ; \end{aligned}$$

- *if $\underline{v}+\varepsilon \le 2^{n_j} \le \overline{v}-\varepsilon$ and $2^{n_{-j}} < \underline{v}+\varepsilon$,*

$$Prob\,(j\ wins\ v) = \frac{1}{2} \cdot \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad ;$$

- *if $\underline{v}+\varepsilon \le 2^{n_j} \le \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,*

$$\begin{aligned} Prob\,(j\ wins\ v) = {} & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} + \\ & + \frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right] \quad ; \end{aligned}$$

- *if* $2^{n_j} < \underline{v}+\varepsilon$ *and* $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$Prob\,(j\ \ wins\ v) = \left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}--\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+\frac{1}{2}\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}} \ ;$$

- *if* $2^{n_j} < \underline{v}+\varepsilon$ *and if* $2^{n_{-j}} < \underline{v}+\varepsilon$,

$$Prob\,(j\ \ wins\ v) = \frac{1}{2} \ ;$$

- *if* $2^{n_j} < \underline{v}+\varepsilon$ *and* $2^{n_{-j}} > \overline{v}-\varepsilon$,

$$Prob\,(j\ \ wins\ v) = 1 \ ;$$

- *if* $2^{n_j} > \overline{v}-\varepsilon$ *and* $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$Prob\,(j\ \ wins\ v) = \frac{1}{2}\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right] \ ;$$

- *if* $2^{n_j} > \overline{v}-\varepsilon$ *and* $2^{n_{-j}} < \underline{v}+\varepsilon$,

$$Prob\,(j\ \ wins\ v) = 0 \ ;$$

- *if* $2^{n_j} > \overline{v}-\varepsilon$ *and* $2^{n_{-j}} > \overline{v}-\varepsilon$,

$$Prob\,(j\ \ wins\ v) = \frac{1}{2} \ .$$

Proof. In the $g_1(v)$, given the contest success function $P_j(X_j, X_{-j})\,\forall j \in \{1,2\}$, the probability of winning the prize v for group $j \in \{1,2\}$ is:

$$\begin{aligned} Prob\,(j \text{ wins } v) =& Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right)=(1,0)\right]+\frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right)=(0,0)\right]+ \\ &+\frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right)=(1,1)\right] \ . \end{aligned}$$

On the other hand, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium $x_{ij}^*(v_{ij})$ depends on whether or not 2^{n_j} belongs to $[\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, where v_{ij} is uniformly distributed. Hence, we will consider all possible cases:

- if $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$, [4]

$$Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right] = \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \ ;$$

$$Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right] = \left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \ ;$$

$$Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right] = \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} \ .$$

[4] Note that for $\gamma_{-j}=0$, it suffices that just one $i-j$ chooses $x_{i-j}(v_{i-j})=0$, due to the weakest-link impact function.

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)] + \\
&+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)] \\
=& \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+ \\
&+\frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right]+ \\
&+\frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}} \quad .
\end{aligned}$$

- If $\underline{v}-\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $2^{n_{-j}} \leq \underline{v}+\varepsilon$,

$$\begin{aligned}
&Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] = 0\ ; \\
&Prob\,[(\gamma_j,\gamma_{-j})=(0,0)] = 0\ ; \\
&Prob\,[(\gamma_j,\gamma_{-j})=(1,1)] = \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)] + \\
&+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)] \\
=& \frac{1}{2}\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad .
\end{aligned}$$

- If $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,

$$\begin{aligned}
&Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] = \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad ; \\
&Prob\,[(\gamma_j,\gamma_{-j})=(0,0)] = 1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad ; \\
&Prob\,[(\gamma_j,\gamma_{-j})=(1,1)] = 0\ .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)] + \\
&+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)] \\
=& \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} + \frac{1}{2}\left[1-\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}\right] \quad .
\end{aligned}$$

- If $2^{n_j} < \underline{v} - \varepsilon$ and $\underline{v} + \varepsilon \leq 2^{n-j} \leq \overline{v} - \varepsilon$,

$$\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] =& 1 - \left(1 - \frac{2^{n-j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n-j} \quad ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] =& 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] =& \left(1 - \frac{2^{n-j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n-j} \quad .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] \\
=& 1 - \left(1 - \frac{2^{n-j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n-j} + \frac{1}{2}\left(1 - \frac{2^{n-j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n-j} \quad .
\end{aligned}$$

- If $2^{n_j} < \underline{v} + \varepsilon$ and $2^{n-j} < \underline{v} + \varepsilon$,

$$\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 1 \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = \frac{1}{2} \ .
\end{aligned}$$

- If $2^{n_j} < \underline{v} + \varepsilon$ and $2^{n-j} > \overline{v} - \varepsilon$,

$$\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 1 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 0 \ .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 1 \ .
\end{aligned}$$

- If $2^{n_j} > \overline{v} - \varepsilon$ and $\underline{v} + \varepsilon \leq 2^{n-j} \leq \overline{v} - \varepsilon$,

$$\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] =& 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] =& 1 - \left(1 - \frac{2^{n-j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n-j} \quad ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] =& 0 \ .
\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]\\ =&\frac{1}{2}\left[1-\left(1-\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}}\right] .\end{aligned}$$

- If $2^{n_j} > \overline{v}-\varepsilon$ and $2^{n_{-j}} \leq \underline{v}+\varepsilon$,

$$\begin{aligned}Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]&=0 ;\\ Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]&=0 ;\\ Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]&=0 .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=0 .\end{aligned}$$

- If $2^{n_j} > \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,

$$\begin{aligned}Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]&=0 ;\\ Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]&=1 ;\\ Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]&=0 .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=\frac{1}{2} .\end{aligned}$$

■

3.2 Incomplete Information à la global games about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C_{ij}\,(x_j\,(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$, that is the $BMMGC^{*b}$ model . We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c},\overline{c}]$, e.g. $[-v,+v]$;
- given the realization c, each player $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ idiosyncratically observes the realization of a random variable C_{ij}, uniform on $[c-\varepsilon, c+\varepsilon]$ for some $0 < \varepsilon < min\left\{\left|\frac{2\overline{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$, so that the players' observation errors $C_{ij}-c$ $\forall ij \in \{1,\ldots,n_j\}$ and $\forall j \in \{1,2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, we refer to this game as $g_2(c)$. Then we are able to obtain the following result.

Proposition 7 *In the $g_2(c)$, there is a unique equilibrium in (monotonic) switching strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$:*

$$x_{ij}^*(c_{ij}) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v. \end{cases}$$

Remark 6 *The presence of both an upward dominance region and a downward dominance region is conducive to the selection of a unique equilibrium in (monotonic) switching strategies.*

Remark 7 *The existence of a unique equilibrium in (monotonic) switching strategies in the $g_2(c)$ is ensured as long as $0 < \varepsilon < min\left\{\left|\frac{2\bar{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$. However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small ε is.*

Remark 8 *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e. $c_{ij} = 2^{-n_j}v$, does not coincide with the one of the risk-dominance region, that is $c_{ij} = \frac{v}{4}$ for any $j \in \{1, 2\}$, differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and van Damme [1993b] for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when we depart from the 2×2 case.*

Proof. In the $g_2(c)$, note that $E(C|c_{ij}) = c_{ij}$, if ij observes $c_{ij} \in [\underline{c} + \varepsilon, \bar{c} - \varepsilon]$ so that $C|c_{ij} \sim U(c_{ij} - \varepsilon, c_{ij} + \varepsilon)$. Furthermore, for $c_{ij} \in [\underline{c} - \varepsilon, \bar{c} + \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$. Hence, $Prob[C_{-ij} < c_{ij}|c_{ij}] = Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$.

Now, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} > \frac{v}{2}$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} > \frac{v}{2}$. Iterating this dominance argument, if players $-ij \in \{1, \ldots, N-1\}$ are forced to play $x_{i-j} = 0$ whenever they observe $c_{-ij} > \frac{v}{2}$, then player ij, observing $c_{ij} = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j - 1 + n_{-j}}$ to $\sum_{-ij=1}^{N-1} x_{-ij} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $\frac{1}{2}(1 - 2^{-n_{-j}})v$, so that $x_{ij} = 1$ can be discarded by iterated dominance for $c_{ij} = \frac{v}{2}$, since the conditionally expected payoff from exerting effort equals $\left(2^{-n_j} - 2^{-n_{-j}-1}\right)v$. Note that we imposed by assumption that $\varepsilon < \left|\frac{2\bar{c}-v}{4}\right|$, so that $c_{ij} + 2\varepsilon < \bar{c}$ for $c_{ij} = \frac{v}{2}$. Let c_{ij}^* be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_{ij}^* = 2^{-n_j}v$. Note that $c_{ij} = \frac{v}{4}$ is the threshold for the risk-dominance regions. As a matter of fact, when $c_{ij} = 2^{-n_j}v$, the conditionally expected payoff from exerting effort equals

$$Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(v - 2^{-n_j}v\right) + \Big(Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot$$

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left((n_j - 1)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big)\left(\frac{v}{2} - 2^{-n_j}v\right) +$$

$$+ Prob\left((n_j - 1)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \left(-2^{-n_j}v\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(v - 2^{-n_j}v\right) + \left(\left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) + \left(\frac{1}{2}\right)^{n_{-j}} \cdot\right.$$

$$\cdot \left(\frac{1}{2}\right)^{n_j-1}\Bigg) \cdot \left(\frac{v}{2} - 2^{-n_j}v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{1}{2}\right)^{n-j} \cdot \left(-2^{-n_j}v\right) = \frac{1}{2}\left(1 - 2^{-n_{-j}}\right) v \ ,$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v}{2} \Leftrightarrow \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} = \frac{1}{2}\left(1 - 2^{-n_{-j}}\right) v \ .$$

The cutoff $c^*_{ij} = 2^{-n_j}v$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_{ij} \in [c - \epsilon, c + \epsilon]$ solving

$$\left(\frac{1}{2}\right)^{n_j-1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot (v - c_{ij}) + \Bigg(\left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_j-1}\right) + \left(\frac{1}{2}\right)^{n_{-j}} \cdot$$

$$\cdot \left(\frac{1}{2}\right)^{n_j-1}\Bigg) \cdot \left(\frac{v}{2} - c_{ij}\right) + \left(1 - \left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{1}{2}\right)^{n-j} \cdot (-c_{ij}) = \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot v$$

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region. Again, let us assume $\varepsilon < -\frac{c}{2}$ and suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ observes $c_{ij} < 0$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is positive and greater than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 1$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ whenever she observes $c_{ij} < 0$. Iterating this dominance argument, if players $-ij \in \{1, \ldots, N-1\}$ are forced to play $x_{-ij} = 1$ whenever they observe $c_{-ij} < 0$, then player ij, observing $c_{ij} = 0$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j-1+n_{-j}}$ to $\sum_{-ij=1}^{N-1} x_{-ij} = N-1$. Thus, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, will be at least $\frac{1}{2}\left(1 - 2^{-n_{-j}} + 2^{1-n_j}\right) v$, so that $x_{ij} = 0$ can be discarded by iterated dominance for $c_{ij} = 0$, since the conditionally expected payoff from not exerting effort equals $\frac{1}{2}\left(1 - 2^{-n_{-j}}\right) v$. Note that we imposed by assumption that $0 < \varepsilon < \left|\frac{c}{2}\right|$, so that $c_{ij} - 2\varepsilon > \underline{c}$ for $c_{ij} = 0$. Let c^{**}_{ij} be the smallest observation such that $x_{ij} = 0$ cannot be excluded by iterated dominance. Then, it is possible to show that $c^{**}_{ij} = 2^{-n_j}v$. Note that $c_{ij} = \frac{v}{4}$ is the threshold for the risk-dominance regions. As a matter of fact, when $c_{ij} = 2^{-n_j}v$, the conditionally expected payoff from exerting effort equals

$$Prob\left(\left(n_j - 1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \left(v - 2^{-n_j}v\right) + \Big(Prob\left(\left(n_j - 1\right)\gamma_j < n_j - 1\right) \cdot$$

$$\cdot Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) + Prob\left(\left(n_j - 1\right)\gamma_j = n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right)\Big) \left(\frac{v}{2} - 2^{-n_j}v\right) +$$

$$+ Prob\left(\left(n_j - 1\right)\gamma_j < n_j - 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = n_{-j}\right) \cdot \left(-2^{-n_j}v\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{2}\right)^{n_j-1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(v - 2^{-n_j}v\right) + \Bigg(\left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_j-1}\right) + \left(\frac{1}{2}\right)^{n_{-j}} \cdot$$

$$\cdot \left(\frac{1}{2}\right)^{n_j-1}\Bigg) \cdot \left(\frac{v}{2} - 2^{-n_j}v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_j-1}\right) \cdot \left(\frac{1}{2}\right)^{n-j} \cdot \left(-2^{-n_j}v\right) = \frac{1}{2}\left(1 - 2^{-n_{-j}}\right) v \ ,$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} < n_{-j}\right) \cdot \frac{v}{2} \Leftrightarrow \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} = \frac{1}{2}\left(1 - 2^{-n_{-j}}\right) v \ .$$

The cutoff $c_{ij}^{**} = 2^{-n_j} v$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_{ij} \in [c - \varepsilon, c + \varepsilon]$ solving

$$\left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot (v - c_{ij}) + \left(\left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) + \left(\frac{1}{2}\right)^{n_{-j}} \cdot \right.$$

$$\left. \cdot \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{v}{2} - c_{ij}\right) + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n - j} \cdot (-c_{ij}) = \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} \ .$$

Hence, $c_{ij}^{*} = c_{ij}^{**}$ and there exists a unique equilibrium in switching strategies in $g_2(x)$ such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$

$$x_{ij}^{*}(c_{ij}) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v. \end{cases}$$

■

Differently from what is obtained by Barbieri et al. [2019] for a deterministic two-group contest with the weakest-link impact function, continuous effort, a public good prize commonly valued across groups and incomplete information about the cost of effort, in the $g_2(c)$ there are no multiple equilibria in pure strategies but a unique equilibrium in (monotonic) switching strategies. The uniqueness result achieved by Barbieri et al. [2019] regards only the class of nondegenerate Bayes-Nash equilibria without mass at the top. As a matter of fact, the authors obtain a continuum of non-degenerate Bayes-Nash equilibria with mass at the top, other than degenerate Bayes-Nash equilibria, where players perfectly align effort choices on the highest cost type: a degeneracy result consinstent with the complete information model due to Chowdhury et al. [2016] . Semidegenerate equilibria, that is equilibria in which effort levels are dispersed just in one group, are found only for the setting with asymmetric prize valuations between the two groups, i.e. $v_1 \neq v_2$.

Moreover, we are able to calculate the probability of winning the prize v for both groups at the unique equilibrium in (monotonic) switching strategies, as shown by the following result.

Proposition 8 *In the $g_2(c)$, the probability of winning the prize v for group $j \in \{1, 2\}$ at the cutoff equilibrium equals:*

- *if $\underline{c} + \varepsilon \leq 2^{-n_j} v \leq \overline{c} - \varepsilon$ and $\underline{c} + \varepsilon \leq 2^{-n_{-j}} v \leq \overline{c} - \varepsilon$,*

$$\begin{aligned} Prob\left(j \ wins \ v\right) = & \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left[1 - \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\ & + \frac{1}{2}\left[1 - \left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] + \\ & + \frac{1}{2}\left(\frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(\frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad ; \end{aligned}$$

- *if* $\underline{c}+\varepsilon \leq 2^{-n_j}v \leq \overline{c}-\varepsilon$ *and* $2^{-n_{-j}}v < \underline{c}+\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j} + \\ +\frac{1}{2}\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\ ;$$

- *if* $\underline{c}+\varepsilon \leq 2^{-n_j}v \leq \overline{c}-\varepsilon$ *and* $2^{-n_{-j}}v > \overline{c}-\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \frac{1}{2}\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)\ ;$$

- *if* $2^{-n_j}v < \underline{c}+\varepsilon$ *and* $\underline{c}+\varepsilon \leq 2^{-n_{-j}}v \leq \overline{c}-\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \frac{1}{2}\left[1-\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]\ ;$$

- *if* $2^{-n_j}v < \underline{c}+\varepsilon$ *and* $2^{-n_{-j}}v < \underline{c}+\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \frac{1}{2}\ ;$$

- *if* $2^{-n_j}v < \underline{c}+\varepsilon$ *and* $2^{-n_{-j}}v > \overline{c}-\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = 0\ ;$$

- *if* $2^{-n_j}v > \overline{c}-\varepsilon$ *and* $\underline{c}+\varepsilon \leq 2^{-n_{-j}}v \leq \overline{c}-\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \left[1-\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right] + \\ +\frac{1}{2}\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\ ;$$

- *if* $2^{-n_j}v > \overline{c}-\varepsilon$ *and* $2^{-n_{-j}}v < \underline{c}+\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = 1\ ;$$

- *if* $2^{-n_j}v > \overline{c}-\varepsilon$ *and* $2^{-n_{-j}}v > \overline{c}-\varepsilon$,

$$Prob\,(j\ \ wins\ \ v) = \frac{1}{2}\ .$$

Proof. In the $g_2\,(c)$, given the contest success function $P_j\,(X_j, X_{-j})\,\forall j \in \{1,2\}$, the probability of winning the prize v for group $j \in \{1,2\}$ is:

$$Prob\,(j \text{ wins } v) = Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (1,0)\right] + \frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (0,0)\right] + \\ +\frac{1}{2}Prob\left[\left(\gamma_j^*, \gamma_{-j}^*\right) = (1,1)\right]\ .$$

On the other hand, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium $x_{ij}^*\,(c_{ij})$ depends on whether or not $2^{-n_j}v$ belongs to $[\underline{c}+\varepsilon, \overline{c}-\varepsilon]$, where c_{ij} is uniformly distributed. Hence, we will consider all possible cases:

- if $\underline{c}+\varepsilon \leq 2^{-n_j}v \leq \bar{c}-\varepsilon$ and $\underline{c}+\varepsilon \leq 2^{-n_{-j}}v \leq \bar{c}-\varepsilon$, [5]

$$Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]=\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left[1-\left(\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right] \ ;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]=\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right] \ ;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]=\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left(\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}} \ .$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]+\\&+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]\\=&\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left[1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]+\\&+\frac{1}{2}\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]+\\&+\frac{1}{2}\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}} \ .\end{aligned}$$

- If $\underline{c}+\varepsilon \leq 2^{-n_j}v \leq \bar{c}-\varepsilon$ and $2^{-n_{-j}}v < \underline{c}+\varepsilon$,

$$Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]=\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j} \ ;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]=\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right] \ ;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]=0 \ .$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]+\\&+\frac{1}{2}Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]\\=&\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}+\frac{1}{2}\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right] \ .\end{aligned}$$

- If $\underline{c}+\varepsilon \leq 2^{-n_j}v \leq \bar{c}-\varepsilon$ and $2^{-n_{-j}}v > \bar{c}-\varepsilon$,

$$Prob\left[(\gamma_j,\gamma_{-j})=(1,0)\right]=0;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(0,0)\right]=0 \ ;$$
$$Prob\left[(\gamma_j,\gamma_{-j})=(1,1)\right]=\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\bar{c}-\underline{c}-2\varepsilon}\right)^{n_j} \ .$$

[5] Note that for $\gamma_{-j}=0$, it suffices that just one $i-j$ chooses $x_{i-j}\,(c_{i-j})=0$, due to the weakest-link impact function.

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]\\ =&\frac{1}{2}\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\ .\end{aligned}$$

- If $2^{-n_j}v<\underline{c}+\varepsilon$ and $\underline{c}+\varepsilon\leq 2^{-n_{-j}}v\leq\overline{c}-\varepsilon$,

$$\begin{aligned}&Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]=0;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]=1-\left(\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\ ;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=0\ .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]\\ =&\frac{1}{2}\left[1-\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\ .\end{aligned}$$

- If $2^{-n_j}v<\underline{c}+\varepsilon$ and $2^{-n_j}v<\underline{c}+\varepsilon$,

$$\begin{aligned}&Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]=0\ ;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]=1\ ;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=0\ .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]\\ =&\frac{1}{2}\ .\end{aligned}$$

- If $2^{-n_j}v<\underline{c}+\varepsilon$ and $2^{-n_{-j}}v>\underline{c}-\varepsilon$,

$$\begin{aligned}&Prob\,[(\gamma_j,\gamma_{-j})=(1,0)]=0\ ;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]=0\ ;\\ &Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=0\ .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) =& Prob\,[(\gamma_j,\gamma_{-j})=(1,0)] + \frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(0,0)]+\\ &+\frac{1}{2}Prob\,[(\gamma_j,\gamma_{-j})=(1,1)]=0\ .\end{aligned}$$

- If $2^{-n_j}v > \overline{c} - \varepsilon$ and $\underline{c} + \varepsilon \leq 2^{-n_{-j}}v \leq \overline{c} - \varepsilon$,

$$
\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] =& 1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] =& \ 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] =& \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] + \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] \\
=& 1 - \left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} + \frac{1}{2}\left(\frac{2^{-n_{-j}}v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} \quad .
\end{aligned}
$$

- If $2^{-n_j}v > \overline{c} - \varepsilon$ and $2^{-n_{-j}}v < \underline{c} + \varepsilon$,

$$
\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 1 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 0 \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Prob\,(\text{j wins v}) = Prob\left[(\gamma_j, \gamma_{-j}) = \right.& \left.(1,0)\right] + \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 1 \ .
\end{aligned}
$$

- If $2^{-n_j}v > \overline{c} - \varepsilon$ and $2^{-n_{-j}}v > \overline{c} - \varepsilon$,

$$
\begin{aligned}
Prob\left[(\gamma_j, \gamma_{-j}) = (1,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] = 0 \ ; \\
Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = 1 \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Prob\,(j \text{ wins } v) = Prob\left[(\gamma_j, \gamma_{-j}) = \right.& \left.(1,0)\right] + \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (0,0)\right] + \\
&+ \frac{1}{2}Prob\left[(\gamma_j, \gamma_{-j}) = (1,1)\right] = \frac{1}{2} \ .
\end{aligned}
$$

■

4 Conclusions

We introduced incomplete information à la global games in a two-group max-min group contest with binary actions, relaxing the complete information assumption about the value of the prize contested and the cost of providing effort, separately. In the first case, there are both an equilibrium in (monotonic) switching strategies and an equilibrium robust to incomplete information in the sense of Kajii and Morris [1997]; in the second one, a unique equilibrium in (monotonic) switching-strategies emerges. Moreover, given the information structure, it is straightforward to calculate the probability of winning for each group at the equilibrium in switching strategies in both cases. Therefore, introducing incomplete information à la global games in max-min group contests with binary actions does not only deliver informational realism, but it also reduces significantly the burden of equilibrium multiplicity, or rather indeterminacy, which affects deterministic group contests with continuous efforts and a public good prize under both complete information and under incomplete information, as in Chowdhury et al. [2016] and Barbieri et al. [2019], respectively. We would like to stress that this selection result could be relevant for applications of deterministic two-group contests with binary actions, among which we emphasized research groups, international alliances, group strikes and military conflict in the introduction.

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