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Davide Bosco, Mario Gilli, Andrea Sorrentino

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Max-Max Group Contests with Incomplete Information à la Global Games

Davide Bosco[*] Mario Gilli[†] Andrea Sorrentino[‡]

Department of Economics, Management and Statistics

University of Milano-Bicocca

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Abstract

In this paper we introduce incomplete information à la global games into a deterministic two-group contest with the best-shot impact function and binary actions and we characterize the set of equilibria. Depending on whether the complete information assumption is relaxed on the value of the prize or on the cost of providing effort, we obtain different results in terms of equilibrium uniqueness: in the first case, there exist an equilibrium in (monotonic) switching strategies which could be not unique, whereas in the second one there exists a unique equilibrium in (monotonic) switching-strategies. Then, we discuss the presence of the group-size paradox for both classes of games.The results are thus extended to the case of M groups, and the properties of Bayes-Nash equilibria for these classes of games are investigated. Finally, we show a limit-uniqueness and a noise independent selection result.

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Keywords: Group contests, incomplete information, global games.
JEL Codes: D74, D71, C72

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[*] davide.bosco@unimib.it
[†] mario.gilli@unimib.it
[‡] a.sorrentino9@campus.unimib.it

1 Introduction

Mirroring the terminology introduced by Chowdhury, Lee, and Topolyan [2016] for deterministic group contests with the weakest-link impact function, we call “max-max” group contests with the auction-type contest success function and the best-shot impact function, so that the aggregated effort within a group equals the maximum of the efforts provided. Group contests with the best-shot impact function can describe competition by groups for the generation of innovative ideas in $R\&D$, military conflict and provision of a discrete public good, as stressed by Chowdhury, Lee, and Sheremeta [2013] and Barbieri, Malueg, and Topolyan [2014]. In these settings we bolster the idea that effort provision choice can be conceived and modeled as a binary variable for generation of innovative projects, delivery of exceptional skill-based output, military attack and duels. Despite being non-standard, the assumption of a binary action set is not new in contest theory, rather it has been adopted by a wide theoretical and experimental literature, spanning from corporate science, to sabotage activities and contests for status, as reviewed by Sheremeta [2018].

Deterministic group contests with the best-shot impact function display multiplicity of equilibria under both complete and payoff-relevant incomplete information, as shown by Barbieri et al. [2014] and Barbieri and Malueg [2016], respectively. However, a recent contribution by Barbieri and Topolyan [2024] shows that group-public randomization delivers equilibrium uniqueness in best-shot group contests. Group-public randomization is conceived by the authors as a source of extrinsic incomplete information, that is not payoff-relevant. On the other hand, in Bosco, Gilli, and Sorrentino [2024], we show that introducing incomplete information à la global games about the value of the prize or the cost of effort in deterministic group contests with the weakest-link impact function delivers interesting equilibrium selection properties. In this paper, as done for deterministic group contests with the weakest-link impact function in Bosco et al. [2024], we follow Carlsson and van Damme [1993a] by introducing payoff-relevant incomplete information à la global games about the value of the prize contested and the cost of effort into a max-max two-group contest with binary actions. In the first case, we prove the existence of an equilibrium in (monotonic) switching-strategies, without proving equilibrium uniqueness, whereas in the second case we are able to show the existence of a unique equilibrium in (monotonic) switching strategies, resulting by iterated deletion of (interim) strictly dominated strategies, closely following Carlsson and van Damme [1993a] for 2 × 2 games. Risk-dominance is shown to be a valid equilibrium selection criterion for two-group contests with two players per group, but fails to be pivotal as the number of team members exceeds two, confirming what shown by Carlsson and van Damme [1993b] for the generalization of stag-hunt games to the n-player case. These results are very similar to what obtained for max-min group contests with incomplete information à la global games, except for the non-existence of an equilibrium robust to incomplete information, in the sense of Kajii and Morris [1997], in which no player exerts effort for any possible private value of the prize. Other dimensions along which max-max group contests differ from the weakest-link counterpart are the presence of the so-called group-size paradox and the generalization to the M-group case. As a matter of fact, whether an increase in group size translates into a lower probability of winning and a lower expected payoff cannot be established generally for this class of games, so that we provide numerical examples, only. Moreover, the generalization of our analysis to the M-group case, highlights that the equilibria in (monotonic) switching strategies in the two settings considered are rooted in different thresholds with respect to the weakest-link M-group counterpart, as shown in Bosco et al. [2024]. Finally, as done in Bosco et al. [2024] for the weakest-link impact function, we deliver limit-uniqueness and noise-independent selection results, but in a narrower sense with respect to the original work by Carlsson and van Damme [1993a].

The paper is structured as follows. The paper is structured as follows. In Section 2 the formal model with both complete information and incomplete information is presented under two different specifications. Section 3 presents two examples which should clarify the parallelism between group contests and the supermodular payoff structure perturbed in the global games à la Carlsson and van Damme [1993a] and how equilibrium selection naturally arises when modelling incomplete information à la global games, stressing the difference incurring with the weakest-link case address in Bosco et al. [2024]. Section 4 derives the set of Nash equilibria of the complete information game, while section 5 is the core of the paper deriving the set of Bayes-Nash equilibria for the two class of incomplete information games. Section 6 addresses whether there is the so-called group-size paradox in the two model specifications delivered in Section 5, providing numerical examples. Section 7 extends the two-group contest model under incomplete information to an M-group contest model. Section 8 delivers results regarding limit-uniqueness and noise independent selection. Finally, Section 9 concludes.

2 The Model

Let us consider a deterministic group contest defined by the following elements:

1. two **groups,** denoted by $j \in \{1, 2\}$;
2. each group has $n_j \geq 2$ members, where $n_1 \geq n_2$ without loss of generality. The total number of agents is $n_1 + n_2 = N$. As notation device, let us write ij or $j(i)$ for **agents** $i \in \{1, \ldots, n_j\}$ of group j;
3. the **choice** of member $i \in \{1, \ldots, n_j\}$ in group $j \in \{1, 2\}$, to increase the possibility of getting the prize, is denoted by $x_j(i) \in \{0, 1\}$. Let $\mathbf{x}_j$ be the vector of all agents' efforts of group j, and $\mathbf{x}$ the vector of all agents' efforts. Moreover, let $x_j(i) = 1$ be denoted by a and $x_j(i) = 0$ by $\overline{a}$; let us define the average exerted effort in group j, or rather the participation rate in group j as

   $$\gamma_j = \frac{1}{n_j} \sum_{i=1}^{n_j} x_{ij} \in [0, 1];$$

   Moreover, when $\gamma_j \in (0, 1)$, denote by

   $$\gamma_j^+ = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} + 1 \right) \in [0, 1]$$

   the share of active agents at a marginal increase and by

   $$\gamma_j^- = \frac{1}{n_j} \left(\sum_{i=1}^{n_j} x_{ij} - 1 \right) \in [0, 1]$$

   the share of active agents at a marginal decrease.
4. a club good **prize** worth $v \in \mathbb{R}$ is allocated to one of the two groups: thus, the prize v can be worth negative utils, which means that it can be a bad;
5. the **impact function** of group j is given by the best-shot technology

   $$X_j = \max \{x_j(i) \in \{0, 1\}, i \in \{1, ..., n_j\}\};$$

6. the **contest success function** is given by the *all-pay auction:*

   $$p_j(X_1, X_2) = \begin{cases} 1 & if \;\; X_j > X_{-j} \\ \frac{1}{2} & if \;\; X_j = X_{-j} \\ 0 & if \;\; X_j < X_{-j}; \end{cases}$$

7. the individual **costs of effort** $C_{ij}\left(x_j\left(i\right)\right)=x_j\left(i\right)$.

As a consequence of these modelling characteristics, player ij has the expected **payoff**

$$\pi_{ij}\left(\mathbf{x}_1,\mathbf{x}_2\right)=p_jv-x_{ij}=$$

$$=\begin{cases} v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}>\max\left\{\mathbf{x}_{-j}\right\} \\ \frac{1}{2}v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}=\max\left\{\mathbf{x}_{-j}\right\} \\ -x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}<\max\left\{\mathbf{x}_{-j}\right\}. \end{cases}$$

Now we are able to provide a formal definition of a binary max-min group contest with a public good prize.

Definition 1 *A* Binary Max-Max Group Contest *$BMMAGC^*$ is a one-stage game $BMMAGC^*=\left\langle\left\{1,2\right\},N,B_{ij},\pi_{ij}\right\rangle$ defined by*

1. *the set of groups $\left\{1,2\right\}$;*
2. *the set of players $N=\left\{1,...,n_1+n_2\right\}$;*
3. *the set of actions $B_{ij}=\left\{0,1\right\}$: for each player ij, the choice of the effort $x_j\left(i\right)$;*
4. *the payoff functions for each player $ij\in N$*

$$\pi_{ij}\left(\mathbf{x}_1,\mathbf{x}_2\right)=p_jv-x_{ij}=$$

$$=\begin{cases} v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}>\max\left\{\mathbf{x}_{-j}\right\} \\ \frac{1}{2}v-x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}=\max\left\{\mathbf{x}_{-j}\right\} \\ -x_j\left(i\right) & if \quad \max\left\{\mathbf{x}_j\right\}<\max\left\{\mathbf{x}_{-j}\right\} \end{cases}.$$

The notation used in this paper is summed up in table 1.

Variable	Meaning
ij or $j\left(i\right)$	agent i of group j
$\left\{1,...,n_j\right\}$	set of agents in group j
$x_j\left(i\right)$ or x_{ji}	effort of agent i in group j
$X_j=\max\left\{x_j\left(i\right)\in\left\{0,1\right\},i\in\left\{1,...,n_j\right\}\right\}$	impact of effort of all agents in group j
$\mathbf{x}=\left(\mathbf{x}_1,\mathbf{x}_2\right)$	vector of efforts of all agents
$C_{ij}\left(x_j\left(i\right)\right)=x_j\left(i\right)$	cost of effort for agent i of group j
$p_j\left(X_1,X_2\right)$	probability of group j of winning the contest
$\pi_{ij}\left(\mathbf{x}_1,\mathbf{x}_2\right)$	payoff function of agent i of group j
$\gamma_j=\frac{1}{n_j}\sum_{i=1}^{n_j}x_{ij}\in\left[0,1\right]$	share of active agents in group j

Table 1

3 An Example[1]

Let us consider a $BMMAGC^*$ with two members for each group. W.l.g., let players $1, 2$ belong to group 1 and players $3, 4$ to group 2. Consider the following geometric representation of the game, where player 3 "moves horizontally", while player 4 "moves vertically":

3 a $\overline{a}$

1/2	a	$\overline{a}$	1/2	a	$\overline{a}$	
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1$	a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}-1$	a
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1$	$0; 0; v-1; v-1$	$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1$	$0; 0; v; v-1$	

1/2	a	$\overline{a}$	1/2	a	$\overline{a}$	
a	$\frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$\frac{v}{2}-1; \frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}$	a	$v-1; v-1; 0; 0$	$v-1; v; 0; 0$	$\overline{a}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2}-1; \frac{v}{2}-1; \frac{v}{2}$	$0; 0; v-1; v$	$\overline{a}$	$v; v-1; 0; 0$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$	

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3.1 The set of Nash Equilibria in the Two-player Two-group Example

It is straightforward to derive the following properties:

- if $v > 2$, there are four strict Nash equilibria in pure strategies

  $$NE = \{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a})\}$$

  and a Nash equilibrium in symmetric strictly-mixed strategies $\sigma_i^*(a) = 1 - \frac{2}{v} \quad \forall i \in \{1, 2, 3, 4\}$;
- if $v = 2$, there are four Nash equilibria in pure strategies

  $$\begin{aligned} NE = &\{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a}); (\overline{a}, \overline{a}, \overline{a}, a)\} \cup \\ &\cup \{(a, \overline{a}, \overline{a}, \overline{a}); (\overline{a}, a, \overline{a}, \overline{a}); (\overline{a}, \overline{a}, a, \overline{a}); (\overline{a}, \overline{a}, \overline{a}, \overline{a})\}; \end{aligned}$$
- if $v < 2$, the unique Nash equilibrium derived by strict-dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;
- if $v > 2$,
  - i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1;[2]
  - ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2;
  - iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3;
  - iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4;
- if $v = 2$,
  - i. $(\overline{a}, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 1;

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[1] This section is a direct application of the example carried out by Carlsson and van Damme [1993a] in their introduction.

[2] For the formulation of payoff-dominace and risk-dominance concepts see Harsanyi and Selten [1988].

- ii. $(a, \overline{a}, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 2;
- iii. $(\overline{a}, \overline{a}, \overline{a}, a)$ is the payoff-dominant equilibrium for player 3;
- iv. $(\overline{a}, \overline{a}, a, \overline{a})$ is the payoff-dominant equilibrium for player 4;

- if $v > 4$,
  - i. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ for player 1. Then,
    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$.

    Hence, for $v > 4$, $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 1;
  - ii. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 2. Then,
    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$.

    Hence, for $v > 4$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are associated with the largest deviation losses for player 2;
  - iii. $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 3. Then,
    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$.

    Hence, for $v > 4$, $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 3;
  - iv. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 4. Then,
    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$.

    Hence, for $v > 4$, $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are associated with the largest deviation losses for player 4;

- if $2 < v < 4$,

i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 3;

iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4.

Clearly this follows from what shown at the previous point for both groups;

- if $v = 2$,

  i. $(\overline{a}, a, \overline{a}, a), (\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, a), (a, \overline{a}, \overline{a}, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 1. Then,

    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
    - $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow v - 1 - \frac{v}{2} = 0$;
    - $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow v - v + 1 = 1$;
    - $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
    - $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

  Hence, for $v = 2$, $(\overline{a}, a, \overline{a}, a), (\overline{a}, a, a, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a})$ are associated with the largest deviation losses for player 1;

  ii. $(a, \overline{a}, \overline{a}, a), (a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, \overline{a}, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, a), (a, \overline{a}, \overline{a}, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 2. Then,

    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
    - $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
    - $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow v - v + 1 = 1$;
    - $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow v - 1 - \frac{v}{2} = 0$;
    - $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
    - $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

  Hence, for $v = 2$, $(a, \overline{a}, \overline{a}, a), (a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, \overline{a}, \overline{a})$ are associated with the largest deviation losses for player 2;

  iii. $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, \overline{a}, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, a), (a, \overline{a}, \overline{a}, \overline{a}), (\overline{a}, a, \overline{a}, \overline{a}), (\overline{a}, \overline{a}, a, \overline{a}), (\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 3. Then,

    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;

- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow v - v + 1 = 1$;
- $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow v - 1 - \frac{v}{2} = 0$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

Hence, for $v = 2$, $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, \overline{a}, \overline{a}, a)$ are associated with the largest deviation losses for player 3;

iv. $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, a)$, $(a, \overline{a}, \overline{a}, \overline{a})$, $(\overline{a}, a, \overline{a}, \overline{a})$, $(\overline{a}, \overline{a}, a, \overline{a})$, $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$ for player 4. Then,

- $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - 1 - 0 = \frac{v}{2} - 1 = 0$;
- $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + 1 = 1$;
- $(\overline{a}, \overline{a}, \overline{a}, a) \rightarrow v - 1 - \frac{v}{2} = 0$;
- $(a, \overline{a}, \overline{a}, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a}, a, \overline{a}, \overline{a}) \rightarrow 0 - \frac{v}{2} + 1 = 0$;
- $(\overline{a}, \overline{a}, a, \overline{a}) \rightarrow v - v + 1 = 1$;
- $(\overline{a}, \overline{a}, \overline{a}, \overline{a}) \rightarrow \frac{v}{2} - v + 1 = 0$.

Hence, for $v = 2$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 4;

- overall, there is a one-sided dominance region: for $v < 2$, a is a strictly dominated action.

Finally, note that, for $v > 4$,

i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1, whereas $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2, whereas $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3, whereas $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3;

iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4, whereas $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4.

Hence, there is a tension between payoff-dominance and risk-dominance.

Now let us consider a slight variation of the game above and let:

- the individual **costs of effort** $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se, while the prize v is always worth positive utils, so that it is a good. [3]

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[3] Clearly, under complete information, for $c \in \mathbb{R}_{++}$ and $v \in \mathbb{R}_{++}$, the cost of effort $C_{ij}(x_j(i))$ can always be normalized to one via a simple change of variables.

Then, we have the following representation of the game, where player 3 “moves horizontally” and player 4 “moves vertically”:

3	a			$\overline{a}$		
1/2	a	$\overline{a}$	1/2	a	$\overline{a}$	
a	$\frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c; \frac{v}{2} - c$	a	$\frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2}; \frac{v}{2} - c$	a
$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c$	$0; 0; v - c; v - c$	$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c$	$0; 0; v; v - c$	
1/2	a	$\overline{a}$	1/2	a	$\overline{a}$	
a	$\frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2}$	$\frac{v}{2} - c; \frac{v}{2}; \frac{v}{2} - c; \frac{v}{2}$	a	$v - c; v - c; 0; 0$	$v - c; v; 0; 0$	$\overline{a}$
$\overline{a}$	$\frac{v}{2}; \frac{v}{2} - c; \frac{v}{2} - c; \frac{v}{2}$	$0; 0; v - c; v$	$\overline{a}$	$v; v - c; 0; 0$	$\frac{v}{2}; \frac{v}{2}; \frac{v}{2}; \frac{v}{2}$	
						4

It is straightforward to derive the following properties:

- if $c < 0$, the unique Nash equilibrium derived by strict dominance is (a, a, a, a);
- if $c > \frac{v}{2}$, the unique Nash equilibrium derived by strict dominance is $(\overline{a}, \overline{a}, \overline{a}, \overline{a})$;
- if $0 < c < \frac{v}{2}$, there are four strict Nash equilibria in pure strategies

  $$NE = \{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a})\}$$

  and an equilibrium in symmetric strictly mixed strategies $\sigma_i^* (a) = 1 - \frac{2c}{v} \quad \forall i \in \{1, 2, 3, 4\}$;
- if $c = 0$, there are nine Nash equilibria in pure strategies

  $$\begin{aligned} NE = & \{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a}); (a, a, a, a)\} \cup \\ & \cup \{(a, \overline{a}, a, a); (\overline{a}, a, a, a); (a, a, a, \overline{a}); (a, a, \overline{a}, a)\}; \end{aligned}$$

- if $c = \frac{v}{2}$, there are nine Nash equilibria in pure strategies

  $$\begin{aligned} NE = & \{(a, \overline{a}, \overline{a}, a); (\overline{a}, a, \overline{a}, a); (a, \overline{a}, a, \overline{a}); (\overline{a}, a, a, \overline{a}); (\overline{a}, \overline{a}, \overline{a}, a)\} \cup \\ & \cup \{(a, \overline{a}, \overline{a}, \overline{a}); (\overline{a}, a, \overline{a}, \overline{a}); (\overline{a}, \overline{a}, a, \overline{a}); (\overline{a}, \overline{a}, \overline{a}, \overline{a})\} \end{aligned}$$

- if $\frac{v}{4} < c < \frac{v}{2}$,
  i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1;[4]
  ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2;
  iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3;
  iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4;
- if $c = 0$, there is no payoff-dominant equilibrium for any player $i \in \{1, 2, 3, 4\}$;
- if $c = \frac{v}{2}$,

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[4] For the formulation of payoff-dominace and risk-dominance concepts see Harsanyi and Selten [1988].

    i. $(\overline{a}, a, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 1;
    ii. $(a, \overline{a}, \overline{a}, \overline{a})$ is the payoff-dominant equilibrium for player 2;
    iii. $(\overline{a}, \overline{a}, \overline{a}, a)$ is the payoff-dominant equilibrium for player 3;
    iv. $(\overline{a}, \overline{a}, a, \overline{a})$ is the payoff-dominant equilibrium for player 4;

- if $0 < c < \frac{v}{4}$,

    i. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ for player 1. Then,

        - $(a, \overline{a}, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(\overline{a}, a, \overline{a}, a) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(a, \overline{a}, a, \overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(\overline{a}, a, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = c$.

    Hence, for $0 < c < \frac{v}{4}$, $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 1;

    ii. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 2. Then,

        - $(a, \overline{a}, \overline{a}, a) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(\overline{a}, a, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(a, \overline{a}, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(\overline{a}, a, a, \overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c.$

    Hence, for $0 < c < \frac{v}{4}$, $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are associated with the largest deviation losses for player 2;

    iii. $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 3. Then,

        - $(a, \overline{a}, \overline{a}, a) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(\overline{a}, a, \overline{a}, a) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(a, \overline{a}, a, \overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(\overline{a}, a, a, \overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$.

    Hence, for $0 < c < \frac{v}{4}$, $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 3;

    iv. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a)$, $(\overline{a}, a, \overline{a}, a)$, $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ for player 4. Then,

        - $(a, \overline{a}, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(\overline{a}, a, \overline{a}, a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - c$;
        - $(a, \overline{a}, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = c$;
        - $(\overline{a}, a, a, \overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = c$.

    Hence, for $0 < c < \frac{v}{4}$, $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are associated with the largest deviation losses for player 4;

- if $\frac{v}{4} < c < \frac{v}{2}$,

  i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

  ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

  iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 3;

  iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4.

  Clearly this follows from what shown at the previous point for both groups;

- if $c = 0$,

  i. $(a, \overline{a}, \overline{a}, a), (a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, a)$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (a, a, a, a), (a, \overline{a}, a, a), (\overline{a}, a, a, a), (a, a, a, \overline{a}), (a, a, \overline{a}, a)$ for player 1. Then,

    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(a, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
    - $(a, \overline{a}, a, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(\overline{a}, a, a, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(a, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
    - $(a, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$.

    Hence, for $c = 0$, $(a, \overline{a}, \overline{a}, a), (a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, a)$ are associated with the largest deviation losses for player 1;

  ii. $(\overline{a}, a, \overline{a}, a), (\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, a, a)$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (a, a, a, a), (a, \overline{a}, a, a), (\overline{a}, a, a, a), (a, a, a, \overline{a}), (a, a, \overline{a}, a)$ for player 2. Then,

    - $(a, \overline{a}, \overline{a}, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(\overline{a}, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(a, \overline{a}, a, \overline{a}) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(\overline{a}, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(a, a, a, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
    - $(a, \overline{a}, a, a) \rightarrow \frac{v}{2} - \frac{v}{2} + c = 0$;
    - $(\overline{a}, a, a, a) \rightarrow \frac{v}{2} - c - 0 = \frac{v}{2}$;
    - $(a, a, a, \overline{a}) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$;
    - $(a, a, \overline{a}, a) \rightarrow \frac{v}{2} - c - \frac{v}{2} = 0$.

    Hence, for $c = 0$, $(\overline{a}, a, \overline{a}, a), (\overline{a}, a, a, \overline{a})$ and $(\overline{a}, a, a, a)$ are associated with the largest deviation losses for player 2;

  iii. $(a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a})$ and $(a, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a, \overline{a}, \overline{a}, a), (\overline{a}, a, \overline{a}, a), (a, \overline{a}, a, \overline{a}), (\overline{a}, a, a, \overline{a}), (a, a, a, a), (a, \overline{a}, a, a), (\overline{a}, a, a, a), (a, a, a, \overline{a}), (a, a, \overline{a}, a)$ for player 3. Then,

- $(a,\overline{a},\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a},a,\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a,\overline{a},a,\overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(\overline{a},a,a,\overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a,a,a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a,\overline{a},a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(\overline{a},a,a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a,a,a,\overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a,a,\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + c = 0$.

Hence, for $c = 0$, $(a,\overline{a},a,\overline{a}), (\overline{a},a,a,\overline{a})$ and $(a,a,a,\overline{a})$ are associated with the largest deviation losses for player 3;

iv. $(a,\overline{a},\overline{a},a), (\overline{a},a,\overline{a},a)$ and $(a,a,\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a), (\overline{a},a,\overline{a},a), (a,\overline{a},a,\overline{a}), (\overline{a},a,a,\overline{a}), (a,a,a,a), (a,\overline{a},a,a), (\overline{a},a,a,a), (a,a,a,\overline{a}), (a,a,\overline{a},a)$ for player 4. Then,

- $(a,\overline{a},\overline{a},a) \to \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(\overline{a},a,\overline{a},a) \to \frac{v}{2} - c - 0 = \frac{v}{2}$;
- $(a,\overline{a},a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(\overline{a},a,a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a,a,a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a,\overline{a},a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(\overline{a},a,a,a) \to \frac{v}{2} - c - \frac{v}{2} = 0$;
- $(a,a,a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = 0$;
- $(a,a,\overline{a},a) \to \frac{v}{2} - c - 0 = \frac{v}{2}$.

Hence, for $c = 0$, $(a,\overline{a},\overline{a},a), (\overline{a},a,\overline{a},a)$ and $(a,a,\overline{a},a)$ are associated with the largest deviation losses for player 4;

- if $c = \frac{v}{2}$,

i. $(\overline{a},a,\overline{a},a), (\overline{a},a,a,\overline{a})$ and $(\overline{a},a,\overline{a},\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a), (\overline{a},a,\overline{a},a), (a,\overline{a},a,\overline{a}), (\overline{a},a,a,\overline{a}), (\overline{a},\overline{a},\overline{a},a), (a,\overline{a},\overline{a},\overline{a}), (\overline{a},a,\overline{a},\overline{a}), (\overline{a},\overline{a},a,\overline{a}), (\overline{a},\overline{a},\overline{a},\overline{a})$ for player 1. Then,

- $(a,\overline{a},\overline{a},a) \to \frac{v}{2} - c - 0 = \frac{v}{2} - \frac{v}{2} = 0$;
- $(\overline{a},a,\overline{a},a) \to \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(a,\overline{a},a,\overline{a}) \to \frac{v}{2} - c - 0 = \frac{v}{2} - \frac{v}{2} = 0$;
- $(\overline{a},a,a,\overline{a}) \to \frac{v}{2} - \frac{v}{2} + c = \frac{v}{2}$;
- $(\overline{a},\overline{a},\overline{a},a) \to 0 - \frac{v}{2} + c = 0$;
- $(a,\overline{a},\overline{a},\overline{a}) \to v - c - \frac{v}{2} = 0$;
- $(\overline{a},a,\overline{a},\overline{a}) \to v - v + c = \frac{v}{2}$;
- $(\overline{a},\overline{a},a,\overline{a}) \to 0 - \frac{v}{2} + c = 0$;
- $(\overline{a},\overline{a},\overline{a},\overline{a}) \to \frac{v}{2} - v + c = 0$.

Hence, for $c = \frac{v}{2}$, $(\overline{a},a,\overline{a},a), (\overline{a},a,a,\overline{a}), (\overline{a},a,\overline{a},\overline{a})$ are associated with the largest deviation losses for player 1;

ii. $(a,\overline{a},\overline{a},a)\,,(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},\overline{a},\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)\,,(\overline{a},a,\overline{a},a)\,,$ $(a,\overline{a},a,\overline{a})\,,(\overline{a},a,a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},a)\,,$ $(a,\overline{a},\overline{a},\overline{a})\,,(\overline{a},a,\overline{a},\overline{a})\,,(\overline{a},\overline{a},a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 2. Then,

  - $(a,\overline{a},\overline{a},a)\rightarrow\frac{v}{2}-\frac{v}{2}+c=\frac{v}{2}$;
  - $(\overline{a},a,\overline{a},a)\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(a,\overline{a},a,\overline{a})\rightarrow\frac{v}{2}-\frac{v}{2}+c=\frac{v}{2}$;
  - $(\overline{a},a,a,\overline{a})\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(\overline{a},\overline{a},\overline{a},a)\rightarrow 0-\frac{v}{2}+c=0$;
  - $(a,\overline{a},\overline{a},\overline{a})\rightarrow v-v+c=\frac{v}{2}$;
  - $(\overline{a},a,\overline{a},\overline{a})\rightarrow v-c-\frac{v}{2}=0$;
  - $(\overline{a},\overline{a},a,\overline{a})\rightarrow 0-\frac{v}{2}+c=0$;
  - $(\overline{a},\overline{a},\overline{a},\overline{a})\rightarrow\frac{v}{2}-v+c=0$.

  Hence, for $c=\frac{v}{2}$, $(a,\overline{a},\overline{a},a)\,,(a,\overline{a},a,\overline{a})$ and $(a,\overline{a},\overline{a},\overline{a})$ are associated with the largest deviation losses for player 2;

iii. $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$ and $(\overline{a},\overline{a},\overline{a},a)$ are the risk-dominant equilibrium strategy profiles for player 3. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)\,,(\overline{a},a,\overline{a},a)\,,$ $(a,\overline{a},a,\overline{a})\,,(\overline{a},a,a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},a)\,,(a,\overline{a},\overline{a},\overline{a})\,,(\overline{a},a,\overline{a},\overline{a})\,,(\overline{a},\overline{a},a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 3. Then,

  - $(a,\overline{a},\overline{a},a)\rightarrow\frac{v}{2}-\frac{v}{2}+c=c$;
  - $(\overline{a},a,\overline{a},a)\rightarrow\frac{v}{2}-\frac{v}{2}+c=c$;
  - $(a,\overline{a},a,\overline{a})\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(\overline{a},a,a,\overline{a})\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(\overline{a},\overline{a},\overline{a},a)\rightarrow v-v+c=c$;
  - $(a,\overline{a},\overline{a},\overline{a})\rightarrow 0-\frac{v}{2}+c=0$;
  - $(\overline{a},a,\overline{a},\overline{a})\rightarrow 0-\frac{v}{2}+c=0$;
  - $(\overline{a},\overline{a},a,\overline{a})\rightarrow v-c-\frac{v}{2}=0$;
  - $(\overline{a},\overline{a},\overline{a},\overline{a})\rightarrow\frac{v}{2}-v+c=0$.

  Hence, for $c=\frac{v}{2}$, $(a,\overline{a},\overline{a},a)$, $(\overline{a},a,\overline{a},a)$ and $(\overline{a},\overline{a},\overline{a},a)$ are associated with the largest deviation losses for player 3;

iv. $(a,\overline{a},a,\overline{a})$, $(\overline{a},a,a,\overline{a})$ and $(\overline{a},\overline{a},a,\overline{a})$ are the risk-dominant equilibrium strategy profiles for player 4. As a matter of fact, let us compute the deviation losses of $(a,\overline{a},\overline{a},a)\,,(\overline{a},a,\overline{a},a)\,,$ $(a,\overline{a},a,\overline{a})\,,(\overline{a},a,a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},a)\,,(a,\overline{a},\overline{a},\overline{a})\,,(\overline{a},a,\overline{a},\overline{a})\,,(\overline{a},\overline{a},a,\overline{a})\,,(\overline{a},\overline{a},\overline{a},\overline{a})$ for player 4. Then,

  - $(a,\overline{a},\overline{a},a)\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(\overline{a},a,\overline{a},a)\rightarrow\frac{v}{2}-c-0=\frac{v}{2}-c=0$;
  - $(a,\overline{a},a,\overline{a})\rightarrow\frac{v}{2}-\frac{v}{2}+c=\frac{v}{2}$;
  - $(\overline{a},a,a,\overline{a})\rightarrow\frac{v}{2}-\frac{v}{2}+c=\frac{v}{2}$;
  - $(\overline{a},\overline{a},\overline{a},a)\rightarrow v-c-\frac{v}{2}=0$;
  - $(a,\overline{a},\overline{a},\overline{a})\rightarrow 0-\frac{v}{2}+c=0$;
  - $(\overline{a},a,\overline{a},\overline{a})\rightarrow 0-\frac{v}{2}+c=0$;
  - $(\overline{a},\overline{a},a,\overline{a})\rightarrow v-v+c=\frac{v}{2}$;
  - $(\overline{a},\overline{a},\overline{a},\overline{a})\rightarrow\frac{v}{2}-v+c=0$.

Hence, for $c = \frac{v}{2}$, $(a, \overline{a}, a, \overline{a})$, $(\overline{a}, a, a, \overline{a})$ and $(\overline{a}, \overline{a}, a, \overline{a})$ are associated with the largest deviation losses for player 4;

- overall, there are two dominance regions: for $c > \frac{v}{2}$, a is a strictly dominated action; for $c < 0$, $\overline{a}$ ia a strictly dominated action.

Finally, note that, for $0 < c < \frac{v}{4}$,

i. $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 1, whereas $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 1;

ii. $(a, \overline{a}, \overline{a}, a)$ and $(a, \overline{a}, a, \overline{a})$ are the payoff-dominant equilibria for player 2, whereas $(\overline{a}, a, \overline{a}, a)$ and $(\overline{a}, a, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 2;

iii. $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the payoff-dominant equilibria for player 3, whereas $(a, \overline{a}, a, \overline{a})$ and $(a, \overline{a}, a, \overline{a})$ are the risk-dominant equilibrium strategy profiles for player 3;

iv. $(a, \overline{a}, a, \overline{a})$ and $(\overline{a}, a, a, \overline{a})$ are the payoff-dominant equilibria for player 4, whereas $(a, \overline{a}, \overline{a}, a)$ and $(\overline{a}, a, \overline{a}, a)$ are the risk-dominant equilibrium strategy profiles for player 4.

Hence, there is a tension between payoff-dominance and risk-dominance.

3.1.1 Introducing Incomplete Information à la Global Games

We would like to draw a possible comparison with the classical example due to Carlsson and van Damme [1993a] about a 2×2 game under complete information, reported in table 1.

	α_2	β_2
α_1	x, x	$x, 0$
β_1	$0, x$	$4, 4$

Table 1: Game $g(x)$ by Carlsson and van Damme [1993a] .

Carlsson and van Damme [1993a] highlight the following properties of this game under complete information:

- if $x > 4$, the unique Nash equilibrium derived by strict dominance is (α_1, α_2);
- if $x < 0$, the unique Nash equilibrium derived by strict dominance is (β_1, β_2);
- if $0 < x < 4$, there are two strict Nash equilibria, that is (α_1, α_2) and (β_1, β_2);
- if $x \in (2, 4)$, (α_1, α_2) is the risk-dominant equilibrium;
- if $x \in (0, 2)$, (β_1, β_2) is the risk-dominant equilibrium;
- overall, there are two dominance regions.

Finally, note that, for $2 < x < 4$, (β_1, β_2) is the payoff-dominant equilibrium, whereas (α_1, α_2) is the risk-dominant equilibrium: there is a tension between payoff-dominance and risk-dominance.

3.1.2 Incomplete Information about the Prize

Let us consider the case where the individual **costs of effort** is $C_{ij}\left(x_j\left(i\right)\right) = x_j\left(i\right)$. Henceforth, we refer to this game as $g\left(v\right)$. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1, 5]$;
- given the realization v, each player $i \in \{1, 2, 3, 4\}$ idiosyncratically observes the realization of a random variable V_i, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $V_1 - v$, $V_2 - v$, $V_3 - v$ and $V_4 - v$ are independent;
- after these idiosyncratic observations, each player $i \in \{1, 2, 3, 4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game of $g\left(v\right)$;
- note that $E\left(V|v_i\right) = v_i$, if i observes $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_i \sim U\left(v_i - \varepsilon, v_i + \varepsilon\right)$;
- furthermore, for $v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around v_i with support $\left[v_i - 2\varepsilon, v_i + 2\varepsilon\right]$. Hence, $Prob\left[V_{-i} < v_i|v_i\right] = Prob[V_{-i} > v_i|v_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon < \left|\frac{v}{2} - 1\right|$ and suppose player $i \in \{1, 2, 3, 4\}$ observes $v_i < 2$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1, 2, 3, 4\}$ whenever she observes $v_i < 2$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2, 3, 4\}$ are forced to play $\overline{a}$ whenever they observe $v_{-i} < 2$, then player i, observing $v_i = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff from not exerting effort, that is choosing $\overline{a}_i$, will be at least $\frac{3}{4}$, so that a_i can be discarded by iterated dominance for $v_i = 2$, since the conditionally expected payoff from exerting effort equals $\frac{1}{4}$. Let v_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $v_i^* = 4$. Note that $v_i = 4$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $v_i = 4$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(\frac{4}{2} - 1\right) + \frac{1}{8}\left(4 - 1\right) + \frac{1}{8}\left(4 - 1\right) = \frac{3}{2},$$

while the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{4}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(4\right) + \frac{1}{8}\left(\frac{4}{2}\right) = \frac{3}{2}.$$

The cutoff $v_i^* = 4$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for $v_i \in [v - \varepsilon, v + \varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(\frac{v_i}{2} - 1\right) + \frac{1}{8}\left(v_i - 1\right) + \frac{1}{8}\left(v_i - 1\right) =$$

$$= \frac{1}{8}\left(\frac{v_i}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{v_i}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(\frac{v_i}{2}\right) + \frac{1}{8} * 0 + \frac{1}{8}\left(v_i\right) + \frac{1}{8}\left(\frac{v_i}{2}\right).$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower dominance region and an upper dominance region. However, in contrast with what we showed in Bosco et al. [2024] for the weakest-link impact function, it is not possible to establish the existence of an equilibrium robust to incomplete information in the sense of Kajii and Morris [1997] in which no player exerts effort, since $x_i(v_i) = 1$ is not a dominated action $\forall v_i \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$ and $\forall i \in \{1,2,3,4\}$.

Hence, we conclude that in $g(v)$ under incomplete information à la global games, for sufficiently small ε, there is an equilibrium in (monotonic) cutoff strategies, such that $\forall i \in \{1,2,3,4\}$:

$$x_i^*(v_i) = \begin{cases} 1 & \text{if } v_i > 4 \\ 0 & \text{if } v_i \leq 4 \ . \end{cases}$$

3.1.3 Incomplete Information about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$. Henceforth, we refer to this game as $g(c)$. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the realization c, each player $i \in \{1,2,3,4\}$ idiosyncratically observes the realization of a random variable C_i, uniform on $[c-\varepsilon, c+\varepsilon]$ for some $\varepsilon > 0$, so that the players' observation errors $C_1 - c$, $C_2 - c$, $C_3 - c$ and $C_4 - c$ are independent;
- after these idiosyncratic observations, each player $i \in \{1,2,3,4\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described by the strategic form game $g(c)$;
- note that $E(C|c_i) = c_i$, if i observes $c_i \in [\underline{c}+\varepsilon, \overline{c}-\varepsilon]$ so that $C|c_i \sim U(c_i - \varepsilon, c_i + \varepsilon)$;
- furthermore, for $c_i \in [\underline{c}+\varepsilon, \overline{c}-\varepsilon]$, the conditional distribution of the teammate's or opponents' observation will be centered around c_i with support $[c_i - 2\varepsilon, c_i + 2\varepsilon]$. Hence, $Prob[C_{-i} < c_i | c_i] = Prob[C_{-i} > c_i | c_i] = \frac{1}{2}$.

Now, let us further assume $\varepsilon < \left|\frac{2\overline{c}-v}{4}\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $c_i > \frac{v}{2}$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is smaller than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1,2,3,4\}$ whenever she observes $c_i > \frac{v}{2}$. Suppose $i = 1$ without loss of generality. Iterating this dominance argument, if players $-i \in \{2,3,4\}$ are forced to play $\overline{a}$ whenever they observe $c_{-i} > \frac{v}{2}$, then player i, observing $c_i = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to $(\overline{a}_2, \overline{a}_3, \overline{a}_4)$. Thus, i's conditionally expected payoff from not exerting effort, that is choosing $\overline{a}_i$ will be at least $\frac{3}{8}v$, so that a_i can be discarded by iterated dominance for $c_i = \frac{v}{2}$, since the conditionally expected payoff from exerting effort equals $\frac{v}{8}$. Let c_i^* be the smallest observation such that a_i cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^* = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$

$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v\ .$$

The cutoff $c_i^* = \frac{v}{4}$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c-\epsilon, c+\epsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(v-c_i\right)=$$

$$=\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v\right)+\frac{1}{8}\left(\frac{v}{2}\right).$$

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region. Again, let us assume $\varepsilon < \left|-\frac{c}{2}\right|$ and suppose player $i \in \{1,2,3,4\}$ observes $c_i < 0$. Then, i's conditionally expected payoff from exerting effort, that is choosing a, is positive and greater than the one from exerting no effort, that is choosing $\overline{a}$. Accordingly, $\overline{a}$ is a conditionally strictly dominant action for player $i \in \{1,2,3,4\}$ whenever she observes $c_i < 0$. Iterating this dominance argument, suppose $i = 1$ without loss of generality. Then, if players $-i \in \{2,3,4\}$ are forced to play a whenever they observe $c_{-i} < 0$, player i, observing $c_i = 0$ has to assign at least probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to (a_2, a_3, a_4). Thus, i's conditionally expected payoff from exerting effort, that is choosing a_i will be at least $\frac{5}{8}v$, so that $\overline{a}_i$ can be discarded by iterated dominance for $c_i = 0$, since the conditionally expected payoff from not exerting effort equals $\frac{3}{8}v$. Let c_i^{**} be the smallest observation such that $\overline{a}_i$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_i^{**} = \frac{v}{4}$. Note that $c_i = \frac{v}{4}$ is the threshold for the risk-dominance regions as well. As a matter of fact, when $c_i = \frac{v}{4}$, the conditionally expected payoff from exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+\frac{1}{8}\left(\frac{v}{2}-\frac{v}{4}\right)+$$

$$+\frac{1}{8}\left(v-\frac{v}{4}\right)+\frac{1}{8}\left(v-\frac{v}{4}\right)=\frac{3}{8}v$$

and the conditionally expected payoff from not exerting effort equals

$$\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v\right)+\frac{1}{8}\left(\frac{v}{2}\right)=\frac{3}{8}v\ .$$

The cutoff $c_i^{**} = \frac{v}{4}$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for $c_i \in [c-\varepsilon, c+\varepsilon]$ solving

$$\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(\frac{v}{2}-c_i\right)+\frac{1}{8}\left(v-c_i\right)+\frac{1}{8}\left(v-c_i\right)=$$

$$=\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(\frac{v}{2}\right)+\frac{1}{8}*0+\frac{1}{8}\left(v\right)+\frac{1}{8}\left(\frac{v}{2}\right).$$

Hence, $c_i^* = c_i^{**}$ and we conclude that in $g(c)$ under incomplete information à la global games, for sufficiently small ε, there exists a unique equilibrium in switching strategies such that $\forall i \in \{1,2,3,4\}$

$$x_i^*(c_i) = \begin{cases} 1 & \text{if } c_i < \frac{v}{4} \\ 0 & \text{if } c_i \geq \frac{v}{4} \end{cases}.$$

3.2 Observations

Overall, we can state some general points from the example above:

- under complete information, there are multiple Nash equilibria in pure strategies in a max-max two-group four-player contest with binary actions and a public good prize, independently from whether the cost of effort equates effort itself or a parameter belonging to the set of real numbers.
- In both examples we highlight a tension between payoff-dominance and risk-dominance, as in the example due to Carlsson and van Damme [1993a].
- Assuming incomplete information à la global games induces the existence of an equilibrium in (monotonic) switching strategies, whose cutoff coincides with the one of the risk-dominance region.
- Equilibrium selection happens even for "a pinch of uncertainty", no matter how small ε is.
- Whether the selection induced delivers uniqueness or not crucially depends on the properties of the payoffs structure under complete information: in particular, the presence of both an upward and a downward dominance region is conducive to a unique equilibrium in (monotonic) switching strategies by deletion of iterated interim-strictly dominated strategies when departing from the complete information assumption in the sense of Carlsson and van Damme [1993a].
- Focusing on mixed-strategies would not affect the equilibria in (monotonic) switching strategies, since they are derived by iterated deletion of conditionally strictly dominated strategies and pure dominated strategies cannot be part of the support of any mixed strategy equilibrium.
- Differently from what shown in Bosco et al. [2024] for the weakest-link impact function, the presence of a one-sided dominance region is not conducive to the coexistence of an equilibrium in (monotonic) switching-strategies and an equilibrium robust to incomplete information à la Kajii and Morris [1997] in which no player exerts effort, since, at $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$, $x_i(v_i) = 1$ is not a strictly dominated action $\forall v_i \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ and $i \in \{1, 2, 3, 4\}$. Moreover, note that in the complete information game with $v \in \mathbb{R}$ and $C_{ij}(x_j(i)) = x_j(i)$ there does not exist any equilibrium strategy profile which is played for every possible value for v, that is $v \in \mathbb{R}$, in contrast with what happens under the weakest-link technology for $(\overline{a}_1, \overline{a}_2, \overline{a}_3, \overline{a}_4)$.
- Finally, note that whether the risk-dominant equilibrium in $g(v_i)$ and $g(c_i)$ coincides with the risk-dominant equilibrium in the actual game selected by Nature, i.e. $g(v)$ and $g(c)$ respectively, or not, depends on whether ε is sufficiently small, that is for $\varepsilon < |v - 2|$ and $\varepsilon < \left|c - \frac{v}{4}\right|$, respectively.

Once highlighted the main properties of our example, the general model and the mechanisms guiding to the related results should be more transparent.

4 The Set of Nash Equilibria of Binary Max-Max Group Contests

To simplify notation and presentation, the NE of the $BMMAGC^*$ will be presented in terms of share of active agents, i.e. $(\gamma_1, \gamma_2) \in [0, 1] \times [0, 1]$. The following results characterize the set of Nas equilibria of $BMMAGC^*$.

Proposition 1 *In the $BMMAGC^*$,*

- *if* $v > 2$*, there are* $n_1 \cdot n_2$ *strict Nash equilibria in pure strategies*

$$NE \equiv \left\{ (\gamma_1, \gamma_2) \ \textit{such that}\ \gamma_j \in (0,1) \ \textit{and}\ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\} \right\}$$

and a Nash equilibrium in within-group symmetric strictly-mixed strategies

$$\sigma^*_{ij}\left(x_{ij} = 1\right) = \left(1 - \frac{2}{v}\right)^{1/(n_j - 1)} \quad \forall\ i \in \{1, \ldots, n_j\} \ \textit{and}\ j \in \{1,2\}\,;$$

- *if* $v = 2$*, there are* $n_1 \cdot n_2 + n_1 + n_2 + 1$ *Nash equilibria in pure strategies*

$$\begin{aligned} NE \equiv & \left\{ (\gamma_1, \gamma_2) \ \textit{such that}\ \gamma_j \in (0,1) \ \textit{and}\ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\} \right\} \cup \\ & \cup \left\{ (\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \ \textit{such that}\ \gamma_j \in (0,1) \ \textit{and}\ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\} \right\} \cup \\ & \cup \left\{ (\gamma_1, \gamma_2) = (0,0) \right\} \end{aligned}$$

- *if* $v < 2$*, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0,0) \ .$$

Overall, there is a one-sided dominance region: for $v < 2$*,* $x_{ij} = 1$ *is a strictly dominated action* $\forall$ $ij \in \{1, \ldots, N\}$*.*

Moreover, it is easy to prove the following result.

Proposition 2 *In the BMMAGC*,*

- *for* $v > 2$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$*;*
- *for* $v > 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$ *;*
- *for* $2 < v < 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$ *;*
- *for* $v = 2$*,* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any player* ij *such that* $x_{ij} = 0$ *;*
- *for* $v = 2$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *and* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 0$ *.*

Remark 1 *Note that, for* $v > 4$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the payoff-dominant equilibria for any* ij *such that* $x_{ij} = 0$*, whereas* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 1$ *: there is a tension between payoff-dominance and risk-dominance.*

Now consider a slight variation of the game above and let:

- the individual cost of effort $C_{ij}\left(x_j\left(i\right)\right) = c$ with $c \in \mathbb{R}$. Thus, costs of effort may be negative, which means that agents could enjoy effort per se;
- the club good prize $v > 0$, i.e. the prize v is always worth positive utils, so that it is a good. [5]

Henceforth, we term this variation as $BMMAGC^{*b}$. Then, it is straightforward to derive the following results.

Proposition 3 *In the $BMMAGC^{*b}$,*

- *if $c < 0$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (1, 1) \ \ ;$$

- *if $c > \frac{v}{2}$, there is a unique Nash equilibrium in pure strategies derived by strict-dominance*

$$(\gamma_1, \gamma_2) = (0, 0) \ \ ;$$

- *if $0 < c < \frac{v}{2}$, there are $n_1 \cdot n_2$ strict Nash equilibria in pure strategies*

$$NE \equiv \left\{ (\gamma_1, \gamma_2) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\} \right\}$$

and an equilibrium in within-group symmetric strictly mixed strategies

$$\sigma_i^* \left(x_{ij} = 1\right) = \left(1 - \frac{2c}{v}\right)^{1/(n_j - 1)} \quad \forall \ i \in \{1, \ldots, n_j\} \ \textit{and} \ j \in \{1, 2\} \,;$$

- *if $c = 0$, the set of Nash equilibria in pure strategies is*

$$\begin{aligned} NE \equiv & \{(\gamma_1, \gamma_2) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ \ \forall j \in \{1, 2\}\} \cup \\ & \cup \{(\gamma_j, \gamma_{-j}) = (\gamma_j, 1) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ \ \forall j \in \{1, 2\}\} \cup \\ & \cup \{(\gamma_1, \gamma_2) = (1, 1)\} \end{aligned}$$

- *if $c = \frac{v}{2}$, there are $n_1 \cdot n_2 + n_1 + n_2 + 1$ Nash equilibria in pure strategies*

$$\begin{aligned} NE \equiv & \left\{ (\gamma_1, \gamma_2) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\} \right\} \cup \\ & \cup \left\{ (\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \ \textit{such that} \ \gamma_j \in (0, 1) \ \textit{and} \ \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1, 2\} \right\} \cup \\ & \cup \{(\gamma_1, \gamma_2) = (0, 0)\} \end{aligned}$$

Overall, there are two dominance regions: for $c < 0$, $x_{ij} = 0$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$: for $c > \frac{v}{2}$, $x_{ij} = 1$ is a strictly dominated action for any $ij \in \{1, \ldots, N\}$.

As before, it is easy prove the following result.

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[5] As stressed previously in our example, under complete information, for $c \in \mathbb{R}_{++}$ and $v \in \mathbb{R}_{++}$, the cost of effort $C_{ij}\left(x_j\left(i\right)\right)$ can always be normalized to one via a simple change of variables.

Proposition 4 *In the $BMMAGC^{*b}$,*

- *for* $0 < c < \frac{v}{2}$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$*;*
- *for* $0 < c < \frac{v}{4}$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$ *;*
- *for* $\frac{v}{4} < c < \frac{v}{2}$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 0$;
- *for* $c = \frac{v}{2}$*,* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the payoff-dominant equilibrium strategy profiles for any player* ij *such that* $x_{ij} = 0$ *;*
- *for* $c = \frac{v}{2}$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *and* $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibria for any* ij *such that* $x_{ij} = 0$ *.*
- *for* $c = 0$*, there is no payoff-dominant equilibrium strategy profile* $\forall ij \in \{1, \ldots, N\}$*;*
- *for* $c = 0$*,* (γ_1, γ_2) *such that* $\gamma_j \in (0,1)$ *and* $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ $\forall j \in \{1,2\}$ *are the risk-dominant equilibrium strategy profiles for any* ij *such that* $x_{ij} = 1$ *.*

Remark 2 *Note that, for* $\frac{v}{4} < c < \frac{v}{2}$ *and any group* $j \in \{1,2\}$*,* $(\gamma_j, \gamma_{-j}) = (1,0)$ *is the payoff-dominant equilibrium strategy profile for group* $j \in \{1,2\}$*, whereas* $(\gamma_j, \gamma_{-j}) = (0,1)$ *and* $(\gamma_j, \gamma_{-j}) = (0,0)$ *are the risk-dominant equilibrium strategy profiles for group* $j \in \{1,2\}$*: there is a tension between payoff-dominance and risk-dominance.*

5 The Set of Bayes-Nash Equilibria with Incomplete Information à la Global Games.

5.1 Incomplete Information à la Global Games about the Prize

Let us consider the case where the individual **costs of effort** $C_{ij}(x_j(i)) = x_j(i)$, that is the $BMMAGC^*$ model. We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1, 5]$;
- given the realization v, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ idiosyncratically observes the realization of a random variable V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$, so that the players' observation errors $V_{ij} - v$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1,2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, we refer to this game as $g_1(v)$. Then, we are able to obtain the following result.

Proposition 5 *In the* $g_1(v)$, *there is an equilibrium in (monotonic) switching strategies, such that* $\forall ij \in \{1, \ldots, n_j\}$ *and* $\forall j \in \{1, 2\}$:

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > 2^{n_j} \\ 0 & \text{if } v_{ij} \leq 2^{n_j} \end{cases} .$$

Remark 3 *Note that proposition 5 delivers an existence result, not a uniqueness result, that is we are able to find an equilibrium in (monotonic) switching strategies in* $g_3(v)$, *which may not be unique.*

Remark 4 *in contrast with what we showed in Bosco et al. [2024] for the weakest-link impact function, it is not possible to establish the existence of an equilibrium robust to incomplete information in the sense of Kajii and Morris [1997] in which no player exerts effort, since* $x_{ij}(v_{ij}) = 1$ *is not a dominated action* $\forall v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ *and* $\forall ij \in \{1, \ldots, N\}$.

Remark 5 *The existence of an equilibrium in (monotonic) switching strategies in the* $g_1(v)$ *is ensured as long as* $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$. *However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small* ε *is.*

Remark 6 *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e.* $v_{ij} = 2^{n_j}$, *does not coincide with the one of the risk-dominance region, that is* $v_{ij} = 4$ *for any* $j \in \{1, 2\}$, *differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and van Damme [1993b] for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when we depart from the 2×2 case.*

5.2 Incomplete Information à la global games about the Cost of Effort

Let us consider the case where the individual **costs of effort** is $C_{ij}(x_j(i)) = c$ with $c \in \mathbb{R}$ and the club good **prize** worth $v > 0$, that is the $BMMAGC^{*b}$ model . We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the realization c, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ idiosyncratically observes the realization of a random variable C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $0 < \varepsilon < min\left\{\left|\frac{2\overline{c} - v}{4}\right|, \left|\frac{c}{2}\right|\right\}$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, we refer to this game as $g_2(c)$. Then we are able to obtain the following result.

Proposition 6 *In the* $g_2(c)$, *there is a unique equilibrium in (monotonic) switching strategies, such that* $\forall ij \in \{1, \ldots, n_j\}$ *and* $\forall j \in \{1, 2\}$*:*

$$x^*_{ij}(c_{ij}) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v \end{cases} .$$

Remark 7 *The presence of both an upward dominance region and a downward dominance region is conducive to the selection of a unique equilibrium in (monotonic) switching strategies.*

Remark 8 *The existence of a unique equilibrium in (monotonic) switching strategies in the* $g_2(c)$ *is ensured as long as* $0 < \varepsilon < min\left\{\left|\frac{2\overline{c} - v}{4}\right|, \left|\frac{c}{2}\right|\right\}$. *However, equilibrium selection happens even for "a pinch of uncertainty", no matter how small* ε *is.*

Remark 9 *Note that the cutoff of the equilibrium in (monotonic) switching strategies, i.e.* $c_{ij} = 2^{-n_j} v$*, does not coincide with the one of the risk-dominance region, that is* $c_{ij} = \frac{v}{4}$ *for any* $j \in \{1, 2\}$*, differently from what happens in the two-group four-player example. This is very close to the point made by Carlsson and van Damme [1993b] for n-player stag hunt games, where the authors stress that risk-dominance fails as an equilibrium selection criterion when we depart from the 2×2 case.*

6 The Group-Size Paradox

In this section we calculate the probabilities of winning the prize v and the expected payoffs for both groups at the unique equilibrium in (monotonic) switching strategies in $g_1(v)$ and $g_2(c)$, in order to assess the presence of the so-called group-size paradox.

6.1 The Group-Size Paradox in $g_1(v)$

Let us consider the $g_1(v)$. Then it is posssible to derive the probability of winning the prize v for group $j \in \{1, 2\}$, as the following proposition shows.

Proposition 7 *In the* $g_1(v)$*, the probability of winning the prize* v *for group* $j \in \{1, 2\}$ *at the cutoff equilibrium equals:*

- *if* $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ *and* $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$,

$$\begin{aligned} Prob\,(j\ \ wins\ v) = & \left[1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\right] \cdot \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} + \\ & + \frac{1}{2}\left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} + \\ & + \frac{1}{2}\left[1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right] \ ; \end{aligned}$$

- *if* $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ *and* $2^{n_{-j}} > \overline{v} - \varepsilon$,

$$Prob\,(j\ \ wins\ v) = 1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_j} + \frac{1}{2}\left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_j} \ ;$$

- *if* $2^{n_j} > \overline{v} - \varepsilon$ *and* $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$,

$$Prob\,(j\ \ wins\ v) = \frac{1}{2}\left[\left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_{-j}}\right] \ ;$$

- *if* $2^{n_j} > \overline{v} - \varepsilon$ *and* $2^{n_{-j}} > \overline{v} - \varepsilon$,

$$Prob\,(j\ \ wins\ v) = \frac{1}{2} \ .$$

Corollary 1 *In the* $g_1(v)$*, the probability of winning the prize* v *for group* j *at the equilibrium in (monotonic) switching strategies is:*

- *decreasing in* n_j *at the threshold* $n_j^* = log(\overline{v} - \varepsilon)/log(2)\ \forall\ 0 < \varepsilon < |\frac{v}{2} - 1|$;
- *increasing in* n_{-j} *at the threshold* $n_{-j}^* = log(\overline{v} - \varepsilon)/log(2)\ \forall\ 0 < \varepsilon < |\frac{v}{2} - 1|$.

Remark 10 *Note that, for* $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ *and* $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$ *, whether the probability of winning the prize* v $\forall j \in \{1, 2\}$ *is increasing or decreasing with respect to group size* n_j *cannot be established analytically for any* $\overline{v}, \underline{v}$ *and* ε*, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

Example 1 *Let us consider:*

- $[\underline{v} + \varepsilon, \overline{v} - \varepsilon] = [1, 1000]$ *;*
- $n_j = 2$ *and* $n_{-j} = 3$ *.*

Then,

$$\Delta_j Prob(\ j \ wins \ v\ ) = Prob(\ j \ wins \ v \ ; n_j + 1) - Prob(\ j \ wins \ v \ ; n_j) =$$

$$= \left(\left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon} \right)^{n_{-j}} - 1 \right) \left(\left(\frac{-2^{n_j+1} + \underline{v} + \varepsilon}{\underline{v} - \overline{v}} \right)^{n_j+1} - \left(\frac{-2^{n_j} + \underline{v} + \varepsilon}{\underline{v} - \overline{v}} \right)^{n_j} \right) ,$$

so that we can find numerically:

- $\Delta_j Prob(\ j \ wins \ v\ ) > 0 \quad if \quad n_j < 5$ *,*
- $\Delta_j Prob(\ j \ wins \ v\ ) < 0 \quad if \quad n_j \geq 5$ *.*

Moreover,

$$\Delta_{-j} Prob(\ j \ wins \ v\ ) = Prob(\ j \ wins \ v \ ; n_{-j} + 1) - Prob(\ j \ wins \ v \ ; n_{-j}) =$$

$$= \left(\left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon} \right)^{n_j} \right) \left(\left(\frac{-2^{n_{-j}+1} + \underline{v} + \varepsilon}{\underline{v} - \overline{v}} \right)^{n_{-j}+1} - \left(\frac{-2^{n_{-j}} + \underline{v} + \varepsilon}{\underline{v} - \overline{v}} \right)^{n_{-j}} \right) ,$$

so that we can find numerically:

- $\Delta_{-j}(\ j \ wins \ v\ ) < 0 \quad if \quad n_{-j} < 5$ *,*
- $\Delta_{-j} Prob(\ j \ wins \ v\ ) > 0 \quad if \quad n_{-j} \geq 5$ *.*

Therefore, in this specific example, the impact of an increase of the group sizes on the probability of winning the prize for a group is non-monotonic. Furthermore, the finite differences with respect to the two group sizes display opposite signs.

Finally, let us check whether the probability for a group winning the prize decreases in n_j *and increases in* n_{-j} *at the thresholds highlighted in the corollary above.*

- $log(\overline{v} - \varepsilon) / log(2) = log(1000) / log(2) = 9.96578$ *;*
- *for* $n_j = n_{-j} = 9 < log(1000) / log(2)$*,* $Prob(\ j \ wins \ v\ ) = 0.5$ *;*
- *for* $n_j = 10 > log(1000) / log(2) > n_{-j} = 9$*,* $Prob(\ j \ wins \ v\ ) = 0.00119858$ *;*
- *for* $n_{-j} = 10 > log(1000) / log(2) > n_j = 9$*,* $Prob(\ j \ wins \ v\ ) = 0.998801$ *.*

Moreover, once computed the probabilities of winning, it is immediate to retrieve the expected payoffs at the equilibrium in (monotonic) switching strategies.

Proposition 8 *In the $g_1(v)$, the expected payoff at the cutoff equilibrium $x^*_{ij}(v_{ij})$ for any $ij \in \{1,\dots,n_j\}$ $\forall j \in \{1,2\}$ equals:* [6]

- *for $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n-j} \leq \overline{v}-\varepsilon$,*

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = {} & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)+\\ & +\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right]\cdot\left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot\frac{\underline{v}+\overline{v}}{2}+\\ & +\left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left(1-\left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right)\cdot\left(\frac{\underline{v}+\overline{v}}{4}-1\right)+\\ & +\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left(1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right)\cdot\left(1-\left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\right)\cdot\\ & \cdot\frac{\underline{v}+\overline{v}}{4}+\\ & +\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot\frac{\underline{v}+\overline{v}}{4}\ ;\end{aligned}$$

- *for $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $2^{n-j} > \overline{v}-\varepsilon$,*

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = {} & \left(1-\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left(\frac{\underline{v}+\overline{v}}{2}-1\right)\\ & +\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)\cdot\left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j-1}\right]\cdot\frac{\underline{v}+\overline{v}}{2}+\\ & +\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\frac{\underline{v}+\overline{v}}{4}\ ;\end{aligned}$$

- *for $2^{n_j} > \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n-j} \leq \overline{v}-\varepsilon$,*

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = \left(\frac{2^{n-j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n-j}\cdot\frac{\underline{v}+\overline{v}}{4}\ ;$$

- *for $2^{n_j} > \overline{v}-\varepsilon$ and $2^{n-j} > \overline{v}-\varepsilon$,*

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = \frac{\underline{v}+\overline{v}}{4}\ .$$

Corollary 2 *In the $g_1(v)$, the expected payoff $\forall ij$ at the equilibrium in (monotonic) switching strategies $x^*_{ij}(v_{ij})$ for any $ij \in \{1,\dots,n_j\}$ $\forall j \in \{1,2\}$ is:*

- *decreasing in n_j at the threshold $n^*_j = log(\overline{v}-\varepsilon)/log(2)\ \forall\ 0<\varepsilon<|\frac{\underline{v}}{2}-1|$;*

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[6] By $\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right]$ we mean the expected payoff at the cutoff equilibrium $x^*_{ij}(v_{ij})$ for any $ij \in \{1,\dots,n_j\}$ $\forall j \in \{1,2\}$, where $\mathbf{x}^*_j,\mathbf{x}^*_{-j}$ are the vectors of equilibrium strategies profiles for the two groups.

- *increasing in* n_{-j} *at the threshold* $n^*_{-j} = log(\overline{v} - \varepsilon)/log(2) \ \forall \ 0 < \varepsilon < |\frac{v}{2} - 1|$.

Remark 11 *Note that, for* $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ *and* $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$ *, whether the expected payoff* $\forall ij \in \{1, \ldots, n_j\} \ \forall j \in \{1, 2\}$ *is increasing or decreasing with respect to group size* n_j *cannot be established analytically for any* $\overline{v}, \underline{v}$ *and* ε*, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

Example 2 *Let us consider:*

- $[\underline{v} + \varepsilon, \overline{v} - \varepsilon] = [1, 1000]$ *;*
- $n_j = 2$ *and* $n_{-j} = 3$ *;*
- $\mathbb{E}[V] = \frac{\underline{v} + \overline{v}}{2}$ *.*

Then,

$$\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] = \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}; n_j + 1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}; n_j\right)\right] =$$

$$= \frac{2^{n_j}}{\overline{v} - \varepsilon - \underline{v} - \varepsilon} + \frac{(\overline{v} - \varepsilon + \underline{v} + \varepsilon)\left(\left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} - \left(\frac{2^{n_j + 1} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j + 1}\right)}{4}$$

so that we can find numerically:

- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] > 0$ *if* $n_j < 8$ *,*
- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] < 0$ *if* $n_j \geq 8$ *.*

Moreover,

$$\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] = \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}; n_{-j} + 1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}; n_{-j}\right)\right]$$

$$= \frac{(\overline{v} - \varepsilon + \underline{v} + \varepsilon)\left(\left(\frac{-2^{n_{-j}+1} + \underline{v} - \varepsilon}{\underline{v} - \overline{v}}\right)^{n_{-j}+1} - \left(\frac{-2^{n_{-j}} + \underline{v} - \varepsilon}{\underline{v} - \overline{v}}\right)^{n_{-j}}\right)}{4}$$

so that we can find numerically:

- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right]$ *if* $n_{-j} < 5$ *,*
- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j, \mathbf{x}^*_{-j}\right)\right] > 0$ *if* $n_{-j} \geq 5$ *.*

Therefore, in this specific example, the impact of an increase in the group sizes on the expected payoff at the cutoff equilibrium $\forall ij \in \{1, \ldots, n_j\} \ \forall j \in \{1, 2\}$ *is non-monotonic. Note that for* $n_j \in \{5, 6, 7\}$*, the expected payoff of group* j *is increasing in* n_j*, notwithstanding the probability of winning for group* j *is decreasing in* n_j*, as shown in the previous numerical example.*

Finally, let us check whether the equilibrium individual expected payoff decreases in n_j *and increases in* n_{-j} *at the thresholds highlighted in the corollary above.*

- $log\,(\overline{v} - \varepsilon)\,/log\,(2) = log\,(1000)\,/log\,(2) = 9.96578$ *;*
- *for* $n_j = n_{-j} = 9 < log\,(1000)\,/log\,(2)$*, Prob (j wins v)* $= 249.762$ *;*
- *for* $n_j = 10 > log\,(1000)\,/log\,(2) > n_{-j} = 9$*, Prob (j wins v)* $= 0.599892$ *;*
- *for* $n_{-j} = 10 > log\,(1000)\,/log\,(2) > n_j = 9$*, Prob (j wins v)* $= 499.412$ *.*

6.2 The Group-Size Paradox in $g_2(c)$

Let us now turn to the $g_2(c)$ model. Then it is posssible to derive the probability of winning the prize v for group $j \in \{1,2\}$, as the following proposition shows.

Proposition 9 *In the $g_2(c)$, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium equals:*

$$\begin{aligned} Prob\,(j\ wins\ v) = & \left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}+ \\ & +\frac{1}{2}\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}+ \\ & +\frac{1}{2}\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]\ . \end{aligned}$$

Remark 12 *Contrary to $g_1(v)$, in the $g_2(c)$ the probability of winning the prize v for group j at the equilibrium in (monotonic) switching strategies is not increasing in n_j at the threshold $n_j^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)\ \forall 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and increasing in n_{-j} at the threshold $n_{-j}^* = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$, since $2^{-n_j}v > \underline{c}+\varepsilon\ \forall\ \underline{c} < 0,\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall n_j \geq 2$ and $2^{-n_j}v < \overline{c}-\varepsilon\ \forall\ \overline{c} > \frac{v}{2},\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall\ n_j \geq 2$.*

Remark 13 *Note that whether the probability of winning the prize $v \forall j \in \{1,2\}$ is increasing or decreasing with respect to group size n_j cannot be established analytically for any $\overline{c}, \underline{c}, \varepsilon$ and v, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution, prize value and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

Example 3 *Let us consider:*

- $[\underline{c}+\varepsilon, \overline{c}-\varepsilon] = [-1, 1000]$;
- $n_j = 2$ *and* $n_{-j} = 3$;
- $v = 800$ *utils ;*
- $\mathbb{E}[C] = \frac{\underline{c}+\overline{c}}{2}$.

Then,

$$\begin{aligned} \Delta_j Prob\,(\ j\ wins\ v\ ) &= Prob\,(\ j\ wins\ v\ ; n_j+1) - Prob\,(\ j\ wins\ v\ ; n_j) \\ &= \frac{1}{4}\left(\frac{2^{-n_j}\left(v-2^{n_j+1}\left(\overline{c}-\varepsilon\right)\right)\left(\frac{\overline{c}-\varepsilon-2^{-n_j-1}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}+\right. \\ &\quad \left. +2\left(\frac{\overline{c}-\varepsilon-2^{-n_j}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\right)\ . \end{aligned}$$

so that we can find numerically:

- $\Delta_j Prob\,(\ j\ wins\ v\ ) < 0$ *if* $n_j \leq 12$,
- $\Delta_j Prob\,(\ j\ wins\ v\ ) > 0$ *if* $n_j > 12$.

Moreover,

$$\begin{aligned}\Delta_{-j} Prob(\ j\ wins\ v\ ) &= Prob(\ j\ wins\ v\ ; n_{-j}+1) - Prob(\ j\ wins\ v\ ; n_{-j}) \\ &= \frac{1}{4}\left(\frac{2^{-n_{-j}}\left(v - 2^{n_{-j}+1}\left(\overline{c}-\varepsilon\right)\right)\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}-1}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}}{\overline{c}-\varepsilon-\underline{c}-\varepsilon} + \right. \\ &\left. -2\left(\frac{\overline{c}-\varepsilon-2^{-n_{-j}}v}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\right) .\end{aligned}$$

so that we can find numerically:

- $\Delta_{-j}(\ j\ wins\ v\ ) > 0\quad if\quad n_{-j} \leq 12,$
- $\Delta_{-j} Prob(\ j\ wins\ v\ ) < 0\quad if\quad n_{-j} > 12\ .$

Therefore, in this specific example, the impact of an increase of group sizes on the probability of winning the prize is non-monotonic. Furthermore, the finite differences with respect to the two group sizes display different signs.

Moreover, once computed the probabilities of winning, it is immediate to retrieve the expected payoffs at the equilibrium in (monotonic) switching strategies.

Proposition 10 *In the* $g_2(c)$*, the expected payoff at the cutoff equilibrium* $x^*_{ij}(c_{ij})$ *for any* $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ *equals:* [7]

$$\begin{aligned}\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right] = &\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\left(v-\frac{\underline{c}+\overline{c}}{2}\right)+ \\ &+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right]\cdot \\ &\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot v + \\ &+\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{-n_{-j}}\right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)\ + \\ &+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right)\cdot \\ &\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\right)\cdot\frac{v}{2}\ + \\ &+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\frac{v}{2}\ .\end{aligned}$$

Remark 14 *Contrary to* $g_1(v)$*, in the* $g_2(c)$ *the expected payoff for any* $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$ *at the equilibrium in (monotonic) switching strategies is not increasing in* n_j *at the threshold* $n^*_j = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$ *and increasing in* n_{-j} *at the threshold* $n^*_{-j} = log(\frac{v}{\overline{c}-\varepsilon})/log(2)$ $\forall\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$, *since* $2^{-n_j}v > \underline{c}+\varepsilon\ \forall\ \underline{c} < 0,\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$ *and* $\forall\ n_j \geq 2$ *and* $2^{-n_j}v < \overline{c}-\varepsilon\ \forall\ \overline{c} > \frac{v}{2},\ 0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|,|\frac{\underline{c}}{2}|\right\}$ *and* $\forall\ n_{-j} \geq 2$.

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[7] By $\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}^*_j,\mathbf{x}^*_{-j}\right)\right]$ we mean the expected payoff at the cutoff equilibrium $x^*_{ij}(c_{ij})$ for any $ij \in \{1,\ldots,n_j\}$ $\forall j \in \{1,2\}$, where $\mathbf{x}^*_j,\mathbf{x}^*_{-j}$ are the vectors of equilibrium strategies profiles for the two groups.

Remark 15 *Note that whether the expected payoff* $\forall\ ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, 2\}$ *is increasing or decreasing with respect to group size* n_j *cannot be established analytically for any* $\overline{c}, \underline{c}, \varepsilon$ *and* v*, so that the presence of the so-called group-size paradox depends on the specific details of the uniform prior distribution, prize value and group sizes. Nonetheless, numerical solutions can be found as the following example clarifies.*

Example 4 *Let us consider:*

- $[\underline{c} + \varepsilon, \overline{c} - \varepsilon] = [-1, 1000]$;
- $n_j = 2$ *and* $n_{-j} = 3$;
- $v = 800$ *utils* ;
- $\mathbb{E}[C] = \frac{\underline{c} + \overline{c}}{2}$.

Then,

$$\begin{aligned}\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j + 1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_j\right)\right] \\ &= \frac{2^{-n_j-2}\left(\left(\underline{c} + \varepsilon + \overline{c} - \varepsilon\right) v - \left(2^{n_j+1}\left(\overline{c} - \varepsilon\right) - v\right)\left(\frac{\overline{c} - \varepsilon - 2^{-n_j-1} v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j}\right)}{\overline{c} - \varepsilon - \underline{c} + \varepsilon} + \\ &+ \frac{2^{-n_j-2}\left(2^{n_j+1}\left(\overline{c} - \varepsilon - \underline{c} - \varepsilon\right) v \left(\frac{\overline{c} - \varepsilon - 2^{-n_j} v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_j}\right)}{\overline{c} - \varepsilon - \underline{c} - \varepsilon} ,\end{aligned}$$

so that we can find numerically:

- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] > 0$ *if* $n_j = 2$ *or* $n_j \geq 12$,
- $\Delta_j \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] < 0$ *if* $2 < n_j < 12$.

Moreover,

$$\begin{aligned}\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] &= \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j} + 1\right)\right] - \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*; n_{-j}\right)\right] \\ &= \frac{v}{4}\left(\frac{2^{-n_{-j}}\left(2^{n_{-j}+1}\left(\overline{c} - \varepsilon\right) - v\right)\left(\frac{\overline{c} - \varepsilon - 2^{-n_{-j}-1} v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}}}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right) + \\ &- \frac{v}{2}\left(\frac{\overline{c} - \varepsilon - 2^{-n_{-j}} v}{\overline{c} - \varepsilon - \underline{c} - \varepsilon}\right)^{n_{-j}} ,\end{aligned}$$

so that we can find numerically:

- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] > 0$ *if* $n_{-j} \leq 12$,
- $\Delta_{-j} \mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] < 0$ *if* $n_{-j} > 12$.

Remark 16 *The need of numerical examples to verify the presence of the so-called group-size paradox constitutes a sharp difference with respect to what obtained for the weakest-link case, as shown in Bosco et al. [2024], where in the general case , i.e.* $log\left(\underline{v} + \varepsilon\right)/log\left(2\right) \leq 2^{n_j} \leq log\left(\overline{v} - \varepsilon\right)/log\left(2\right) - 1$ *, an increase in group size translates into a lower probability of winning and a lower expected payoff for sufficiently high value of the prize and a sufficiently low cost effort in the two cases considered.*

7 Extension to an M-Group Model

In this section we assess the robustness of the results obtained under the two-group assumption by extending our model to the M-group case with $M \geq 2$. We will directly inspect the two incomplete information cases separately as done for the two-group setting, that is about the prize contested and the cost of exerting effort.

7.1 Incomplete Information à la Global Games about the Prize and M Groups

Let us define the $BMMAMGC^*$ as the $BMMAGC^*$ with M groups, where $M \geq 2$, and $n_1 \geq \ldots \geq n_M \geq 2$ without loss of generality.

We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the prize v as follows:

- let V be a random variable which is uniform on some interval $[\underline{v}, \overline{v}]$ including the dominance region and the threshold for the risk-dominance, e.g. $[1, 5]$;
- given the realization v, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ idiosyncratically observes the realization of a random variable V_{ij}, uniform on $[v - \varepsilon, v + \varepsilon]$ for some $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$, so that the players' observation errors $V_{ij} - v$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above.

Henceforth, we refer to this game as $g_3(v)$. Then, we are able to obtain the following result.

Proposition 11 *In the $g_3(v)$, there is an equilibrium in (monotonic) switching strategies such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:*

$$x_{ij}^*(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > v_j^* \\ 0 & \text{if } v_{ij} \leq v_j^*, \end{cases}$$

where $v_j^* = \frac{A}{B}$, with

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) ,$$

$$B = 2^{1-n_j-\sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} ,$$

$$Q_k = \{-J \in \{\{1, \ldots, M\} - \{j\}\} \mid |-J| = k\} .$$

Remark 17 *Note that proposition 11 delivers an existence result, not a uniqueness result, that is we are able to find an equilibrium in (monotonic) switching strategies in $g_3(v)$, which may not be unique.*

Remark 18 *The cutoff $v_j^* = \frac{A}{B}$ is different for $M > 2$ with respect to the the one obtained under the weakest-link impact function in Bosco et al. [2024] . Hence, the parallelism holds just in the two-group setting.*

7.2 Incomplete Information à la Global Games about the Cost of Effort and M Groups

Let us define the $BMMAMGC^{*b}$ as the $BMMAGC^{*b}$ with M groups, where $M \geq 2$, and $n_1 \geq \ldots \geq n_M \geq 2$ without loss of generality.

We closely follow Carlsson and van Damme [1993a] introducing incomplete information about the cost of effort c as follows:

- let C be a random variable which is uniform on some interval $[\underline{c}, \overline{c}]$ including both dominance regions, e.g. $[-v, +v]$;
- given the realization c, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ idiosyncratically observes the realization of a random variable C_{ij}, uniform on $[c - \varepsilon, c + \varepsilon]$ for some $0 < \varepsilon < min \left\{ \left| \frac{2\overline{c} - v}{4} \right|, \left| \frac{\underline{c}}{2} \right| \right\}$, so that the players' observation errors $C_{ij} - c$ $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$ are independent;
- after these idiosyncratic observations, each player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ simultaneously and independently decides whether to exert effort or not and gets a payoff as described above;

Henceforth, we refer to this game as $g_4(c)$. Then we are able to obtain the following result.

Proposition 12 *In $g_4(c)$ under incomplete information à la global games there is a unique equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:*

$$x^*_{ij}(c_{ij}) = \begin{cases} 1 & \textit{if } c_{ij} < c^*_j \\ 0 & \textit{if } c_{ij} \geq c^*_j, \end{cases}$$

where $c^*_j = \frac{B}{A} v$, with

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \ ,$$

$$B = 2^{1 - n_j - \sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1 - n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k + 1} \ ,$$

$$Q_k = \{-J \in \{\{1, \ldots, M\} - \{j\}\} \mid |-J| = k\} \ .$$

Remark 19 *Note that proposition 12 delivers both an existence and a uniqueness result, that is we are able to find a unique equilibrium in (monotonic) switching strategies in $g_3(v)$, differently from what achieved in $g_4(c)$. This result is intimately related to the presence of both an upper and a lower dominance region, as the the proof should clarify.*

Remark 20 *The cutoff $c^*_j = \frac{B}{A} v$ is different for $M > 2$ with respect to the the one obtained under the weakest-link impact function in Bosco et al. [2024] . Hence, the parallelism holds just in the two-group setting.*

8 Further Results

8.1 Limit-Uniqueness Result

One of the main results due to Carlsson and van Damme [1993a] is about the robustness of the equilibrium in (monotonic) switching strategies as the noise tends to zero, i.e. the so-called limit-uniqueness result. We can easily show that, under our uniform information structure, the main results obtained so far hold as the noise fades away, as stated by the following result.

Proposition 13 *As the size of noise tends to zero, i.e.* $\varepsilon \to 0$*, propositions 5, 6, 11, 12 hold.*

8.2 Noise Independent Selection

The strength of the equilibrium selection phenomenon in global games highlighted by Carlsson and van Damme [1993a] is also due to its invariance with respect to both the prior distribution and the distribution of the noise, as long as the the support of the latter is sufficiently small and all other assumptions about independence and continuity in the information structure are preserved. Moreover, if it is employed a uniform improper prior, then exact results hold, as shown by Carlsson and van Damme [1993a], even without assuming a sufficiently small support for the noise. We will focus on the latter case to prove the invariance of our results with respect to the choice of the noise distribution.

Proposition 14 *Under the uniform improper prior distribution for* V *and* C*, the results in propositions 5-11 and 6-11, respectively, are invariant to the exact distribution of the noise, provided that it is symmetric, with mean zero and unit variance.*

9 Conclusions

We introduced incomplete information à la global games in a two-group max-max group contest with binary actions, relaxing the complete information assumption about the value of the prize contested and the cost of providing effort, separately. In the first case, we prove the existence of an equilibrium in (monotonic) switching strategies which may be not unique; in the second one, a unique equilibrium in (monotonic) switching-strategies emerges. These results are extended to the general M-group case. Moreover, in the two-group setting, given the uniform information structure, it is straightforward to calculate the probability of winning for each group and the expected payoffs at the equilibrium in switching strategies in both incomplete information cases, but numerical examples are needed to assess the presence of the group-size paradox. Finally we somehow replicate the limit-uniqueness and noise-independent selection results due to Carlsson and van Damme [1993a], but in a more restrictive fashion. Therefore, introducing incomplete information à la global games in deterministic group contests with binary actions and the best-shot impact function does not only deliver informational realism, but it also reduces significantly the burden of equilibrium multiplicity, or rather indeterminacy, which affects deterministic group contests with continuous efforts and a public good prize under both complete information and under incomplete information, as in Barbieri et al. [2014] and Barbieri and Malueg [2016], respectively. Therefore, we find a setting in which payoff-relevant incomplete information, that is intrinsic private information, can lead to equilibrium uniqueness, complementing the results by Barbieri and Topolyan [2024] obtained through the adoption of a group-public randomization device. We would like to stress that this selection result could be relevant for applications of deterministic two-group contests with binary actions, among which we emphasized the classical examples stemming from the related literature in the introduction, such as *R&D* competition and military conflict.

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10 APPENDIX: PROOFS

10.1 Proof of Proposition 1

Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \;.^{8}$$

Then,

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\} \;.$$

Suppose $2 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$ and $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \;.$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is an incentive to deviate $\forall v \in R$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \; \forall v \in \mathbb{R} \;.$$

On the other hand, suppose $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$ and $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \;.$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_1\right\} < max\left\{\mathbf{x}_2\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = 0.$$

Hence, for player ij there is no incentive to deviate if and only if $v \geq 2$, since

$$\frac{v}{2} - 1 \geq 0 \; \forall v \geq 2 \;.$$

Moreover, suppose $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0$ and $x_j\left(i\right) = 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} \;.$$

If agent ij deviates to $x_j\left(i\right) = 1$, then

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

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[8] $\mathbb{1}_{x_{ij}=1}$ stands for the Indicator random variable taking value 1 when player ij chooses $x_{ij} = 1$, that is she exerts effort.

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R$, since

$$\frac{v}{2} - 1 < \frac{v}{2} \ \forall v \in \mathbb{R} \ .$$

Thus,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$$

is a Nash equilibrium in pure strategies $\forall v \geq 2$.

Suppose

$$(\gamma_1, \gamma_2) = (1,1) \ .$$

Then,

$$max\left\{\mathbf{x}_1\right\} = max\left\{\mathbf{x}_2\right\}$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = \frac{v}{2} - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} = max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for player ij there is an incentive to deviate $\forall v \in R$ since

$$\frac{v}{2} - 1 < \frac{v}{2} \forall v \in \mathbb{R} \ .$$

Thus,

$$(\gamma_1, \gamma_2) = (1,1)$$

is not a Nash equilibrium in pure strategies $\forall v \in R$.

Suppose

$$(\gamma_j, \gamma_{-j}) = (1,0) \, \forall j \in \{1,2\} \ .$$

Then,

$$max\left\{\mathbf{x}_j\right\} = 1 > max\left\{\mathbf{x}_{-j}\right\} = 0$$

so that

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - 1 \ .$$

If agent ij deviates to $x_j\left(i\right) = 0$, then

$$max\left\{\mathbf{x}_j\right\} > max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^-, \gamma_{-j}\right) = v \ .$$

Hence, for agent ij there is an incentive to deviate $\forall v \in R$, since

$$v - 1 < v \forall v \in \mathbb{R} \ .$$

On the other hand, for completeness sake, if agent $i-j$ deviates to $x_{-j}\left(i\right)=1$, then

$$max\left\{\mathbf{x}_{j}\right\}=max\left\{\mathbf{x}_{-j}\right\}=1$$

so that the deviation payoff is

$$\pi_{i-j}^{D}\left(\gamma_{j},\gamma_{-j}^{+}\right)=\frac{v}{2}-1\ .$$

Hence, for player $i-j$ there are no incentives to deviate if and only if

$$\pi_{i-j}\left(\gamma_{j},\gamma_{-j}\right)=0\geq\pi_{i-j}^{D}\left(\gamma_{j},\gamma_{-j}^{+}\right)=\frac{v}{2}-1\Leftrightarrow v\leq2\ .$$

Thus,

$$\left(\gamma_{j},\gamma_{-j}\right)=\left(1,0\right)\ \forall j\in\left\{1,2\right\}$$

is not a Nash equilibrium in pure strategies for any $v\in R$.

Suppose

$$\left(\gamma_{1},\gamma_{2}\right)=\left(0,0\right)\ .$$

Then

$$max\left\{\mathbf{x}_{1}\right\}=max\left\{\mathbf{x}_{2}\right\}$$

so that

$$\pi_{ij}\left(\gamma_{1},\gamma_{2}\right)=\frac{v}{2}\ \forall j\in\left\{1,2\right\}\ .$$

If agent ij deviates to $x_{j}\left(i\right)=1$, then

$$max\left\{\mathbf{x}_{j}\right\}>max\left\{\mathbf{x}_{-j}\right\}$$

so that

$$\pi_{ij}^{D}\left(\gamma_{j}^{+},\gamma_{-j}\right)=v-1\ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\pi_{ij}\left(\gamma_{j},\gamma_{-j}\right)=\frac{v}{2}\geq\pi_{ij}^{D}\left(\gamma_{j}^{D},\gamma_{-j}\right)=v-1\Leftrightarrow v\leq2\ \forall j\in\left\{1,2\right\}\ .$$

Thus,

$$\left(\gamma_{1},\gamma_{2}\right)=\left(0,0\right)$$

is a Nash equilibrium in pure strategies for any $v\leq2$.

Let $\sigma_{ij}\left(x_{ij}=1\right)$ be the within-group symmetric randomization over pure strategy $x_{ij}=1$ for player ij, then

$$EU_{ij}\left(x_{ij}=1\right)=EU_{ij}\left(x_{ij}=0\right)\Leftrightarrow$$

$$\Leftrightarrow Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot\left(v-1\right)+Prob\left(n_{-j}\gamma_{-j}\geq1\right)\cdot\left(\frac{v}{2}-1\right)=$$

$$=\Big(Prob\left(n_{j}\gamma_{j}\geq1\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq1\right)+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{v}{2}+$$

$$+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq1\right)\cdot\left(0\right)+Prob\left(n_{j}\gamma_{j}\geq1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot v\Leftrightarrow$$

$$\Leftrightarrow\left(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\right)\cdot\left(v-1\right)+\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\right)\cdot\left(\frac{v}{2}-1\right)=$$

$$=\Bigg(\left(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_{j}-1}\right)\cdot\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\right)+$$

$$+\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_{j}-1}\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Bigg)\cdot\frac{v}{2}+$$

$$+\Big(1-(1-\sigma_{ij}(x_{ij}=1))^{n_j-1}\Big)\cdot(1-\sigma_{i-j}(x_{i-j}=1))^{n-j}\cdot v$$

$$\Leftrightarrow \sigma^*_{ij}(x_{ij}=1)=\left(1-\frac{2}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\} \ .$$

Thus,

$$\sigma^*_{ij}(x_{ij}=1)=\left(1-\frac{2}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\} \ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall v>2$.

■

10.2 Proof of Proposition 2

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten [1988], it is straightforward to state that:

- for $v>2$ and any ij such that $x_{ij}=0$, any (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ payoff-dominate (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ in which $x_{ij}=1, since$

  $$\pi_{ij}((\gamma_j,\gamma_{-j}) \text{ s.t. } x_{ij}=0)=\frac{v}{2}>\pi_{ij}((\gamma_j,\gamma_{-j}) \text{ s.t. } x_{ij}=1)=\frac{v}{2}-1\forall v>2 \text{ and } \forall j\in\{1,2\} \ .$$

- for $v>4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=1$. As a matter of fact, let us compare the deviation losses of (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ and (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=0$. Then, for any ij:

  $$\left(\frac{v}{2}-1-0\right)>\left(\frac{v}{2}-\left(\frac{v}{2}-1\right)\right)\Leftrightarrow v>4,$$

  that is, for $v>4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ are associated with the largest Nash difference; [9]

- for $2<v<4$, (γ_1,γ_2) such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=0$. Clearly this follows from what shown at the previous point for both groups;

- for $v=2$, $(\gamma_j,\gamma_{-j})=(\gamma_j,0)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the payoff-dominant equilibria for any player ij such that $x_{ij}=0$, since

  $$\pi_{ij}((\gamma_j,0) \text{ s.t. } x_{ij}=0)=v \ ,$$

  which is the highest attainable payoff;

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[9] Here we do not employ deviation losses to determine the risk-dominant equilibrium, since we define it at a single-player level and not at group level. In the latter case, for $v>2$ all Nash equilibria in pure strategies would be clearly equivalent in terms of risk-dominance.

- for $v = 2$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \; \forall j \in \{1,2\}$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \; \forall j \in \{1,2\}$ are the risk-dominant equilibria for any ij such that $x_{ij} = 0$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \; \forall j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \; \forall j \in \{1,2\}$ for any ij such that $x_{ij} = 1$ and $(\gamma_1, \gamma_2) = (0,0)$. Then,

$$\left(\frac{v}{2} - \left(\frac{v}{2} - 1\right)\right) = (v - (v-1)) > \left(\frac{v}{2} - 1 - 0\right) = \left(v - 1 - \frac{v}{2}\right) = \left(0 - \left(\frac{v}{2} - 1\right)\right) = \left(\frac{v}{2} - (v-1)\right) \; .$$

■

10.3 Proof of Proposition 3

- Suppose

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \; \forall j \in \{1,2\} \; .$$

Then,

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\} \; .$$

Suppose $x_j(i) = 1$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \; .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} < max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^-, \gamma_{-j}) = 0 \; .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$\frac{v}{2} - c \geq 0 \Leftrightarrow c \leq \frac{v}{2} \; .$$

Moreover, suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \; .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j}) = \frac{v}{2} - c \; .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \; .$$

Conversely, suppose $x_j(i) = 1$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^-, \gamma_{-j}) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

On the other hand, suppose $x_j(i) = 0$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

so that the deviation payoff is

$$\pi_{ij}^D(\gamma_j^+, \gamma_{-j}) = \frac{v}{2} - c \ .$$

Hence, for player ij there is no incentive to deviate if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

Thus, for $0 < c \leq \frac{v}{2}$,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1)$$

is a Nash equilibrium if and only if $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$.

Moreover,

$$(\gamma_1, \gamma_2) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\}$$

is a Nash equilibrium if and only if $c = 0$.

- Suppose

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 0) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\} \ .$$

Then,

$$max\{\mathbf{x}_j\} > max\{\mathbf{x}_{-j}\} \ .$$

Suppose $x_j(i) = 1$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = v - c \text{ and } \pi_{i-j}(\gamma_j, \gamma_{-j}) = 0 \ .$$

If agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-, \gamma_{-j}\right) = \frac{v}{2}.$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$v - c \geq \frac{v}{2} \Leftrightarrow c \leq \frac{v}{2} \ .$$

Moreover, if agent $i-j$ deviates to $x_{-j}(i) = 1$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j, \gamma_{-j}^-\right) = \frac{v}{2} - c \ .$$

Hence, for player $i-j$ there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$0 \geq \frac{v}{2} - c \Leftrightarrow c \geq \frac{v}{2} \ .$$

Conversely, suppose $x_j(i) = 1$ and $1 < \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1$, then

$$\pi_{ij}\left(\gamma_j, \gamma_{-j}\right) = v - c \text{ and } \pi_{i-j}\left(\gamma_j, \gamma_{-j}\right) = 0 \ .$$

For player $i-j$ everything remains unchanged from the previous case, whereas if agent ij deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} > max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}\left(\gamma_j^-, \gamma_{-j}\right) = v \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$v - c \geq v \Leftrightarrow c \leq 0 \ .$$

Clearly, for players ij such that $x_{ij} = 0$, everything remains unchanged with respect to the previous case.

Thus,

$$\left(\gamma_j, \gamma_{-j}\right) = \left(\gamma_j, 0\right) \text{ such that } \gamma_j \in (0,1) \text{ and } \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\} \ .$$

is a Nash equilibrium if and only if $c = \frac{v}{2}$.

- Suppose

$$\left(\gamma_j, \gamma_{-j}\right) = \left(\gamma_j, 1\right) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\} \ .$$

Then,

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\} \ .$$

Suppose $x_j\left(i\right)=1$ and $1<\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c \text{ and } \pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

If agent ij deviates to $x_j\left(i\right)=0$, then

$$max\left\{\mathbf{x}_j\right\}=max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-,\gamma_{-j}\right)=\frac{v}{2}\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in R_{++}$ if and only if

$$\frac{v}{2}-c\geq\frac{v}{2}\Leftrightarrow c\leq 0\ .$$

Conversely, suppose $x_j\left(i\right)=0$ and $1<\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}\leq n_j-1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c \text{ and } \pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}\ .$$

If agent ij deviates to $x_j\left(i\right)=1$, then

$$max\left\{\mathbf{x}_j\right\}=max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-,\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in R_{++}$ if and only if

$$\frac{v}{2}\geq\frac{v}{2}-c\Leftrightarrow c\geq 0\ .$$

The same arguments hold for player $i-j$, so that for player ij there is no incentive to deviate $\forall v\in R_{++}$ if and only if

$$\frac{v}{2}-c\geq\frac{v}{2}\Leftrightarrow c\leq 0\ .$$

On the other hand, suppose $x_j\left(i\right)=1$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1$, then

$$\pi_{ij}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c \text{ and } \pi_{i-j}\left(\gamma_j,\gamma_{-j}\right)=\frac{v}{2}-c\ .$$

If agent ij deviates to $x_j\left(i\right)=0$, then

$$max\left\{\mathbf{x}_j\right\}<max\left\{\mathbf{x}_{-j}\right\}$$

so that the deviation payoff is

$$\pi_{ij}^D\left(\gamma_j^-,\gamma_{-j}\right)=0\ .$$

Hence, for player ij there is no incentive to deviate $\forall v\in R_{++}$ if and only if

$$\frac{v}{2}-c\geq 0\Leftrightarrow c\leq\frac{v}{2}\ .$$

Conversely, suppose $x_j(i) = 0$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1$, then

$$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \text{ and } \pi_{i-j}(\gamma_j, \gamma_{-j}) = \frac{v}{2} \ .$$

If agent ij deviates to $x_j(i) = 1$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2} - c \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$\frac{v}{2} \geq \frac{v}{2} - c \Leftrightarrow c \geq 0 \ .$$

If agent $i-j$ deviates to $x_j(i) = 0$, then

$$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

so that the deviation payoff is

$$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2} \ .$$

Hence, for player ij there is no incentive to deviate $\forall v \in R_{++}$ if and only if

$$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

Thus,

$$(\gamma_j, \gamma_{-j}) = (\gamma_j, 1) \text{ such that } \gamma_j \in (0,1) \text{ and } 1 \leq \sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} \leq n_j - 1 \ \forall j \in \{1,2\}$$

is a Nash equilibrium in pure strategies if and only if $c = 0$.

- Suppose

  $$(\gamma_1, \gamma_2) = (1,1) \ .$$

  Then,

  $$max\{\mathbf{x}_1\} = max\{\mathbf{x}_2\}$$

  so that

  $$\pi_{ij}(\gamma_j, \gamma_{-j}) = \frac{v}{2} - c \ .$$

  If agent ij deviates to $x_j(i) = 0$, then

  $$max\{\mathbf{x}_j\} = max\{\mathbf{x}_{-j}\}$$

  so that the deviation payoff is

  $$\pi_{ij}^{D}\left(\gamma_j^{-}, \gamma_{-j}\right) = \frac{v}{2} \ .$$

  Hence, for player ij there is no incentive to deviate if and only if

  $$\frac{v}{2} - c \geq \frac{v}{2} \Leftrightarrow c \leq 0 \ .$$

  Thus,

  $$(\gamma_1, \gamma_2) = (1,1)$$

  is a Nash equilibrium in pure strategies if and only if $c \leq 0$.

- Suppose
$$(\gamma_j, \gamma_{-j}) = (1, 0)\, \forall j \in \{1, 2\} \ .$$
Then,
$$max\, \{\mathbf{x}_j\} > max\, \{\mathbf{x}_{-j}\}$$
so that
$$\pi_{ij}\, (\gamma_j, \gamma_{-j}) = v - c \ .$$
If agent ij deviates to $x_j\,(i) = 0$, then
$$max\, \{\mathbf{x}_j\} > max\, \{\mathbf{x}_{-j}\}$$
so that the deviation payoff is
$$\pi^D_{ij}\, \left(\gamma_j^-, \gamma_{-j}\right) = v \ .$$
Hence, for agent ij there is no incentive to deviate if and only if
$$v - c \geq v \Leftrightarrow c \leq 0 \ .$$
On the other hand, if agent $i - j$ deviates to $x_{-j}\,(i) = 1$, then
$$max\, \{\mathbf{x}_j\} = max\, \{\mathbf{x}_{-j}\} = 0$$
so that the deviation payoff is
$$\pi^D_{i-j}\, \left(\gamma_j, \gamma^+_{-j}\right) = \frac{v}{2} - c \ .$$
Hence, for player $i - j$ there is no incentive to deviate if and only if
$$\pi_{i-j}\, (\gamma_j, \gamma_{-j}) = 0 \geq \pi^D_{i-j}\, \left(\gamma_j, \gamma^+_{-j}\right) = \frac{v}{2} - c \Leftrightarrow c \geq \frac{v}{2} \ .$$
Thus,
$$(\gamma_j, \gamma_{-j}) = (1, 0)\ \ \forall j \in \{1, 2\}$$
is a not a Nash equilibrium in pure strategies for any $c \in R$.

- Suppose
$$(\gamma_1, \gamma_2) = (0, 0) \ .$$
Then
$$max\, \{\mathbf{x}_1\} = max\, \{\mathbf{x}_2\}$$
so that
$$\pi_{ij}\, (\gamma_1, \gamma_2) = \frac{v}{2}\ \forall j \in \{1, 2\} \ .$$
If agent ij deviates to $x_j\,(i) = 1$, then
$$max\, \{\mathbf{x}_1\} > max\, \{\mathbf{x}_2\}$$
so that
$$\pi^D_{ij}\, \left(\gamma^+_j, \gamma_{-j}\right) = v - c \ .$$
Hence, for player ij there is no incentive to deviate if and only if
$$\pi_{ij}\, (\gamma_j, \gamma_{-j}) = \frac{v}{2} \geq \pi^D_{ij}\, \left(\gamma^D_j, \gamma_{-j}\right) = v - c \Leftrightarrow c \geq \frac{v}{2}\ \forall j \in \{1, 2\} \ .$$
Thus,
$$(\gamma_1, \gamma_2) = (0, 0)$$
is a Nash equilibrium in pure strategies for any $c \geq \frac{v}{2}$.

- Let $\sigma_{ij}\left(x_{ij}=1\right)$ be the within-group symmetric randomization over pure strategy $x_{ij}=1$ for player ij $\forall j \in \{1,2\}$, then

$$EU_{ij}\left(x_{ij}=1\right)=EU_{ij}\left(x_{ij}=0\right) \Leftrightarrow$$

$$\Leftrightarrow Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot(v-c)+Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(\frac{v}{2}-c\right)=$$

$$=\Big(Prob\left(n_{j}\gamma_{j}\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{v}{2}+$$

$$+Prob\left(n_{j}\gamma_{j}=0\right)\cdot Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot(0)+Prob\left(n_{j}\gamma_{j}\geq 1\right)\cdot Prob\left(n_{-j}\gamma_{-j}=0\right)\cdot v \Leftrightarrow$$

$$\Leftrightarrow\Big(1-\left(\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot(v-c)+\Big(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot\left(\frac{v}{2}-c\right)=$$

$$=\Big(\Big(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\Big)\cdot\left(1-\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\right)+$$

$$+\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\Big)\cdot\frac{v}{2}+$$

$$+\Big(1-\left(1-\sigma_{ij}\left(x_{ij}=1\right)\right)^{n_j-1}\Big)\cdot\left(1-\sigma_{i-j}\left(x_{i-j}=1\right)\right)^{n_{-j}}\cdot v$$

$$\Leftrightarrow \sigma^*_{ij}\left(x_{ij}=1\right)=\left(1-\frac{2c}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\}\ .$$

Thus,

$$\sigma^*_{ij}\left(x_{ij}=1\right)=\left(1-\frac{2c}{v}\right)^{1/(n_j-1)} \quad \forall i\in\{1,\ldots,n_j\} \text{ and } j\in\{1,2\}\ .$$

is a Nash equilibrium in within-group symmetric strictly-mixed strategies $\forall\ 0<c<\frac{v}{2}$.

■

10.4 Proof of Proposition 4

Following the formulation of payoff-dominance and risk-dominance concepts by Harsanyi and Selten (1988), it is straightforward to state that in the $BMMAGC^{*b}$:

- for $0<c<\frac{v}{2}$ and any ij such that $x_{ij}=0$, any $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ payoff-dominate $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ in which $x_{ij}=1, since$

  $$\pi_{ij}\left(\left(\gamma_j,\gamma_{-j}\right)\text{ s.t. } x_{ij}=0\right)=\frac{v}{2}>\pi_{ij}\left(\left(\gamma_j,\gamma_{-j}\right)\text{ s.t. } x_{ij}=1\right)=\frac{v}{2}-c\ \forall\ 0<c<\frac{v}{2} \text{ and } \forall j\in\{1,2\}\,.$$

- for $0<c<\frac{v}{4}$, $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij}=1$. As a matter of fact, let us compare the deviation losses of $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ and $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=0$. Then, for any ij:

  $$\left(\frac{v}{2}-c-0\right)>\left(\frac{v}{2}-\left(\frac{v}{2}-c\right)\right)\Leftrightarrow 0<c<\frac{v}{4},$$

  that is, for $0<c<\frac{v}{4}$, $\left(\gamma_1,\gamma_2\right)$ such that $\gamma_j\in(0,1)$ and $\sum_{i=1}^{n_j}\mathbb{1}_{x_{ij}=1}=1\ \forall j\in\{1,2\}$ for any ij such that $x_{ij}=1$ are associated with the largest Nash difference; [10]

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[10] Here we do not employ Nash products to determine the risk-dominant equilibrium, since we define it at a single-player level and not at group level. In the latter case, for $0<c<\frac{v}{2}$ all Nash equilibria in pure strategies would be clearly equivalent in terms of risk-dominance.

- for $\frac{v}{4} < c < \frac{v}{2}$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij} = 0$. Clearly this follows from what shown at the previous point for both groups.

- for $c = \frac{v}{2}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the payoff-dominant equilibria for any player ij such that $x_{ij} = 0$, since

  $$\pi_{ij} \left((\gamma_j, 0) \text{ s.t. } x_{ij} = 0 \right) = v \ ,$$

  which is the highest attainable payoff;

- for $c = \frac{v}{2}$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ and $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$ are the risk-dominant equilibria for any ij such that $x_{ij} = 0$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 0 \ \forall j \in \{1,2\}$, $(\gamma_j, \gamma_{-j}) = (\gamma_j, 0)$ such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ for any ij such that $x_{ij} = 1$ and $(\gamma_1, \gamma_2) = (0,0)$. Then,

$$\left(\frac{v}{2} - \left(\frac{v}{2} - c \right) \right) = (v - (v - c)) > \left(\frac{v}{2} - c - 0 \right) = \left(v - c - \frac{v}{2} \right) = \left(0 - \left(\frac{v}{2} - c \right) \right) = \left(\frac{v}{2} - (v - c) \right) \ .$$

- for $c = 0$, there is no payoff-dominant equilibrium strategy profile $\forall ij \in \{1, \dots, N\}$. As a matter of fact,

  $$\pi_{ij} \left(\forall \left(\gamma_1, \gamma_2 \right) \in NE \right) = \frac{v}{2} \ .$$

- for $c = 0$, (γ_1, γ_2) such that $\gamma_j \in (0,1)$ and $\sum_{i=1}^{n_j} \mathbb{1}_{x_{ij}=1} = 1 \ \forall j \in \{1,2\}$ are the risk-dominant equilibrium strategy profiles for any ij such that $x_{ij} = 1$. As a matter of fact, let us compare the deviation losses of the set of equilibria above with the ones of $\{(\gamma_1, \gamma_2)$ such that $\gamma_j \in (0,1)$ and $\forall j \in \{1,2\}\}$ in which $x_{ij} = 0$,$\{(\gamma_j, \gamma_{-j}) = (\gamma_j, 1)$ such that $\gamma_j \in (0,1)$ and $\forall j \in \{1,2\}\}$ and $\{(\gamma_1, \gamma_2) = (1,1)\}$. Then,

  $$\left(\frac{v}{2} - c - 0 \right) > \left(\frac{v}{2} - \left(\frac{v}{2} - c \right) \right) = \left(\frac{v}{2} - c - \frac{v}{2} \right) = \left(\frac{v}{2} - \left(\frac{v}{2} - c \right) \right) = \left(\frac{v}{2} - c - \frac{v}{2} \right) \ .$$

■

10.5 Proof of Proposition 5

In the $g_1(v)$, note that $E(V|v_{ij}) = v_{ij}$, if ij observes $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$ so that $V|v_{ij} \sim U(v_{ij} - \varepsilon, v_{ij} + \varepsilon)$. Furthermore, for $v_{ij} \in [\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with support $[v_{ij} - 2\varepsilon, v_{ij} + 2\varepsilon]$. Hence, $Prob\left[V_{-ij} < v_{ij} | v_{ij}\right] = Prob[V_{-ij} > v_{ij} | v_{ij}] = \frac{1}{2} \ \forall ij \in \{1, \dots, n_j\}$ and $j \in \{1,2\}$.

Now, suppose player $ij \in \{1, \dots, n_j\} \ \forall j \in \{1,2\}$ observes $v_{ij} < 2$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \dots, n_j\} \ \forall j \in \{1,2\}$ whenever she observes $v_{ij} < 2$. Iterating this dominance argument, if players $-ij \in \bigcup_{q=1}^{2} \bigcup_{l=1}^{n_q} \{lq\} - \{ij\}$ are forced to play $x_{-ij} = 0$ whenever they observe $v_{-ij} < 2$, then player ij, observing $v_{ij} = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j - 1 + n_{-j}}$ to $\sum_{-ij \neq ij} x_{-ij} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $1 - 2^{1-n_j} + 2^{-n_{-j}}$, so that $x_{ij}=1$ can be discarded by iterated dominance for $v_{ij}=2$, since the conditionally expected payoff from exerting effort equals $2^{-n_{-j}}$. Note that we imposed by assumption that $0<\varepsilon<\left|\frac{v}{2}-1\right|$, so that $v_{ij}-2\varepsilon>\underline{v}$ for $v_{ij}=2$. Let v_{ij}^{*} be the smallest observation such that $x_{ij}=1$ cannot be excluded by iterated dominance. Then, it is possible to show that $v_{ij}^{*}=2^{n_j}$. Note that $v_{ij}=4$ is the threshold for the risk-dominance regions. As a matter of fact, when $v_{ij}=2^{n_j}$, the conditionally expected payoff from exerting effort equals

$$\mathit{Prob}\left(n_{-j}\gamma_{-j}=0\right)\cdot\left(2^{n_j}-1\right)+\mathit{Prob}\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(\frac{2^{n_j}}{2}-1\right)=$$

$$=\left(\frac{1}{2}\right)^{n_{-j}}\cdot\left(2^{n_j}-1\right)+\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(\frac{2^{n_j}}{2}-1\right)=2^{n_j-1}+2^{n_j-n_{-j}-1}-1,$$

while the conditionally expected payoff from not exerting effort equals

$$\Big(\mathit{Prob}\left(n_j\gamma_j\geq 1\right)\cdot\mathit{Prob}\left(n_{-j}\gamma_{-j}\geq 1\right)+\mathit{Prob}\left(n_j\gamma_j=0\right)\cdot\mathit{Prob}\left(n_{-j}\gamma_{-j}=0\right)\Big)\cdot\frac{2^{n_j}}{2}+$$
$$+\mathit{Prob}\left(n_j\gamma_j=0\right)\cdot\mathit{Prob}\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot(0)+\mathit{Prob}\left(n_j\gamma_j\geq 1\right)\cdot\mathit{Prob}\left(n_{-j}\gamma_{-j}=0\right)\cdot 2^{n_j}\Leftrightarrow$$
$$\Leftrightarrow\left(\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)+\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\frac{2^{n_j}}{2}+$$
$$+\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot 0+\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{1}{2}\right)^{n_{-j}}\cdot 2^{n_j}=2^{n_j-1}+2^{n_j-n_{-j}-1}-1\ .$$

The cutoff $v_{ij}^{*}=2^{n_j}$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for v_{ij} solving

$$\left(\frac{1}{2}\right)^{n_{-j}}\cdot\left(v_{ij}-1\right)+\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\left(\frac{v_{ij}}{2}-1\right)=\left(\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)+\right.$$
$$\left.+\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot\frac{v_{ij}}{2}+\left(\frac{1}{2}\right)^{n_j-1}\cdot\left(1-\left(\frac{1}{2}\right)^{n_{-j}}\right)\cdot 0+\left(1-\left(\frac{1}{2}\right)^{n_j-1}\right)\cdot\left(\frac{1}{2}\right)^{n_{-j}}\cdot v_{ij}$$

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information about the cost of effort itself. As a matter of fact, in the latter there are both a lower and an upper dominance region.

Hence, in $g_1(v)$ under incomplete information à la global games there is an equilibrium in (monotonic) cutoff strategies, such that $\forall ij\in\{1,\ldots,n_j\}$ and $\forall j\in\{1,2\}$:

$$x_{ij}^{*}\left(v_{ij}\right)=\begin{cases}1 & \text{if } v_{ij}>2^{n_j}\\ 0 & \text{if } v_{ij}\leq 2^{n_j}\ .\end{cases}$$

■

10.6 Proof of Proposition 6

In the $g_2(c)$, note that $E\left(C|c_{ij}\right)=c_{ij}$, if ij observes $c_{ij}\in\left[\underline{c}+\varepsilon,\overline{c}-\varepsilon\right]$ so that $C|c_{ij}\sim U\left(c_{ij}-\varepsilon,c_{ij}+\varepsilon\right)$. Furthermore, for $c_{ij}\in\left[\underline{c}-\varepsilon,\overline{c}+\varepsilon\right]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with support $\left[c_{ij}-2\varepsilon,c_{ij}+2\varepsilon\right]$. Hence, $\mathit{Prob}\left[C_{-ij}<c_{ij}|c_{ij}\right]=\mathit{Prob}[C_{-ij}>c_{ij}|c_{ij}]=\frac{1}{2}$.

Now, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ observes $c_{ij} > \frac{v}{2}$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ whenever she observes $c_{ij} > \frac{v}{2}$. Iterating this dominance argument, if players $-ij \in \bigcup_{q=1}^{2} \bigcup_{l=1}^{n_q} \{lq\} - \{ij\}$ are forced to play $x_{i-j} = 0$ whenever they observe $c_{-ij} > \frac{v}{2}$, then player ij, observing $c_{ij} = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j - 1 + n_{-j}}$ to $\sum_{-ij \neq ij} x_{-ij} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $\frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v$, so that $x_{ij} = 1$ can be discarded by iterated dominance for $c_{ij} = \frac{v}{2}$, since the conditionally expected payoff from exerting effort equals $2^{-n_{-j}-1} v$. Note that we imposed by assumption that $\varepsilon < \left|\frac{2\overline{c} - v}{4}\right|$, so that $c_{ij} + 2\varepsilon < \overline{c}$ for $c_{ij} = \frac{v}{2}$. Let c^*_{ij} be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $c^*_{ij} = 2^{-n_j} v$. Note that $c_{ij} = \frac{v}{4}$ is the threshold for the risk-dominance regions. As a matter of fact, when $c_{ij} = 2^{-n_j} v$, the conditionally expected payoff from exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} = 0\right) \cdot \left(v - 2^{-n_j} v\right) + Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) =$$

$$= \left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(v - 2^{-n_j} v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) = \frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v,$$

while the conditionally expected payoff from not exerting effort equals

$$\Big(Prob\left(n_j\gamma_j \geq 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} \geq 1\right) + Prob\left(n_j\gamma_j = 0\right) \cdot Prob\left(n_{-j}\gamma_{-j} = 0\right)\Big) \cdot \frac{v}{2} +$$

$$+ Prob\left(n_j\gamma_j = 0\right) \cdot Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot (0) + Prob\left(n_j\gamma_j \geq 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = 0\right) \cdot v \Leftrightarrow$$

$$\Leftrightarrow \left(\left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} +$$

$$+ \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot 0 + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n_{-j}} \cdot v = \frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v \ .$$

The cutoff $c^*_{ij} = 2^{-n_j} v$ is the unique threshold that can be established from the upper dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} solving

$$\left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(v - 2^{-n_j} v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) = \left(\left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) +\right.$$

$$\left. + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot 0 + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n_{-j}} \cdot v \ .$$

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region. Again, let us assume $\varepsilon < -\frac{\underline{c}}{2}$ and suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ observes $c_{ij} < 0$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is positive and greater than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 1$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1,2\}$ whenever she observes $c_{ij} < 0$. Iterating this dominance argument, if players $-ij \in \bigcup_{q=1}^{2} \bigcup_{l=1}^{n_q} \{lq\} - \{ij\}$ are forced to play $x_{-ij} = 1$ whenever they observe $c_{-ij} < 0$, then player ij, observing $c_{ij} = 0$ has to assign at least probability $\left(\frac{1}{2}\right)^{n_j - 1 + n_{-j}}$ to $\sum_{-ij \neq ij} x_{-ij} = N - 1$. Thus, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, will be at least $\frac{1}{2}\left(1 + 2^{-n_{-j}}\right) v$, so that $x_{ij} = 0$ can be discarded by iterated dominance for $c_{ij} = 0$, since the conditionally expected payoff from not exerting effort equals $\frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v$. Note that we imposed by assumption that $0 < \varepsilon < \left|\frac{\underline{c}}{2}\right|$, so that $c_{ij} - 2\varepsilon > \underline{c}$ for $c_{ij} = 0$. Let c_{ij}^{**} be the smallest observation such that $x_{ij} = 0$ cannot be excluded by iterated dominance. Then, it is possible to show that $c_{ij}^{**} = 2^{-n_j} v$. Note that $c_{ij} = \frac{v}{4}$ is the threshold for the risk-dominance regions. As a matter of fact, when $c_{ij} = 2^{-n_j} v$, the conditionally expected payoff from exerting effort equals

$$Prob\left(n_{-j}\gamma_{-j} = 0\right) \cdot \left(v - 2^{-n_j} v\right) + Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) =$$

$$= \left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(v - 2^{-n_j} v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) = \frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v,$$

while the conditionally expected payoff from not exerting effort equals

$$\Big(Prob\left(n_j\gamma_j \geq 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} \geq 1\right) + Prob\left(n_j\gamma_j = 0\right) \cdot Prob\left(n_{-j}\gamma_{-j} = 0\right)\Big) \cdot \frac{v}{2} +$$

$$+ Prob\left(n_j\gamma_j = 0\right) \cdot Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot (0) + Prob\left(n_j\gamma_j \geq 1\right) \cdot Prob\left(n_{-j}\gamma_{-j} = 0\right) \cdot v \Leftrightarrow$$

$$\Leftrightarrow \left(\left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{2^{n_j}}{2} +$$

$$+ \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot 0 + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n_{-j}} \cdot 2^{n_j} = \frac{1}{2}\left(1 - 2^{1-n_j} + 2^{-n_{-j}}\right) v \ .$$

The cutoff $c_{ij}^{**} = 2^{-n_j} v$ is the unique threshold that can be established from the lower dominance region by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} solving

$$\left(\frac{1}{2}\right)^{n_{-j}} \cdot \left(v - 2^{-n_j} v\right) + \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \left(\frac{v}{2} - 2^{-n_j} v\right) = \left(\left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) +\right.$$

$$\left. + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot \frac{v}{2} + \left(\frac{1}{2}\right)^{n_j - 1} \cdot \left(1 - \left(\frac{1}{2}\right)^{n_{-j}}\right) \cdot 0 + \left(1 - \left(\frac{1}{2}\right)^{n_j - 1}\right) \cdot \left(\frac{1}{2}\right)^{n_{-j}} \cdot v \ .$$

Hence, $c_{ij}^{*} = c_{ij}^{**}$ and there exists a unique equilibrium in switching strategies in $g_2(c)$ such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, 2\}$

$$x_{ij}^{*}\left(c_{ij}\right) = \begin{cases} 1 & \text{if } c_{ij} < 2^{-n_j} v \\ 0 & \text{if } c_{ij} \geq 2^{-n_j} v \ . \end{cases}$$

■

10.7 Proof of Proposition 7

In the $g_1(v)$, given the contest success function $P_j\left(X_j, X_{-j}\right) \forall j \in \{1, 2\}$, the probability of winning the prize v for group $j \in \{1, 2\}$ is: [11]

$$Prob\left(j \text{ wins } v\right) = Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 0)\right] + \frac{1}{2} Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (0, 0)\right] +$$

$$+ \frac{1}{2} Prob\left[\left(X_j^{*}, X_{-j}^{*}\right) = (1, 1)\right] \ .$$

On the other hand, the probability of winning the prize v for group $j \in \{1, 2\}$ at the cutoff equilibrium $x_{ij}^{*}\left(v_{ij}\right)$ depends on whether or not 2^{n_j} belongs to $[\underline{v} + \varepsilon, \overline{v} - \varepsilon]$, where v_{ij} is uniformly distributed. Hence, we will consider all possible cases:

---

[11] X_j^{*} $\forall j \in \{1, 2\}$ stands for the impact function of group j at the equilibrium $x_{ij}^{*}\left(v_{ij}\right) \forall ij \in \{1, \ldots, n_j\}$.

- if $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$, [12]

$$\begin{aligned}
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,0)\right] &= \left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\right]\cdot\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(0,0)\right] &= \left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,1)\right] &= \left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \; .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) &= Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,0)\right]+\frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right)=(0,0)\right]+\\
&+\frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,1)\right]\\
&=\left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\right]\cdot\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}+\\
&+\frac{1}{2}\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\cdot\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}+\\
&+\frac{1}{2}\left[1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\varepsilon-\underline{v}-\varepsilon}\right)^{n_{-j}}\right] \; .
\end{aligned}$$

- If $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $2^{n_{-j}} > \overline{v}-\varepsilon$,

$$\begin{aligned}
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,0)\right] &=1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(0,0)\right] &=\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,1)\right] &=0 \; .
\end{aligned}$$

Hence,

$$\begin{aligned}
Prob\left(j \text{ wins } v\right) &= Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,0)\right]+\frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right)=(0,0)\right]+\\
&+\frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,1)\right]\\
&=1-\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j}+\frac{1}{2}\left(\frac{2^{n_j}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_j} \quad .
\end{aligned}$$

- If $2^{n_j} > \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$\begin{aligned}
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,0)\right] &=0 \; ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(0,0)\right] &=\left(\frac{2^{n_{-j}}-\underline{v}-\varepsilon}{\overline{v}-\underline{v}-2\varepsilon}\right)^{n_{-j}} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right)=(1,1)\right] &=0 \; .
\end{aligned}$$

---

[12] Note that for $X_j^*=1$, it suffices that just one ij chooses $x_{i-j}\left(v_{i-j}\right)=1$, due to the best-shot impact function.

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) &= Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] + \frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] + \\ &+ \frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] \\ &= \frac{1}{2}\left[\left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \underline{v} - 2\varepsilon}\right)^{n_{-j}}\right] \ .\end{aligned}$$

- If $2^{n_j} > \overline{v} - \varepsilon$ and $2^{n_{-j}} > \overline{v} - \varepsilon$,

$$\begin{aligned}Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] = 0 \ ; \\ Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] = 1 \ ; \\ Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] = 0 \ .\end{aligned}$$

Hence,

$$\begin{aligned}Prob\,(j \text{ wins } v) &= Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] + \frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] + \\ &+ \frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] = \frac{1}{2} \ .\end{aligned}$$

Finally, note that $2^{n_j} \geq \underline{v} + \varepsilon \ \forall \ \underline{v} < 2$ and $0 < \varepsilon < |\frac{v}{2} - 1|$.

■

10.8 Proof of Proposition 8

First of all, in the $g_1(v)$ the expected value of the prize according to the uniform prior distribution is $E[V] = \frac{\underline{v}+\overline{v}}{2}$.
Then, let us consider all the distinct cases:

- for $\underline{v}+\varepsilon \leq 2^{n_j} \leq \overline{v}-\varepsilon$ and $\underline{v}+\varepsilon \leq 2^{n_{-j}} \leq \overline{v}-\varepsilon$,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = {} & Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) \ \cdot \\
& \cdot\, Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v}+\overline{v}}{2} - 1\right) + \\
& + Prob\,(\text{ ij receives a signal lower than or equal to } 2^{n_j}\ ) \ \cdot \\
& \cdot\, Prob\,(\text{ at least one -ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot\, Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v}+\overline{v}}{2} + \\
& + Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot\, Prob\,(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v}+\overline{v}}{4} - 1\right) + \\
& + Prob\,(\text{ ij receives a signal smaller than } 2^{n_j}\ ) \cdot \\
& \cdot\, Prob\,(\text{ at least one -ij receives a signal higher than } 2^{n_j}) \cdot \\
& \cdot\, Prob\,(\text{ at a least one i-j receives a signal higher than or equal to } 2^{n_{-j}}\ ) \cdot \\
& \cdot\, \frac{\underline{v}+\overline{v}}{4} + \\
& + Prob\,(\text{ no ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot\, Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v}+\overline{v}}{4}\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
& Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) = 1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\ , \\
& Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) = \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\ , \\
& Prob\,(\text{ ij receives a signal lower than or equal to } 2^{n_j}) = \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\ , \\
& Prob\,(\text{ at least one agent -ij receives a signal higher than } 2^{n_j}) = 1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\ , \\
& Prob\,(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) = 1 - \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \\
& Prob\,(\text{ no ij receives a signal higher than } 2^{n_j}\ ) = \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\ .
\end{aligned}
$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = & \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) + \\
& + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\right] \cdot \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \cdot \frac{\underline{v} + \overline{v}}{2} + \\
& + \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left(1 - \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right) \cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1\right) + \\
& + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left(1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\right) \cdot \left(1 - \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\right) \cdot \\
& \cdot \frac{\underline{v} + \overline{v}}{4} + \\
& + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \cdot \frac{\underline{v} + \overline{v}}{4} \ ;
\end{aligned}$$

- for $\underline{v} + \varepsilon \leq 2^{n_j} \leq \overline{v} - \varepsilon$ and $2^{n_{-j}} \geq \overline{v} - \varepsilon$,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = & \mathit{Prob}\left(\text{ ij receives a signal higher than } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ \right) \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) + \\
& + \mathit{Prob}\left(\text{ ij receives a signal lower than or equal to } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ at least one -ij receives a signal higher than } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ \right) \cdot \frac{\underline{v} + \overline{v}}{2} + \\
& + \mathit{Prob}\left(\text{ ij receives a signal higher than } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ \right) \cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1\right) + \\
& + \mathit{Prob}\left(\text{ ij receives a signal smaller than } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ at least one -ij receives a signal higher than } 2^{n_j}\right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ at a least one i-j receives a signal higher than or equal to } 2^{n_{-j}}\ \right) \cdot \\
& \cdot \frac{\underline{v} + \overline{v}}{4} + \\
& + \mathit{Prob}\left(\text{ no ij receives a signal higher than } 2^{n_j}\ \right) \cdot \\
& \cdot \mathit{Prob}\left(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ \right) \cdot \frac{\underline{v} + \overline{v}}{4} \ ,
\end{aligned}$$

where

$$Prob(\text{ ij receives a signal higher than } 2^{n_j}\ ) = 1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\ ,$$

$$Prob(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) = 1\ ,$$

$$Prob(\text{ ij receives a signal lower than or equal to } 2^{n_j}) = \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\ ,$$

$$Prob(\text{ at least one agent -ij receives a signal greater than } 2^{n_j}) = 1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\ ,$$

$$Prob(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) = 0$$

$$Prob(\text{ no ij receives a signal higher than } 2^{n_j}\ ) = \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j}\ .$$

Hence,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = & \left(1 - \frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) \\
& + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right) \cdot \left[1 - \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j - 1}\right] \cdot \frac{\underline{v} + \overline{v}}{2} + \\
& + \left(\frac{2^{n_j} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_j} \cdot \frac{\underline{v} + \overline{v}}{4}\ ;
\end{aligned}$$

- for $2^{n_j} > \overline{v} - \varepsilon$ and $\underline{v} + \varepsilon \leq 2^{n_{-j}} \leq \overline{v} - \varepsilon$,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = & Prob(\text{ ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) + \\
& + Prob(\text{ ij receives a signal lower than or equal to } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ at least one -ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v} + \overline{v}}{2} + \\
& + Prob(\text{ ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1\right) + \\
& + Prob(\text{ ij receives a signal smaller than } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ at least one -ij receives a signal higher than } 2^{n_j}) \cdot \\
& \cdot Prob(\text{ at a least one i-j receives a signal higher than or equal to } 2^{n_{-j}}\ ) \cdot \\
& \cdot \frac{\underline{v} + \overline{v}}{4} + \\
& + Prob(\text{ no ij receives a signal higher than } 2^{n_j}\ ) \cdot \\
& \cdot Prob(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v} + \overline{v}}{4}\ ,
\end{aligned}$$

where

$$\begin{aligned}
&Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) = 0\ ,\\
&Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) = \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\ ,\\
&Prob\,(\text{ ij receives a signal lower than or equal to } 2^{n_j}) = 1\ ,\\
&Prob\,(\text{ at least one agent -ij receives a signal greater than } 2^{n_j}) = 0\ ,\\
&Prob\,(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) = 1 - \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}}\\
&Prob\,(\text{ no ij receives a signal higher than } 2^{n_j}\ ) = 1\ .
\end{aligned}$$

Hence,

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \left(\frac{2^{n_{-j}} - \underline{v} - \varepsilon}{\overline{v} - \varepsilon - \underline{v} - \varepsilon}\right)^{n_{-j}} \cdot \frac{\underline{v} + \overline{v}}{4}\ ;$$

- for $2^{n_j} > \overline{v} - \varepsilon$ and $2^{n_{-j}} > \overline{v} - \varepsilon$,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = &\ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ )\ \cdot\\
&\cdot Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v} + \overline{v}}{2} - 1\right) +\\
&+ Prob\,(\text{ ij receives a signal lower than or equal to } 2^{n_j}\ )\ \cdot\\
&\cdot Prob\,(\text{ at least one -ij receives a signal higher than } 2^{n_j}\ ) \cdot\\
&\cdot Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v} + \overline{v}}{2} +\\
&+ Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) \cdot\\
&\cdot Prob\,(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \left(\frac{\underline{v} + \overline{v}}{4} - 1\right) +\\
&+ Prob\,(\text{ ij receives a signal smaller than } 2^{n_j}\ ) \cdot\\
&\cdot Prob\,(\text{ at least one -ij receives a signal higher than } 2^{n_j}) \cdot\\
&\cdot Prob\,(\text{ at a least one i-j receives a signal higher than or equal to } 2^{n_{-j}}\ ) \cdot\\
&\cdot \frac{\underline{v} + \overline{v}}{4} +\\
&+ Prob\,(\text{ no ij receives a signal higher than } 2^{n_j}\ ) \cdot\\
&\cdot Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) \cdot \frac{\underline{v} + \overline{v}}{4}\ ,
\end{aligned}$$

where

$$\begin{aligned}
&Prob\,(\text{ ij receives a signal higher than } 2^{n_j}\ ) = 0\ ,\\
&Prob\,(\text{ no i-j receives a signal higher than } 2^{n_{-j}}\ ) = 1\ ,\\
&Prob\,(\text{ ij receives a signal lower than or equal to } 2^{n_j}) = 1\ ,\\
&Prob\,(\text{ at least one agent -ij receives a signal greater than } 2^{n_j}) = 0\ ,\\
&Prob\,(\text{ at least one i-j receives a signal higher than } 2^{n_{-j}}\ ) = 0\\
&Prob\,(\text{ no ij receives a signal higher than } 2^{n_j}\ ) = 1\ .
\end{aligned}$$

Hence,

$$\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] = \frac{\underline{v} + \overline{v}}{4}\ .$$

■

10.9 Proof of Proposition 9

In the $g_2\,(c)$, given the contest success function $P_j\,(X_j, X_{-j})\,\forall j \in \{1,2\}$, the probability of winning the prize v for group $j \in \{1,2\}$ is: [13]

$$\begin{aligned}
Prob\,(j \text{ wins } v) = &Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] + \frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] +\\
&+\frac{1}{2}Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right]\ .
\end{aligned}$$

On the other hand, the probability of winning the prize v for group $j \in \{1,2\}$ at the cutoff equilibrium $x_{ij}^*\,(c_{ij})$ depends on whether or not $2^{-n_j}v$ belongs to $[\underline{c}+\varepsilon, \overline{c}-\varepsilon]$, where c_{ij} is uniformly distributed. However, note that $2^{-n_j}v > \underline{c}+\varepsilon\ \forall\ \underline{c} < 0$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall\ n_j \geq 2$. Moreover, $2^{-n_j}v < \overline{c}-\varepsilon\ \forall\ \overline{c} > \frac{v}{2}$, $0 < \varepsilon < min\left\{|\frac{2\overline{c}-v}{4}|, |\frac{\underline{c}}{2}|\right\}$ and $\forall\ n_{-j} \geq 2$. Therefore, we will consider restrict our attention to the unique possible case, i.e. $\underline{c}+\varepsilon < 2^{-n_j}v < \overline{c}-\varepsilon$ and $\underline{c}+\varepsilon < 2^{-n_{-j}}v < \overline{c}-\varepsilon$.

Accordingly,

$$\begin{aligned}
Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] &= \left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] &= \left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}} \quad ;\\
Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] &= \left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_j}\right]\cdot\left[1-\left(1-\frac{2^{n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\underline{c}-2\varepsilon}\right)^{n_{-j}}\right]\ .
\end{aligned}$$

---

[13] Note that for $X_j = 1$, it suffices that just one ij chooses $x_{ij}\,(c_{ij}) = 1$, due to the best-shot impact function.

Hence,

$$\begin{aligned}
Prob\,(j \text{ wins } v) =& Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,0)\right] + \frac{1}{2} Prob\left[\left(X_j^*, X_{-j}^*\right) = (0,0)\right] + \\
&+ \frac{1}{2} Prob\left[\left(X_j^*, X_{-j}^*\right) = (1,1)\right] \\
=& \left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} + \\
+& \frac{1}{2}\left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j} \cdot \left(1 - \frac{2^{-n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}} + \\
+& \frac{1}{2}\left[1 - \left(1 - \frac{2^{-n_j} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_j}\right] \cdot \left[1 - \left(1 - \frac{2^{n_{-j}} v - \underline{c} - \varepsilon}{\overline{c} - \underline{c} - 2\varepsilon}\right)^{n_{-j}}\right] .
\end{aligned}$$

■

10.10 Proof of Proposition 10

First of all, in the $g_2\left(c\right)$ the expected value of the cost of effort according to the uniform prior distribution is $E\left[C\right] = \frac{\underline{c}+\overline{c}}{2}$. Moreover, $2^{-n_j} v > \underline{c} + \varepsilon \; \forall \; \underline{c} < 0, 0 < \varepsilon < min\left\{\left|\frac{2\overline{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$ and $\forall n_j \geq 2$ and $2^{-n_j} v < \overline{c} - \varepsilon$ $\forall \; \overline{c} > \frac{v}{2}, 0 < \varepsilon < min\left\{\left|\frac{2\overline{c}-v}{4}\right|, \left|\frac{\underline{c}}{2}\right|\right\}$ and $\forall \; n_{-j} \geq 2$.
Then,

$$\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*, \mathbf{x}_{-j}^*\right)\right] =& Prob\,(\text{ ij receives a signal smaller than } 2^v \text{ }) \cdot \\
&\cdot Prob\left(\text{ no i-j receives a signal smaller than } 2^{-n_{-j}} v \text{ }\right) \cdot \left(\frac{v}{2} - \frac{\underline{c}+\overline{c}}{2}\right) + \\
&+ Prob\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j} v \text{ }\right) \cdot \\
&\cdot Prob\left(\text{ at least one -ij receives a signal lower than } 2^{-n_j} v \text{ }\right) \cdot \\
&\cdot Prob\left(\text{ no i-j receives a signal lower than } 2^{-n_{-j}} v \text{ }\right) \cdot \frac{v}{2} + \\
&+ Prob\left(\text{ ij receives a signal lower than } 2^{-n_j} v \text{ }\right) \cdot \\
&\cdot Prob\left(\text{ at least one i-j receives a signal lower than } 2^{-n_{-j}} v \text{ }\right) \cdot \left(\frac{v}{2} - \frac{\underline{c}+\overline{c}}{2}\right) + \\
&+ Prob\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j} v \text{ }\right) \cdot \\
&\cdot Prob\left(\text{ at least one -ij receives a signal smaller than } 2^{-n_j} v\right) \cdot \\
&\cdot Prob\left(\text{ at a least one i-j receives a signal smaller than } 2^{-n_{-j}} v \text{ }\right) \cdot \frac{v}{2} + \\
&+ Prob\left(\text{ no ij receives a signal smaller than } 2^{-n_j} v \text{ }\right) \cdot \\
&\cdot Prob\left(\text{ no i-j receives a signal smaller than } 2^{-n_{-j}} v \text{ }\right) \cdot \frac{v}{2} \text{ ,}
\end{aligned}$$

where

$$
\begin{aligned}
&Prob\left(\text{ ij receives a signal lower than } 2^{-n_j}v\ \right) = \frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\ ,\\
&Prob\left(\text{ no i-j receives a signal lower than } 2^{-n_{-j}}v\ \right) = \left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\ ,\\
&Prob\left(\text{ ij receives a signal higher than or equal to } 2^{-n_j}v\right) = 1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\ ,\\
&Prob\left(\text{ at least one agent -ij receives a signal lower than } 2^{-n_j}v\right) = 1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\ ,\\
&Prob\left(\text{ at least one i-j receives a signal lower than } 2^{-n_{-j}}v\ \right) = 1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\\
&Prob\left(\text{ no ij receives a signal smaller than } 2^{-n_j}v\ \right) = \left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[\pi_{ij}\left(\mathbf{x}_j^*,\mathbf{x}_{-j}^*\right)\right] = &\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\left(v-\frac{\underline{c}+\overline{c}}{2}\right)+\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left[1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right]\cdot\\
&\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot v+\\
&+\left(\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{-n_{-j}}\right)\cdot\left(\frac{v}{2}-\frac{\underline{c}+\overline{c}}{2}\right)+\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)\cdot\left(1-\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j-1}\right)\cdot\\
&\cdot\left(1-\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\right)\cdot\frac{v}{2}+\\
&+\left(1-\frac{2^{-n_j}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_j}\cdot\left(1-\frac{2^{-n_{-j}}v-\underline{c}-\varepsilon}{\overline{c}-\varepsilon-\underline{c}-\varepsilon}\right)^{n_{-j}}\cdot\frac{v}{2}\ .
\end{aligned}
$$

■

10.11 Proof of Proposition 11

In the $g_3(v)$, note that $E(V|v_{ij}) = v_{ij}$, if ij observes $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$ so that $V|v_{ij} \sim U(v_{ij}-\varepsilon, v_{ij}+\varepsilon)$. Furthermore, for $v_{ij} \in [\underline{v}+\varepsilon, \overline{v}-\varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around v_{ij} with support $[v_{ij}-2\varepsilon, v_{ij}+2\varepsilon]$. Hence, $Prob\left[V_{-ij} < v_{ij}|v_{ij}\right] = Prob[V_{-ij} > v_{ij}|v_{ij}] = \frac{1}{2}\ \forall ij \in \{1,\ldots,n_j\}$ and $j \in \{1,\ldots,M\}$. Moreover, let us define Q_k as the set of all subsets of cardinality k formed by groups different from j, that is $Q_k = \{-J \in \{\{1,\ldots,M\}-\{j\}\} \mid |-J| = k\}$.

Now, suppose player $ij \in \{1,\ldots,n_j\}\ \forall j \in \{1,\ldots,M\}$ observes $v_{ij} < 2$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player

$ij \in \{1, \ldots, n_j\}\ \forall j \in \{1, \ldots, M\}$ whenever she observes $v_{ij} < 2$. Iterating this dominance argument, if players $-ij \in \bigcup_{q=1}^{M} \bigcup_{l=1}^{n_q} \{lq\} - \{ij\}$ are forced to play $x_{-ij} = 0$ whenever they observe $v_{-ij} < 2$, then player ij, observing $v_{ij} = 2$ has to assign at least probability $\left(\frac{1}{2}\right)^{-1+\sum_{j=1}^{M} n_j}$ to $\sum_{-i \neq i} x_{-ij} + \sum_{-j \neq j} \sum_i x_{i-j} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $2^{-n_j+2-\sum_{-j\neq j} n_{-j}} \frac{1}{M} + 2^{-\sum_{-j \neq j} n_{-j}} \left(1 - 2^{1-n_j}\right) 2 + 2 \left(1 - 2^{1-n_j}\right) \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1}$, so that $x_{ij} = 1$ can be discarded by iterated dominance for $v_{ij} = 2$, since the conditionally expected payoff from exerting effort equals $2^{-\sum_{-j\neq j} n_{-j}} + \sum_{k=1}^{M-1} \left(\frac{2}{k+1} - 1\right) \left(1 - 2^{-n_{-j}}\right)$. Note that we imposed by assumption that $0 < \varepsilon < \left|\frac{v}{2} - 1\right|$, so that $v_{ij} - 2\varepsilon > \underline{v}$ for $v_{ij} = 2$.

Let v_{ij}^* be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $v_{ij}^* = A/B$ where

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) ,$$

$$B = 2^{1-n_j-\sum_{-j\neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} .$$

As a matter of fact, following iterated dominance, the conditionally expected payoff from exerting effort equals[14]

$$Prob\left(\sum_{-j \neq j} n_{-j}\gamma_{-j} = 0\right) \cdot (v_{ij} - 1) + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot \left(\frac{v_{ij}}{k+1} - 1\right) =$$

$$= 2^{-\sum_{-j \neq j} n_{-j}} (v_{ij} - 1) + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \left(\frac{v_{ij}}{k+1} - 1\right) ,$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(\sum_{j=1}^{M} n_j\gamma_j = 0\right) \cdot \frac{v_{ij}}{M} + Prob\left(n_j\gamma_j \geq 1\right) \cdot Prob\left(\sum_{-j \neq j} n_{-j}\gamma_{-j} = 0\right) \cdot v_{ij} +$$

$$+ Prob\left(n_j\gamma_j \geq 1\right) \cdot \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} Prob\left(n_{-j}\gamma_{-j}\right) \cdot \frac{v_{ij}}{k+1}$$

$$= 2^{-n_j+1-\sum_{-j\neq j} n_{-j}} \cdot \frac{v_{ij}}{M} + 2^{-\sum_{-j \neq j} n_{-j}} \left(1 - 2^{1-n_j}\right) \cdot v_{ij} +$$

$$+ \left(1 - 2^{1-n_j}\right) \sum_{k=1}^{M-1} \frac{v_{ij} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} .$$

Hence, by equating the two conditionally expected payoffs we obtain the cutoff value $v_{ij}^* = A/B$, as above.

The cutoff $v_{ij}^* = A/B$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for v_{ij} equating the conditionally expected payoff from exerting effort and the conditionally expected payoff from not exerting effort.

The same kind of reasoning cannot be carried out for large observations of v, since it does not exist an upper dominance region. Conversely, this is possible in our second setting in which there is incomplete information

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[14] Due the auction-type contest success function and the presence of $M \geq 2$ groups, we have to consider the possibility of a tie with up to $M - 1$ groups.

about the cost of effort itself. As a matter of fact, in the latter there are both a lower and an upper dominance region.

Hence, in $g_3(v)$ under incomplete information à la global games there is an equilibrium in (monotonic) cutoff strategies, such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:

$$x^*_{ij}(v_{ij}) = \begin{cases} 1 & \text{if } v_{ij} > v^*_j \\ 0 & \text{if } v_{ij} \leq v^*_j, \end{cases}$$

where $v^*_j = \frac{A}{B}$, with

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right),$$

$$B = 2^{1-n_j - \sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1}.$$

■

10.12 Proof of Proposition 12

In the $g_4(c)$, note that $E(C|c_{ij}) = c_{ij}$, if ij observes $c_{ij} \in [\underline{c} + \varepsilon, \overline{c} - \varepsilon]$ so that $C|c_{ij} \sim U(c_{ij} - \varepsilon, c_{ij} + \varepsilon)$. Furthermore, for $c_{ij} \in [\underline{c} - \varepsilon, \overline{c} + \varepsilon]$, the conditional distribution of the teammates' or opponents' observation will be centered around c_{ij} with support $[c_{ij} - 2\varepsilon, c_{ij} + 2\varepsilon]$. Hence, $Prob\,[C_{-ij} < c_{ij}|c_{ij}] = Prob[C_{-ij} > c_{ij}|c_{ij}] = \frac{1}{2}$. Moreover, let us define Q_k as the set of all subsets of cardinality k formed by groups different from j, that is $Q_k = \{-J \in \{\{1, \ldots, M\} - \{j\}\} \mid |-J| = k\}$.

Now, suppose player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ observes $c_{ij} > \frac{v}{2}$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij} = 1$, is smaller than the one from exerting no effort, that is choosing $x_{ij} = 0$. Accordingly, $x_{ij} = 0$ is a conditionally strictly dominant action for player $ij \in \{1, \ldots, n_j\}$ $\forall j \in \{1, \ldots, M\}$ whenever she observes $c_{ij} > \frac{v}{2}$. Iterating this dominance argument, if players $-ij \in \bigcup_{q=1}^{M} \bigcup_{l=1}^{n_q} \{lq\} - \{ij\}$ are forced to play $x_{-ij} = 0$ whenever they observe $c_{-ij} > \frac{v}{2}$, then player ij, observing $c_{ij} = \frac{v}{2}$ has to assign at least probability $\left(\frac{1}{2}\right)^{-1+\sum_{j=1}^{M} n_j}$ to $\sum_{-i \neq i} x_{-ij} + \sum_{-j \neq j} \sum_i x_{i-j} = 0$. Thus, ij's conditionally expected payoff from not exerting effort, that is choosing $x_{ij} = 0$ will be at least $2^{-n_j+1-\sum_{-j \neq j} n_{-j}} \cdot \frac{v}{M} + 2^{-\sum_{-j \neq j} n_{-j}} \left(1 - 2^{1-n_j}\right) \cdot v + \left(1 - 2^{1-n_j}\right) \sum_{k=1}^{M-1} \frac{v \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1}$, so that $x_{ij} = 1$ can be discarded by iterated dominance for $c_{ij} = \frac{v}{2}$, since the conditionally expected payoff from exerting effort equals $2^{-\sum_{-j \neq j} n_{-j}} \left(v - \frac{v}{2}\right) + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \left(\frac{v}{k+1} - \frac{v}{2}\right)$. Note that we imposed by assumption that $\varepsilon < \left|\frac{2\overline{c} - v}{4}\right|$, so that $c_{ij} + 2\varepsilon < \overline{c}$ for $c_{ij} = \frac{v}{2}$. Let c^*_{ij} be the smallest observation such that $x_{ij} = 1$ cannot be excluded by iterated dominance. Then, it is possible to show that $c^*_{ij} = (Bv)/A$, where

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right),$$

$$B = 2^{1-n_j - \sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1}.$$

As a matter of fact, following iterated dominance, the conditionally expected payoff from exerting effort equals[15]

$$Prob\left(\sum_{-j\neq j} n_{-j}\gamma_{-j}=0\right)\cdot(v-c_{ij})+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J} Prob\left(n_{-j}\gamma_{-j}\geq 1\right)\cdot\left(\frac{v}{k+1}-c_{ij}\right)=$$

$$=2^{-\sum_{-j\neq j} n_{-j}}\left(v-c_{ij}\right)+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)\left(\frac{v}{k+1}-c_{ij}\right) \ ,$$

while the conditionally expected payoff from not exerting effort equals

$$Prob\left(\sum_{j=1}^{M} n_j\gamma_j=0\right)\cdot\frac{v}{M}+Prob\left(n_j\gamma_j\geq 1\right)\cdot Prob\left(\sum_{-j\neq j} n_{-j}\gamma_{-j}=0\right)\cdot v+$$

$$+Prob\left(n_j\gamma_j\geq 1\right)\cdot\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J} Prob\left(n_{-j}\gamma_{-j}\right)\cdot\frac{v}{k+1}$$

$$=2^{-n_j+1-\sum_{-j\neq j} n_{-j}}\cdot\frac{v}{M}+2^{-\sum_{-j\neq j} n_{-j}}\left(1-2^{1-n_j}\right)\cdot v+$$

$$+\left(1-2^{1-n_j}\right)\sum_{k=1}^{M-1}\frac{v\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)}{k+1} \ .$$

Hence, by equating the two conditionally expected payoffs we obtain the cutoff value $c_{ij}^{*}=A/B$, as above. The cutoff $c_{ij}^{*}=(Bv)/A$ is the unique threshold that can be established from the lower dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} equating the conditionally expected payoff from exerting effort and the conditionally expected payoff from not exerting effort.

The same kind of reasoning can be carried out for small observations of c, since it does exist a lower dominance region. Again, let us assume $\varepsilon< -\frac{\underline{c}}{2}$ and suppose player $ij\in\{1,\ldots,n_j\}$ $\forall j\in\{1,\ldots,M\}$ observes $c_{ij}<0$. Then, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij}=1$, is positive and greater than the one from exerting no effort, that is choosing $x_{ij}=0$. Accordingly, $x_{ij}=1$ is a conditionally strictly dominant action for player $ij\in\{1,\ldots,n_j\}$ $\forall j\in\{1,\ldots,M\}$ whenever she observes $c_{ij}<0$. Iterating this dominance argument, if players $-ij\in\bigcup_{q=1}^{M}\bigcup_{l=1}^{n_q}\{lq\}-\{ij\}$ are forced to play $x_{-ij}=1$ whenever they observe $c_{-ij}<0$, then player ij, observing $c_{ij}=0$ has to assign at least probability $\left(\frac{1}{2}\right)^{-1+\sum_{j=1}^{M} n_j}$ to $\sum_{-i\neq i} x_{-ij}+\sum_{-j\neq j}\sum_{i} x_{i-j}=N-1$. Thus, ij's conditionally expected payoff from exerting effort, that is choosing $x_{ij}=1$, will be at least

$$2^{-\sum_{-j\neq j} n_{-j}}\left(v-0\right)+\sum_{k=1}^{M-1}\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)\left(\frac{v}{k+1}-0\right) \ ,$$

so that $x_{ij}=0$ can be discarded by iterated dominance for $c_{ij}=0$, since the conditionally expected payoff from not exerting effort equals

$$2^{-n_j+1-\sum_{-j\neq j} n_{-j}}\cdot\frac{v}{M}+2^{-\sum_{-j\neq j} n_{-j}}\left(1-2^{1-n_j}\right)\cdot v+\left(1-2^{1-n_j}\right)\sum_{k=1}^{M-1}\frac{v\sum_{-J\in Q_k}\prod_{-j\in -J}\left(1-2^{-n_{-j}}\right)}{k+1} \ .$$

Note that we imposed by assumption that $0<\varepsilon<\left|\frac{\underline{c}}{2}\right|$, so that $c_{ij}-2\varepsilon>\underline{c}$ for $c_{ij}=0$. Let c_{ij}^{**} be the smallest observation such that $x_{ij}=0$ cannot be excluded by iterated dominance. Then, it is possible to show

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[15] Due the auction-type contest success function and the presence of $M\geq 2$ groups, we have to consider the possibility of a tie with up to $M-1$ groups.

that $c_{ij}^{**} = (Bv)/A$, where

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \ ,$$

$$B = 2^{1-n_j - \sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} \ .$$

As a matter of fact, the conditionally expected payoff from exerting effort equals [16]

$$Prob \left(\sum_{-j \neq j} n_{-j} \gamma_{-j} = 0 \right) \cdot (v - c_{ij}) + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} Prob\left(n_{-j}\gamma_{-j} \geq 1\right) \cdot \left(\frac{v}{k+1} - c_{ij} \right) =$$

$$= 2^{-\sum_{-j \neq j} n_{-j}} \left(v - c_{ij}\right) + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \left(\frac{v}{k+1} - c_{ij} \right) \ ,$$

while the conditionally expected payoff from not exerting effort equals

$$\begin{aligned} & Prob \left(\sum_{j=1}^{M} n_j \gamma_j = 0 \right) \cdot \frac{v}{M} + Prob\left(n_j \gamma_j \geq 1\right) \cdot Prob \left(\sum_{-j \neq j} n_{-j} \gamma_{-j} = 0 \right) \cdot v + \\ & + Prob\left(n_j \gamma_j \geq 1\right) \cdot \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} Prob\left(n_{-j}\gamma_{-j}\right) \cdot \frac{v}{k+1} \\ & = 2^{-n_j + 1 - \sum_{-j \neq j} n_{-j}} \cdot \frac{v}{M} + 2^{-\sum_{-j \neq j} n_{-j}} \left(1 - 2^{1-n_j}\right) \cdot v + \\ & + \left(1 - 2^{1-n_j}\right) \sum_{k=1}^{M-1} \frac{v \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} \ . \end{aligned}$$

Hence, by equating the two conditionally expected payoffs we obtain the cutoff value $c_{ij}^{**} = (Bv)/A$, as above. The cutoff $c_{ij}^{**} = (Bv)/A$ is the unique threshold that can be established from the upper dominance regions by iterated deletion of strictly dominated strategies, since it is the unique value for c_{ij} equating the conditionally expected payoff from exerting effort and the conditionally expected payoff from not exerting effort.

Hence, $c_{ij}^{*} = c_{ij}^{**}$ and there exists a unique equilibrium in switching strategies in $g_4(c)$ such that $\forall ij \in \{1, \ldots, n_j\}$ and $\forall j \in \{1, \ldots, M\}$:

$$x_{ij}^{*}\left(c_{ij}\right) = \begin{cases} 1 & \text{if } c_{ij} < c_j^{*} \\ 0 & \text{if } c_{ij} \geq c_j^{*} \ . \end{cases}$$

where $c_j^{*} = \frac{Bv}{A}$, with

$$A = 2^{-\sum_{-j \neq j} n_{-j}} + \sum_{k=1}^{M-1} \sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right) \ ,$$

$$B = 2^{1-n_j - \sum_{-j \neq j} n_{-j}} \left(1 - \frac{1}{M}\right) + 2^{1-n_j} \sum_{k=1}^{M-1} \frac{\sum_{-J \in Q_k} \prod_{-j \in -J} \left(1 - 2^{-n_{-j}}\right)}{k+1} \ .$$

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[16] Due the auction-type contest success function and the presence of $M \geq 2$ groups, we have to consider the possibility of a tie with up to $M - 1$ groups.

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10.13 Proof of Proposition 13

Under the uniform information structure of $g_1(v)$ and $g_3(v)$, as the noise ε tends to zero,it holds for any ij:

$$\lim_{\varepsilon \to 0} Prob\left[V_{-ij} < v_{ij} | v_{ij}\right] = \lim_{\varepsilon \to 0} Prob\left[V_{-ij} > v_{ij} | v_{ij}\right] = \lim_{\varepsilon \to 0} \frac{v_{ij} - v_{ij} + 2\varepsilon}{v_{ij} + 2\varepsilon - v_{ij} + 2\varepsilon} = \frac{1}{2} \ .$$

Then, iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_1(v)$ and $g_3(v)$, as proved for propositions 5 and 11.

On the other hand, under the uniform information structure of $g_2(c)$ and $g_4(c)$, as the noise ε tends to zero,it holds for any ij:

$$\lim_{\varepsilon \to 0} Prob\left[C_{-ij} < c_{ij} | c_{ij}\right] = \lim_{\varepsilon \to 0} Prob\left[C_{-ij} > c_{ij} | c_{ij}\right] = \lim_{\varepsilon \to 0} \frac{c_{ij} - c_{ij} + 2\varepsilon}{c_{ij} + 2\varepsilon - c_{ij} + 2\varepsilon} = \frac{1}{2} \ .$$

Then, iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_1(v)$ and $g_3(v)$, as proved for propositions 6 and 12.

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10.14 Proof of Proposition 14

We first consider incomplete information about the prize, as follows:

- let V be a random variable which is uniformly distributed on R ;
- each player $ij \in \{1, \ldots, n_j\} \, \forall j \in \{1, \ldots, M\}$, with $M \geq 2$, idiosyncratically observes the realization of a random variable $V_{ij} = V + \varepsilon E_{ij}$, where E_{ij} are independent and symmetric random variables with mean zero and variance one and $\varepsilon > 0$ is a scale parameter, so that players' observation errors are independent.
- Then, the prior distribution for V is the uniform improper, so that $f(V) \propto 1$, that is it is constant over R .
- Moreover, the likelihood distribution is

  $$f\left(V_{ij} = v_{ij} | V\right) = f\left(E_{ij} = \frac{v_{ij} - V}{\varepsilon}\right) \ .$$

  Note that the likelihood function is maximized for $V = v_{ij}$, since E_{ij} is symmetric around 0.
- Using Bayes' rule, the posterior for V given v_{ij}, is proportional to the likelihood

  $$f\left(V | v_{ij}\right) \propto f\left(V_{ij} = v_{ij} | V\right) \cdot f\left(V\right) \ .$$

  Since $f(V) \propto 1$, this simplifies to

  $$f\left(V | v_{ij}\right) \propto f\left(E_{ij} = \frac{v_{ij} - V}{\varepsilon}\right) \ .$$

  The posterior is symetric around $V = v_{ij}$, so that

  $$\mathbb{E}\left[V | v_{ij}\right] = v_{ij} \ .$$

- Accordingly, to evaluate the conditional distribution of $V_{-ij}|v_{ij}$, we can write $V_{-ij} = V + \varepsilon E_{-ij}$ as

  $$V_{-ij} = (v_{ij} - \varepsilon E_{ij}) + \varepsilon E_{-ij} = v_{ij} + \varepsilon (E_{-ij} - E_{ij}) \ .$$

  Conditional on v_{ij}, the term $E_{-ij} - E_{ij}$ is symmetric around 0, since E_{-ij} and E_{ij} are symmetric independent random variables, so that $f(V_{-ij}|v_{ij})$ is symmetric around v_{ij}.

- It follows,

  $$Prob\,(V_{-ij} < v_{ij}|v_{ij}) = Prob\,(V_{-ij} > v_{ij}|v_{ij}) = \frac{1}{2} \ .$$

- Clearly, $Prob\,[V_{i-j} < v_{ij}|v_{ij}] = Prob\,[V_{i-j} > v_{ij}|v_{ij}] = \frac{1}{2}\ \forall i \in \{1, \ldots, n_{-j}\}$ and $\forall -j \neq j$.

- Iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_1(v)$ and $g_3(v)$, as proved for propositions 5 and 11.

We then consider incomplete information about the cost of effort, as follows:

- let C be a random variable which is uniformly distributed on R ;

- each player $ij \in \{1, \ldots, n_j\} \forall j \in \{1, \ldots, M\}$, with $M \geq 2$, idiosyncratically observes the realization of a random variable $C_{ij} = C + \varepsilon E_{ij}$, where E_{ij} are independent and symmetric random variables with mean zero and variance one and $\varepsilon > 0$ is a scale parameter, so that players' observation errors are independent.

- Then, the prior distribution for C is the uniform improper, so that $f(C) \propto 1$, that is it is constant over R .

- Moreover, the likelihood distribution is

  $$f(C_{ij} = c_{ij}|C) = f\left(E_{ij} = \frac{c_{ij} - C}{\varepsilon}\right) \ .$$

  Note that the likelihood function is maximized for $C = c_{ij}$, since E_{ij} is symmetric around 0.

- Using Bayes' rule, the posterior for C given c_{ij}, is proportional to the likelihood

  $$f(C|c_{ij}) \propto f(C_{ij} = c_{ij}|C) \cdot f(C) \ .$$

  Since $f(C) \propto 1$, this simplifies to

  $$f(C|c_{ij}) \propto f\left(E_{ij} = \frac{c_{ij} - C}{\varepsilon}\right) \ .$$

  The posterior is symetric around $C = c_{ij}$, so that

  $$\mathbb{E}[C|c_{ij}] = c_{ij} \ .$$

- Accordingly, to evaluate the conditional distribution of $C_{-ij}|c_{ij}$, we can write $C_{-ij} = C + \varepsilon E_{-ij}$ as

  $$C_{-ij} = (c_{ij} - \varepsilon E_{ij}) + \varepsilon E_{-ij} = c_{ij} + \varepsilon (E_{-ij} - E_{ij}) \ .$$

  Conditional on c_{ij}, the term $E_{-ij} - E_{ij}$ is symmetric around 0, since E_{-ij} and E_{ij} are symmetric independent random variables, so that $f(C_{-ij}|c_{ij})$ is symmetric around c_{ij}.

- It follows,
$$Prob\left(C_{-ij} < c_{ij}|c_{ij}\right) = Prob\left(C_{-ij} > c_{ij}|c_{ij}\right) = \frac{1}{2}\ .$$
- Clearly, $Prob\left[C_{i-j} < c_{ij}|c_{ij}\right] = Prob\left[C_{i-j} > c_{ij}|c_{ij}\right] = \frac{1}{2}\ \forall i \in \{1, \ldots, n_{-j}\}$ and $\forall - j \neq j$.
- Iterated deletion of conditionally/interim strictly-dominated strategies follows until the threshold obtained in $g_2\left(c\right)$ and $g_4\left(c\right)$, as proved for propositions 6 and 12.

■