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Dynamical analysis of an OLG model with interacting epidemiological and environmental domains

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Abstract

We study a model encompassing economic, epidemiological and environmental domains, which feature reciprocal interactions. The economy is described by an overlapping generations model in which productivity and agents' preferences are affected by the epidemiological situation. The evolution of an epidemic is represented through a susceptible-infected-susceptible model, in which the disease spread depends on the pollution level and can be reduced through the government expenditure. The pollution level increases during the production process and can be reduced by allocating resources to its abatement. Resources are collected through the capital taxation and the regulator must decide how to share them between healthcare and environmental protection. For the resulting model, we show the possible existence of a unique steady state, either characterized by the presence of epidemics or disease-free. We study its comparative statics depending on the policy parameter regulating the share of resources that is devoted to improve the epidemiological situation with respect to the environmental one. We investigate the emergence of dynamics non convergent toward the equilibrium, with possible complex and quasi-periodic trajectories.

Keywords: OLG model, Epidemiological and environmental domains, Dynamical analysis, Bifurcations.

JEL Classification: C61 , O11 , Q56

1 Introduction

During the last decades it has become increasingly evident that limiting the study of economic growth to the economic domain alone cannot provide a reliable setting to develop effective policy measures. It is crucial to adopt an integrated approach, which allows assessing the reciprocal influences of different domains. For example, the strong impact on economies of the latest pandemic highlighted the need of carefully planning suitable healthcare policies to prevent epidemic emergencies. Similarly, the environmental degradation has both negative effects on the economic growth and on health situation.

Research started to tackle these challenges by developing modelling approaches that investigate the effects of the interaction among different domains. The effect of consumption on the evolution of the environmental quality was studied in the seminal work by by John and Pecchenino (1994). Coupling an overlapping generations model (OLG) with an equation describing the evolution of the environmental quality, they described how the environmental quality is deteriorated by the consumption of the produced good. They also showed that this effect can be counteracted if the young agents invest for environmental maintenance. The improvements of environmental quality can be addressed also from the public investment perspective, through taxation (Horan et al. (1998)) and/or the promotion of environmental awareness (Constant and Davin (2019); Grassetti et al. (2024)). The literature investigates the interrelation between economic and environmental domains both from macroeconomic (Wei and Aadland (2021); Menuet et al. (2023); Cavalli et al. (2024a)) and microeconomic (Matsumoto and Szidarovszky (2020); Matsumoto et al. (2022); Naimzada and Pireddu (2023)) perspectives, taking into account dynamical aspects as well as the evolutionary approaches to green transition by Zeppini (2015); Cavalli et al. (2024b). For further contributions we refer to the surveys by Brock and Scott Taylor (2005)and Levin and Xepapadeas (2021). Similarly, it is evident that an epidemic outbreak can have severe effects on the productivity, as well as, influencing the agents expectations for the future, can affect the behavior of the agents. A first direct effect is that infected agents may not be able to work, with a negative repercussion on production and hence on economic growth. This was studied by Goenka and Liu (2012); Bell and Gersbach (2013). Goenka and Liu (2019) reconsidered the problem with respect to the risk of being locked in poverty traps. Other research strands have been devoted to study effective policies to counteract the spread of diseases (Anderson et al. (2012)), in particular through preventive measures, as Momota et al. (2005); Gori et al. (2021a,b).

The present contribution expands upon that in Cavalli et al. (2024c), where the interaction between the economic and epidemiological sides was considered, and takes inspiration from the work by Davin et al. (2022), in which the economic, epidemiological and environmental spheres are simultaneously considered. In Davin et al. (2022) the economic side is described by an OLG model. It is assumed that only healthy agents are able to work, and their saving preferences for retired age are negatively affected by the spread of epidemics. The epidemiological domain consists of a susceptible-infected-susceptible (SIS) model, in which the contact rate can be mitigated by a suitable use of resources collected from taxation. Moreover, the contact rate is negatively affected by the pollution, which is quantified through the output level, considered a proxy measure of the pollution level. Government can also issue a debt to provide subsidies for agents who are unable to work as ill. The goal of Davin et al. (2022) is to study the effectiveness of redistributive measures on the overall welfare of the population. One of the drawbacks of the resulting model is the very stylized description of the environmental side. Inherently, taxes can be used only for healthcare and no intervention to improve the environmental quality is possible. Finally, the dynamical analysis of the model is quite limited.

The aim of the present research is to overcome the previous issues. First of all, the model is enriched by a more detailed description of the dynamics of pollution, which is emitted by firms during the production process and can be partially reduced through natural absorption or by means of suitable investments in abatement. The regulator can decide how to distribute collected resources between healthcare and environment. The choice to privilege healthcare has a direct positive effect on the control of epidemic spread, with consequent benefits on the economic side, but likewise has the disadvantage to limit the pollution abatement, which in turn has a negative effect on the epidemiological side. The converse occurs for the opposite choice. The proposed setting allows addressing the problem of developing a balanced policy that takes into account all these reciprocal effects, without disregarding their dynamical consequences.

Firstly, we study a baseline model in which the government does not issue a debt and does not provide subsidies for non working agents. After showing that, as in Davin et al. (2022), a unique steady state characterized by epidemic exists together with a disease-free steady state, we focus on the effects on stability of the parameter regulating the distribution of resources between healthcare and environment. In particular, we show that a flip bifurcation can occur if the investments on healthcare are too reduced, due to instabilities arising from the epidemiological side. Conversely, a Neimark-Sacker bubbling phenomenon can take place as resources for the pollution abatement are decreased, as a consequence of instabilities arising in the environmental dynamics. We then show that the outcomes for the baseline model are reliable to describe the results obtained when debt and subsidies are taken into account.

The remainder of the manuscript is organized as follows. In Section 2 we present the general model. We start studying the limit case without subsidies in Section 3, both from the static and dynamical point of view. We then consider a particular case study for it in Section 4, together with numerical investigations. The complete model with debt and subsidies is investigated in Section 5. Conclusions and future research perspectives are reported in Section 6. Proofs of propositions are collected in Appendix.

2 The model

The baseline model is built along the lines of what is proposed by Davin et al. (2022) and consists of three interacting domains: epidemiological, economic, and environmental. Before delving into the mathematical description of each domain, we briefly outline the main constitutive elements. The evolution occurs in discrete time periods, $t \in \mathbb{N}$.

The focus is on a population composed of overlapping generations of adult and elderly agents, with each group having a constant size over time, normalized to 1. The fraction $s_t \geq 0$ of healthy adult agents works and supplies one unit of labor. Hence, the fraction i_t of infected individuals amounts to $1-s_t$. A healthy agent can become infected depending on the characteristics of the epidemic, the environmental quality, and the government's healthcare policy.

At time t, healthy adult agents produce a homogeneous output y_t, which can be used for consumption or saved as capital k_t. The preference for saving or consumption is influenced by the epidemiological situation: the better the situation, the greater the preference for saving, whereas the opposite occurs when the healthcare scenario deteriorates.

The government taxes production and allocates a fraction $\omega \in [0, 1]$ of the collected resources to healthcare, while the remaining fraction $1 - \omega$ is devoted to environmental policies. Finally, the environmental situation is described by the evolution of the pollution stock p_t, which is generated through the production process, naturally decays, and is reduced through abatement policies funded by government resources.

In what follows, we provide details on how each domain is modelled.

The environmental sphere

The pollutant stock p_t present in the environment at each time period evolves due to three factors:

1. the production process emits pollutants at a constant rate α, which is proportional to output y_t.
2. pollution naturally decays at a constant rate $\delta \in (0, 1)$.
3. public environmental expenditures contribute to pollution abatement, with the effectiveness of the abatement process measured by the constant $\lambda > 0$.

This process is described by the equation:

$$p_{t+1} = \max\{(1-\delta)p_t + \alpha y_t - \lambda(1-\omega)g_t, 0\}, \tag{1}$$

where $(1-\omega)g_t$ represents the fraction of resources allocated by the government to environmental improvement.

In modelling the environmental side, we depart from Davin et al. (2022), where pollution is simply assumed to be equal to the output level.[1] This approach allows us to account for the dynamic evolution of pollution and its influence on other spheres.

Moreover, in Davin et al. (2022), taxation was solely used to improve the epidemiological sector, whereas the introduction of equation (1) enables an investigation of the trade-off in allocating resources between healthcare and environmental policies.

Finally, we note that a zero pollution level represents the so-called "virgin state", in which the environment is completely unpolluted. This occurs when abatement is so effective that it eliminates all pollution. This also explains the use of the max function in equation (1).

The epidemiological sphere

As in Davin et al. (2022), we assume that the epidemic follows a classic SIS model. In each period, there are three generations of agents: children, adults, and retired individuals. The population size for each generation remains constant over time and is equal to N. When children come into contact with other generations, they can become susceptible or infected upon reaching adulthood.[2]

The number of infected (respectively susceptible) adults and retired agents is the same and is denoted by I_t (respectively S_t). Thus, at each time t, we have $2N = 2I_t + 2S_t$, from which we define the fractions $s_t = S_t/N$ and $i_t = I_t/N$, representing the proportions of susceptible and infected agents, respectively. Since the entire adult and retired population is either susceptible or infected, it follows that $i_t + s_t = 1$.

The infection rate θ depends on healthcare policies and the environmental situation. Specifically, it is modelled as a decreasing function $\theta : [0, +\infty) \to (0, +\infty)$ that depends on the ratio $\omega g_t/p_t$, where ωg_t represents the portion of tax revenue allocated to public healthcare. A higher value of this ratio leads to a lower infection (or contact) rate θ, slowing the spread of the disease. Conversely, the inverse dependence on the pollution stock p_t reflects the worsening health conditions as environmental pollution increases.

The resulting SIS model is given by

$$\begin{cases} s_{t+1} = s_t \left(1 - \theta\left(\frac{\omega g_t}{p_t}\right) i_t\right) + \gamma i_t \\ i_{t+1} = (1-\gamma)i_t + \theta i_t s_t \\ s_0, i_0 > 0, \quad s_0 + i_0 = 1. \end{cases} \tag{2}$$

where $0 < \gamma \leq 1$ is the recovery rate.

To ensure the positivity of the trajectories, we impose the condition $\theta\left(\frac{\omega g_t}{p_t}\right) \in \left(0, \left(1+\sqrt{\gamma}\right)^2\right)$ (see Allen (1994)). Finally, we note that only one of the two equations in (2) is needed to model the evolution of the epidemic. Therefore, in what follows, we focus on the equation describing the dynamics of s_t, from which the number of infected agents can be obtained using $i_t = 1 - s_t$.

[1] More precisely, it corresponds to y_{t+1}. Unlike Davin et al. (2022), we assume that emissions are proportional to y_t. The resulting pollution dynamics in equation (1) align with the related existing literature (see, e.g., Cavalli et al. (2024a) and the references therein).

[2] For more details, we refer the interested reader to Section 2.2 in Davin et al. (2022).

The economic sphere

The economy is described by an OLG model with production, along the lines of that in Davin et al. (2022). The main difference from the model in Davin et al. (2022) lies in the alternative characterization of public intervention, following the framework in Cavalli et al. (2024c).

Every adult $i \in \{1, \ldots, N\}$ decides how to allocate consumption between adulthood $c_{i,t}$ and retirement $d_{i,t+1}$ by solving the following maximization problem

$$\begin{cases} \max_{c_{i,t}, d_{i,t+1}} u(c_{i,t}, d_{i,t+1}) = \max_{c_{i,t}, d_{i,t+1}} (\ln c_{i,t} + \beta(s_t) \ln d_{i,t+1}) \\ \text{subject to} \\ \sigma_{i,t} + c_{i,t} = \Omega_{i,t} + \tau_{i,t} \\ d_{i,t+1} = \frac{r_{t+1}}{\beta(s_t)} \sigma_{i,t} \end{cases} \tag{3}$$

where function $u(c_{i,t}, d_{i,t+1})$ captures the preferences of agent i and $\beta : [0, 1] \to (0, 1]$ the survival probability, which in turn influences the willingness to save for old age. Function β is increasing ($\beta'(x) \geq 0$ for any $x \in [0, 1]$) and concave[3]. Utility is maximized subject to the budget constraints given by the last two conditions in (3). Moreover, in the former constraint in (3), $\Omega_{i,t}$ represents the labour income while $\tau_{i,t}$ is the subsidy paid by the government. If adult agent i is able to work, we have $\Omega_{i,t} = w_t > 0$ and $\tau_{i,t} = 0$. Conversely, if adult agent i is unable to work and hence $\Omega_{i,t} = 0$, we assume that the government supplies an amount $\tau_{i,t} > 0$, which represents a subsidy and can be interpreted as a health insurance or a paid sick leave. The labour/subsidy income is split between savings $\sigma_{i,t}$ and consumption $c_{i,t}$, while the latter constraint expresses the relationship between savings at period t and the consumption at period $t+1$. Specifically, future consumption is directly proportional to the marginal productivity of capital r_{t+1} and inversely proportional to the survival probability.

Solving (3) provides

$$c_{i,t} = \frac{1}{1 + \beta(s_t)} (\Omega_{i,t} + \tau_{i,t}), \ \sigma_{i,t} = \frac{\beta(s_t)}{1 + \beta(s_t)} (\Omega_{i,t} + \tau_{i,t}), \ d_{i,t+1} = \frac{r_{t+1}}{1 + \beta(s_t)} (\Omega_{i,t} + \tau_{i,t}).$$

We assume for firms the neoclassical production function with constant return to scale

$$Y_t(L_t, K_t) = A(s_t) L_t^{1-a} K_t^a,$$

where factor inputs L_t and K_t are respectively the labour and the capital, $a \in (0, 1)$, and increasing function $A : [0, 1] \to (0, 1]$ represents the normalized productivity factor. The per capita production function is obtained as

$$y_t(l_t, k_t) = A(s_t) l_t^{1-a} k_t^a,$$

where factor inputs l_t and k_t are respectively the labour and the capital per capita. In line with Davin et al. (2022), we assume that only healthy adult agents work, so $l_t = s_t$, and that productivity is increasing with respect to the fraction of healthy people[4]. If the whole adult population is healthy, we assume $A(1) = 1$.

Total factor productivity (TFP) changes with respect to the number of workers, as discussed in Davin et al. (2022) and its references. However, the relationship between A and s_t is complex and difficult to estimate accurately, as it involves the impact of

[3] Unlike the present model, in Davin et al. (2022) the utility function depends on $\beta(s_{t+1})$. This would imply that agents anticipate the future evolution of the epidemic when making decisions.In contrast, we assume a more realistic scenario where agents only observe the current epidemic situation and base their consumption and saving choices on that The constraint regarding $d_{i,t+1}$ is changed accordingly.

[4] We note that, like in Davin et al. (2022), in (3) we do not take into account a direct negative effect of pollution on productivity. This aspect is left to future developments of the model.

changes in human capital on TFP[5]. Firms can consequently face uncertainty when trying to approximate it, so we assume that firms can accurately estimate TFP but are unable to estimate marginal TFP. Therefore, for small variations in s_t, firms treat TFP as constant, and we adopt the same framework as Cavalli et al. (2024c). Note that the explicit expressions for function $A(s_t)$ used in Davin et al. (2022); Cavalli et al. (2024c) and in the present work exhibits constant behavior over certain ranges.

The government charges taxes on the firms at a rate $\tau \in [0, 1]$, which is assumed exogenous and constant in time. The maximization of firm profits, which correspond to

$$\pi(s_t, k_t) = (1-\tau)y_t(s_t, k_t) - r_t k_t - w_t s_t,$$

provides the marginal productivity of capital and the wage[6]

$$r_t = (1-\tau)aA(s_t)\left(\frac{s_t}{k_t}\right)^{1-a}, \quad w_t = (1-\tau)(1-a)A(s_t)\left(\frac{s_t}{k_t}\right)^{-a}$$

Now we detail the modelling of the subsidy. It is reasonable to link its amount to the economic performance. For this reason, we consider a constant, basic level of subsidy, described by an exogenous parameter $\tau^{ex} > 0$, which is integrated by an amount corresponding to a fraction of the resources obtained from taxation. This latter contribution is described by $\tau^{end}Y_t$, in which $0 \leq \tau^{end} < \tau$, and the goal is to provide the fraction $\tau^{end}Y_t/N$ to each infected adult, with an overall expenditure that then amounts to $\tau^{end}Y_t I_t/N = \tau^{end}Y_t i_t$. For each infected agent the government allocates a subsidy equal to $\tau^{ex} + \tau^{end}Y_t/N$.

If we take into account the possibility to issue a time constant debt $b > \tau^{ex}$ the government budget constraint can be written as

$$B_{t+1} + \tau Y_t = r_t B_t + G_t + \tau^{ex} I_t + \tau^{end} Y_t i_t,$$

where $r_t B_t$ corresponds to the interest on public debt and G_t are the public expenditures. We can obtain the residual resources g_t, net of subsidies, that the government can allocate to healthcare and environment as

$$g_t = b(1-r_t) + \tau y_t - \tau^{ex}(1-s_t) - \tau^{end} y_t(1-s_t), \tag{4}$$

which corresponds to $g_t = b(1-r_t) + \tau A(s_t)s_t^{1-a}k_t^a - \tau^{ex}(1-s_t) - \tau^{end}A(s_t)s_t^{1-a}k_t^a(1-s_t)$.

Finally, the intertemporal equilibrium condition becomes $k_{t+1} + b = \sum_{i=1}^{N} \sigma_{i,t}/N$, from which we obtain

$$\begin{aligned} k_{t+1} =& \frac{\beta(s_t)}{1+\beta(s_t)}\left[(1-\tau)(1-a)A(s_t)s_t^{1-a}k_t^a \right. \\ & \left. + \tau^{ex}(1-s_t) + \tau^{end}A(s_t)s_t^{1-a}k_t^a(1-s_t)\right] - b \end{aligned} \tag{5}$$

We note that $\beta(s_t)/(1+\beta(s_t))$ represents the saving propensity (see Chakraborty (2004)). Finally, in the remainder of the paper, we assume that functions θ, β and A are sufficiently regular to be able to compute all the involved derivatives. For this, since up to third order derivatives are involved, it is sufficient to have $\theta, \beta, A \in C^3$, at least almost everywhere on their domains.

[5] Although the effect of human capital on TFP is well-established (see, e.g., Miller and Upadhyay (2000)), it is hardly realistic to assume that firms have precise knowledge of how changes in human capital affect TFP. A variation of s_t determines in turn a change in the human capital of the workforce, and hence it influences TFP through alterations in the stock of knowledge and skills, social interactions, and individual capabilities. For an investigation on the methods used to measure the impact of human capital on TFP, we refer to Männasoo et al. (2018), which also outlines the challenges in this process.

[6] Accordingly to the assumption of the knowledge of $A(s_t)$ by firms, we have that they act as if $A'(s_t) = 0$.

3 Analysis of the baseline model without subsidies

We study the static and dynamical properties of the model in which $\tau^{ex} = \tau^{end} = b = 0$. This represents the limit situation in which infected agents, who are not able to work, do not have any income during the second period of their life. The reason for which we start focusing on this scenario is related to the analytical tractability of the resulting model. Moreover, in Section 5 we reconsider the model by taking into account government debt and subsidies, showing that the outcomes found for the baseline model still hold true for the improved one.

The resulting model is described by function $M : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,1] \times [0,+\infty) \times [0,+\infty), \boldsymbol{\xi}_t = (s_t, p_t, k_t) \mapsto M(s_t, p_t, k_t)$ defined by means of the first equation in (2) (in which i_t is replaced by $1 - s_t$), and equations (5) and (1), so that we have

$$\begin{cases} s_{t+1} = M_1(\boldsymbol{\xi}_t) = s_t \left[1 - \theta\left(\frac{\omega g_t}{p_t}\right)(1 - s_t)\right] + \gamma(1 - s_t) \\ k_{t+1} = M_2(\boldsymbol{\xi}_t) = \frac{\beta(s_t)}{1+\beta(s_t)}(1 - \tau)(1 - a)A(s_t)s_t^{1-a}k_t^a \\ p_{t+1} = M_3(\boldsymbol{\xi}_t) = (1 - \delta)p_t + \alpha A(s_t)s_t^{1-a}k_t^a - \lambda(1 - \omega)g_t \end{cases} \tag{6}$$

in which we introduced functions $M_i, i = 1, 2, 3$ to represent each component of $M(s_t, p_t, k_t)$. We note that even if the dynamics of the stock of pollution can become null, we avoid to focus on this case, studying situations characterized by $p_t > 0$. The reason is that the occurrence of the virgin state is quite unrealistic, and would presume quite extreme conditions to realize. In what follows, we will analytically take into account the pollution positivity in the steady state investigation, while from the dynamical point of view we will keep track of $p_t > 0$ in the numerical simulations. Finally note that if $p_t > 0$, the argument of function θ is well-defined.

3.1 Static analysis

In this section we study the possible steady states $\boldsymbol{\xi}^* = (s^*, p^*, k^*)$ of (6) and how they change with respect to the most relevant parameters of the model. Consistently with the possible steady states of the SIS model, we speak of endemic steady state if $s^* > 0$ and of disease-free steady state if $s^* = 1$. We avoid to discuss steady states characterized by null capital level, since they are economically irrelevant.

To formulate the results, in what follows we make use of

$$\eta = \frac{\omega g}{p} = \frac{\omega \tau A(s)s^{1-a}k^a}{p} \tag{7}$$

which, as in Davin et al. (2022), represents the relative government expenditure for healthcare with respect to the pollution level. Moreover, we define

$$\eta^* = \frac{\delta \omega \tau}{\alpha - \lambda(1 - \omega)\tau}. \tag{8}$$

Proposition 1 *Model (6) always has a disease-free steady state* $\boldsymbol{\xi}^*_{df} = (s^*_{df}, p^*_{df}, k^*_{df})$

$$\begin{cases} s^*_{df} = 1, \\ k^*_{df} = \left[(1 - \tau)(1 - a)\frac{\beta(1)}{1+\beta(1)}\right]^{\frac{1}{1-\alpha}}, \\ p^*_{df} = \frac{\alpha - \lambda(1-\omega)\tau}{\delta}(k^*_{df})^a. \end{cases}$$

provided that

$$\alpha - \lambda(1 - \omega)\tau > 0. \tag{9}$$

If $\theta(\eta^*) > \gamma$ and if (9) holds, there exists a unique endemic steady state $\boldsymbol{\xi}^* = (s^*, p^*, k^*)$ with positive components at which

$$\begin{cases} s^* = \frac{\gamma}{\theta(\eta^*)}, \\ k^* = \left[(1-\tau)(1-a)\frac{\beta(s^*)}{1+\beta(s^*)}A(s^*)\right]^{\frac{1}{1-a}} s^*, \\ p^* = \frac{\alpha-\lambda(1-\omega)\tau}{\delta}A(s^*)(s^*)^{1-a}(k^*)^a. \end{cases} \tag{10}$$

If $\theta(\eta^*) \leq \gamma$, no endemic steady state exists.

Note that η^* in (8) actually corresponds to the value of (7) when the system is at an endemic steady state.

The disease-free steady state always exists in case (9) holds, and can coexist with a unique endemic steady state. We note that the result of Proposition 1 is in line with that obtained by Davin et al. (2022), while much more complicated steady state scenarios can arise in the simplified setting studied by Cavalli et al. (2024c), in which the environmental side was not considered. The motivation of this can be ascribed to the influence of the pollution level on the contact rate, in particular to the same specific way it is described in both (6) and by Davin et al. (2022). The effect is to eliminate the possibility to have more than one single endemic steady state[7]. The existence of the endemic steady state is guaranteed under two conditions. The former one, $\theta(\eta^*) > \gamma$, is a generalization of the similar requirement in the classic SIS model, while the latter one, $\alpha - \lambda(1-\omega)\tau > 0$, allows preserving the positivity of p^*.

Since the focus of the present research is on the role of the policy parameter ω, we restrict to situations in which we can study its behavior on its whole range of values. For this reason, in what follows we always assume that (9) holds true for any $\omega \in [0, 1]$, which requires $\alpha > \lambda\tau$. We note that this condition is automatically fulfilled if $\alpha > \lambda$, which is an agreeable and realistic setting, since it means that the rate of abatement is smaller than the emission rate. So we make the following assumption

Assumption 1 The rate of abatement is smaller than the rate of emission of new pollutant, i.e. $\alpha > \lambda$.

From now on, all the results are presented and proved under Assumption 1.

In the next proposition we study comparative statics of $\boldsymbol{\xi}^*$. To this end, let us define $g^* : [0, 1] \to [0, 1]$ as

$$g^*(s) = \tau(A(s))^{\frac{1}{1-a}} \left((1-\tau)(1-a)\frac{\beta(s)}{1+\beta(s)}\right)^{\frac{a}{1-a}} s \tag{11}$$

which represents the government expenditure when the economic domain is at the steady state k^* depending on an exogenous fraction s of susceptible agents[8]. In what follows, for a given function f depending on a variable x we denote by $E_f(x)$ the elasticity of f at x. Let us introduce function $E_\theta(g)$, defined where θ is differentiable and representing the elasticity of θ at g, i.e.

$$E_\theta(g) = \frac{g\theta'(g)}{\theta(g)}.$$

[7]In particular, by simulative investigations, the endemic steady states that were observed in the economic-epidemiological model in Cavalli et al. (2024c) and that are ruled out when the environmental side is considered are those characterized by higher levels of susceptible agents. The effect of pollution on the epidemic spread is then to select the least desirable one.

[8]Expression (11) can be obtained by replacing in (4) the equilibrium expression of k for a generic s (see the second equation in (10) or (A2)).

In the suitable domain, we can also define

$$E_{\frac{\beta}{1+\beta}}(s) = \frac{s\left(\frac{\beta(s)}{1+\beta(s)}\right)'}{\frac{\beta(s)}{1+\beta(s)}} = \frac{s\beta'(s)}{\beta(s)(\beta(s)+1)}, \quad E_A(s) = \frac{sA'(s)}{A(s)},$$

respectively representing the elasticity of the saving propensity and of the total factor productivity with respect to the fraction of susceptible agents. Similarly, function $E_{g^*}(s)$, defined where g^* is differentiable and representing the elasticity of g^* at s, can be written

$$\begin{aligned} E_{g^*}(s) = \frac{sg^{*\prime}(s)}{g^*(s)} &= \frac{a}{1-a}\frac{\beta'(s)}{s\beta(s)(1+\beta(s))} + \frac{1}{1-a}\frac{sA'(s)}{A(s)} + 1 \\ &= \frac{a}{1-a}E_{\frac{\beta}{1+\beta}}(s) + \frac{1}{1-a}E_A(s) + 1 \end{aligned} \tag{12}$$

We remark that since θ is strictly decreasing, we have $E_\theta(g) < 0$ for $g > 0$, while since both β and A are non decreasing, we have $E_{g^*}(s) \geq 1$.

Proposition 2 *Let us consider the endemic steady state $\boldsymbol{\xi}^* = (s^*, p^*, k^*)$. Under Assumption 1, on increasing ω, we have that s^*, k^* and p^* increase.*

The main outcome of Proposition 2 is that, in the present setting, by moving resources from environmental protection to healthcare necessarily induces a deterioration of the environmental quality for an improvement of the epidemiological situation and of the economic growth. This latter aspect is an effect of the direct negative influence of the disease spread on the productivity. As a consequence, balanced welfare concerns should bear in mind the trade-off effects of the distribution of resources.

3.2 Dynamical analysis

We recall (see e.g. Allen (1994)) that in the SIS model with exogenous contact rate the endemic steady state is locally asymptotically stable provided that $\theta - \gamma < 2$. Similarly, if s and k are assumed exogenous in the dynamical equation for the pollution, the unique steady state is always locally asymptotically stable.

Concerning the stability of the disease-free steady state we have the following result, which basically confirms what happens in the SIS model.

Proposition 3 *Let $\gamma > \theta(\eta^*)$ so that the endemic steady state $\boldsymbol{\xi}^* = (s^*, p^*, k^*)$ does not exist. We then have that $\boldsymbol{\xi}^*_{df}$ is locally asymptotically stable.*

We remark that when conversely the endemic steady state exists, $\boldsymbol{\xi}^*_{df}$ is unstable. Now we turn our attention to the stability of the endemic steady state. We note that if the endemic steady state exists for $\omega = 0$ (i.e. if $\gamma < \theta(0)$), since as ω increases, we have that η^* increases and hence $\theta(\eta^*)$ decreases, so condition $\gamma < \theta(\eta^*)$ may be violated for suitably large ω. This leads to the disappearance of $\boldsymbol{\xi}^*$ (which actually becomes unfeasible, with $s^* \geq 1$) and $\boldsymbol{\xi}^*_{df}$ becomes stable. This behavior actually recalls that of a transcritical bifurcation, which will be confirmed by the numerical simulations of the next sections.

Proposition 4 *Let $\gamma < \theta(\eta^*)$ so that the endemic steady state $\boldsymbol{\xi}^* = (s^*, p^*, k^*)$ exists. Under Assumption 1, we have that $\boldsymbol{\xi}^*$ is locally asymptotically stable provided that*

$$\begin{cases} c_{11}(\theta(\eta^*) - \gamma) + c_{10} > 0 \\ c_{22}(\theta(\eta^*) - \gamma)^2 + c_{21}(\theta(\eta^*) - \gamma) + c_{20} > 0 \\ c_{31}(\theta(\eta^*) - \gamma) + c_{30} > 0 \end{cases} \tag{13}$$

where

$$\begin{aligned}
c_{11} &= -(a+1)(2-\delta) - 2E_\theta(\eta^*)\left(E_{g^*}\left(\frac{\gamma}{\theta(\eta^*)}\right)(1-a) - 2aE_{\beta/1+\beta}\left(\frac{\gamma}{\theta(\eta^*)}\right)\right),\\
c_{10} &= 2(a+1)(2-\delta),\\
c_{22} &= a\left(1-\delta-E_\theta(\eta^*)E_{\beta/1+\beta}\left(\frac{\gamma}{\theta(\eta^*)}\right)\right)\left(a\delta+1-a+(1-a)E_\theta(\eta^*)E_{g^*}\left(\frac{\gamma}{\theta(\eta^*)}\right)\right)\\
c_{21} &= (1-a)E_\theta(\eta^*)\left((1-a+a\delta)E_{g^*}\left(\frac{\gamma}{\theta(\eta^*)}\right) - a\delta E_{\beta/1+\beta}\left(\frac{\gamma}{\theta(\eta^*)}\right)\right)\\
&\quad + 4a\delta - \delta - 2a - a\delta^2 - 3a^2\delta + a^2 + 2a^2\delta^2 + 1\\
c_{20} &= \delta(1-a)(a\delta+1-a),\\
c_{31} &= 1+a-\delta+(1-a)E_\theta(\eta^*)\left(E_{g^*}\left(\frac{\gamma}{\theta(\eta^*)}\right) - \frac{2a}{1-a}E_{\beta/1+\beta}\left(\frac{\gamma}{\theta(\eta^*)}\right)\right),\\
c_{30} &= 3-(1+a)(1-\delta)-a.
\end{aligned}$$

A flip bifurcation can occur only when the first condition in (13) is violated, while a Neimark-Sacker bifurcation can occur only when the second condition in (13) is violated.

The expressions of stability conditions (13) are very convoluted, as the involved coefficients c_{ij} can depend on η^* and it is not possible to make explicit the role of each parameter, in particular of ω. However, some remarks are possible. Firstly, we recall that the last condition in (13) is not involved in the possible emergence of bifurcations, so we avoid to take it into account it (see e.g. Lines et al. (2020), in which an equivalent formulation of this condition is considered).

A key point is that, as ω changes, the main source of a potential bifurcation is the change in the contact rate $\theta(\eta^*)$. This may suggest that, in the present setting, instabilities arise from the epidemiological side and are then transmitted to the other domains. This can be directly ascribed to the presence of the environmental side.[9] Let us focus on the first condition in (13). The role of the contact rate $\theta(\eta^*)$ can be amplified or damped by its elasticity and that of the government expenditure and of the saving propensity. Let us assume constant total factor productivity and survival probability. If we consider an exogenous contact rate, we find the same stability condition related to the SIS model. If θ is endogenous, $\theta(\eta^*) - \gamma$ decreases as ω increases. In the case of an isoelastic contact rate[10] we have that c_{11} is constant and increasing ω we can have that the first condition in (13) is either always/never fulfilled or it becomes true for sufficiently large ω, which is then stabilizing. Indeed, this behavior may be altered in the case of general contact rate functions or by endogenous A and β.

Similarly, the second condition in (13) seems to suggest that ω can lead to a double stability change for the endemic steady state[11] even if also in this case the role of coefficients c_{22} and c_{21} may alter the occurrence of this scenario. Finally, we note that for $\theta(\eta^*) - \gamma = 0$, both conditions in (13) are fulfilled, and this guarantees that the endemic steady state must become stable as it sufficiently approaches the disease free one. This

[9] In studying dynamical properties, Davin et al. (2022) imposed restrictive conditions in order to obtain stable steady states, since their goal is to focus on this situation. For this reason, and for the introduction in the present contribution of a detailed description of the environmental side, it is awkward to compare the dynamical outcomes of the two models.

[10] Due to the upper bound on the contact rate to preserve the positivity of trajectories, function θ can be isoelastic just for $\eta^* > \bar{\eta}$ with $\theta(\bar{\eta}) < (1 + \sqrt{\gamma})^2$. However, it is possible to consider a piecewise isoelastic contact rate $\theta(\eta) = \min\{\theta_0, \theta_0/\eta^\xi\}$, to which we can apply the next discussion.

[11] For example by considering the same setting sketched for the first stability condition, with isoelastic θ and exogenous A and β.

means that if $\theta(\eta^*) - \gamma = 0$ occurs for some $\omega_{tr} \in (0,1)$, we have that $\boldsymbol{\xi}^*$ is stable on a left neighborhood of ω_{tr}. In the next sections, we look for simplified settings that provide sufficient conditions on the endogenous elements of the model for the occurrence of at most one stability change (respectively two stability changes) arising from the first (respectively second) condition in (13). To this end we will deal with a case of study, in view of which we reformulate Proposition 4 for scenarios of increasing complexity.

4 Case study with increasing complexity

In this section we examine stability by subsequently introducing one by one the endogenous effects. The goal is to obtain specialized versions of Proposition 4 that allow investigating the role of each sphere in being the source of instabilities. Moreover, for each framework, we provide sufficient conditions for which the arising scenarios are those discussed after Proposition 4.

4.1 Endogenous $\boldsymbol{\theta}$

When A and β are constant, conditions (13) simplify as follows. Since the calculations are straightforward, we do not provide a proof.

Corollary 5 *Under Assumption 1, if $A(s) \equiv A$ and $\beta(s) \equiv \beta$ and $\gamma < \theta(\eta^*)$ conditions (13) become*

$$\begin{cases} E_\theta(\eta^*) < \dfrac{(a+1)(2-\delta)[2-(\theta(\eta^*)-\gamma)]}{2(1-a)(\theta(\eta^*)-\gamma)} \\ E_\theta(\eta^*) > -\dfrac{(1-a+a\delta)[1-a+a(\theta(\eta^*)-\gamma)][\delta+(1-\delta)(\theta(\eta^*)-\gamma)]}{(\theta(\eta^*)-\gamma)(1-a)[1-a(1-\delta)+a(1-\delta)(\theta(\eta^*)-\gamma)]} \\ E_\theta(\eta^*) > \dfrac{(1+a)(1-\delta)+a-3-(1+a-\delta)(\theta(\eta^*)-\gamma)}{(1-a)(\theta(\eta^*)-\gamma)} \end{cases} \tag{14}$$

We remark that, since total factor productivity does not depend on s_t, the marginal total factor productivity is null, as assumed by the firms, and therefore this case of study is not affected by such an element of uncertainty. Note that the right hand side of the first condition in (14) is positive when $\theta(\eta^*) - \gamma < 2$, which corresponds to the stability conditions of the endemic steady state in the SIS model, while the left hand side is negative. This means that when the endemic state is stable in an SIS model characterized by a contact rate $\theta_0 = \theta(0)$, the first condition in (14) is fulfilled for any ω. However, since the right hand side in the second inequality in (14) is negative, the stability condition related to the emergence of a Neimark-Sacker can be violated also when the epidemiological sphere is not source of instabilities. Depending on θ, many different scenarios can arise when (14) is violated. A simple situation is described in the next proposition. In what follows, we denote bifurcation thresholds for ω of potential transcritical, flip and Neimark-Sacker bifurcations by subscripts tr, f and ns.

Proposition 6 *If $\theta'(\eta) \neq 0$, $E'_\theta(\eta) < 0$ and*

$$E''_\theta(\eta) > E'_\theta(\eta)\left(\frac{\theta''(\eta)}{\theta'(\eta)} - \frac{2\theta'(\eta)}{\theta(\eta)-\gamma}\right), \tag{15}$$

then we have that depending on the parameters defining the model and function θ, the endemic steady state $\boldsymbol{\xi}^*$ for $\omega \in [0, \omega_{tr}] \cap [0,1]$ can incur

- *at most one period halving bifurcation at ω_f;*
- *at most a couple of Neimark-Sacker bifurcations at $\omega_{1,ns}, \omega_{2,ns}$.*

Different sequences of bifurcations are possible, and, increasing ω, are characterized by a recover of stability at ω_f and $\omega_{2,ns}$, and a loss of stability at $\omega_{1,ns}$.

From Proposition 6, we have that if at the equilibrium the elasticity of the contact rate with respect to the relative government expenditure is decreasing and "not too concave", we can have up to three stability inversions for $\boldsymbol{\xi}^*$, with a possible final transcritical bifurcation. We note that ω_{tr} is the threshold at which the endemic steady state can disappear and disease free steady state can become stable; ω_f is the threshold value at which a period-halving bifurcation can take place, while we can have up to two Neimark-Sacker bifurcations at $\omega_{1,ns}$ and $\omega_{2,ns}$.

To help in the discussion of these scenarios, we also rely on numerical simulations. To this end we introduce

$$\theta(\eta) = \theta_0 e^{-\theta_1 \eta^{\theta_2}}, \tag{16}$$

where $\theta_0, \theta_2 > 0$ and $\theta_1 \geq 0$. We note that setting $\theta_1 = 0$ we have an exogenous contact rate θ_0, which we consider as a limit case. In what follows we focus on $\theta_1 > 0$, in which case θ_0 represents the maximum possible contact rate, occurring when the government expenditure is null. Parameters θ_1 and θ_2 determine the steepness and concavity of function θ. Function (16) fulfills the requirements of Proposition 6, as shown in the next corollary.

Corollary 7 *Let θ be defined by (16). For $\eta \in \left(0, \min\left\{\frac{\delta\tau}{\alpha}, \theta^{-1}\{\gamma\}\right\}\right)$, we have that $E_\theta(\eta)$ is strictly decreasing and fulfills condition (15).*

We start investigating the occurrence of all the possible[12] scenarios arising from Proposition 6 by means of numerical investigations using function (16). In what follows, we set $A = 1, \beta = 1, \theta_1 = 1, a = 0.3, \gamma = 0.999, \alpha = 0.1, \delta = 0.3$ and $\lambda = 0.075$. In Figure 1 we report in the first row the two dimensional bifurcation diagrams in (ω, τ) plane. The color of each point depends on the cardinality of the attractor reached with the corresponding parameter combination. In particular, white color is used for convergence toward the steady state, blue color highlights the occurrence of a period-2 cycle and so on, with cyan color that points out either a very large period cycle or quasi-periodic/chaotic dynamics. Crossing the dashed black line in the upper right parts of the panels, $\boldsymbol{\xi}^*$ disappears and $\boldsymbol{\xi}^*_{df}$ becomes stable, and hence in white regions to its left and below it the trajectories converge to $\boldsymbol{\xi}^*$, while otherwise convergence is toward $\boldsymbol{\xi}^*_{df}$. Below each two dimensional bifurcation diagram we report three examples of one dimensional bifurcation diagrams related to it, obtained for different values of τ on increasing ω.

The simulations reported in the first column of Figure 1 are obtained for a large θ_0, such that the endemic steady state is unstable for an isolated SIS model characterized by such a contact rate. We note that numerical evidence from the simulations indicates that the first condition in (14) is violated for small values of τ and ω, whereas the second condition is satisfied for all $\omega \in [0, 1]$. The arising scenarios are in line with those occurring in an isolated SIS model, and exhibit a stabilizing role for taxation, according to what has been shown by Cavalli et al. (2024c). Similarly, ω is stabilizing as well. For small taxation rates we have that $\boldsymbol{\xi}^*$ is unstable for any $\omega \in [0, 1]$ (panel (d)), while as τ increases it is possible to stabilize dynamics if the share of resources devoted to healthcare is suitably large (panel (g)), and even recover the disease-free steady state (panel (j)). Note that the bifurcation diagram in Figure 1 (j) provides a numerical evidence of a transcritical bifurcation occurring for $\omega = \omega_{tr}$ the endemic steady state merges and swaps its stability with the disease-free one. This is confirmed also by several other bifurcation diagrams, both in Figure 1 and in subsequent figures. In the simulations reported in the first column of Figure 1 the interaction with the environmental side does not introduce new scenarios with respect to those observed in an SIS model or in Cavalli et al. (2024c).

[12] Among the possible combinations, those in which $\omega_{1,ns} = 0$ or $0 < \omega_{1,ns} < \omega_f < \omega_{2,ns}$ or $0 < \omega_{1,ns} < \omega_{2,ns} < \omega_f$ seem not to be possible, but they would provide scenarios characterized by degrees of complexity qualitatively similar to others that occur.

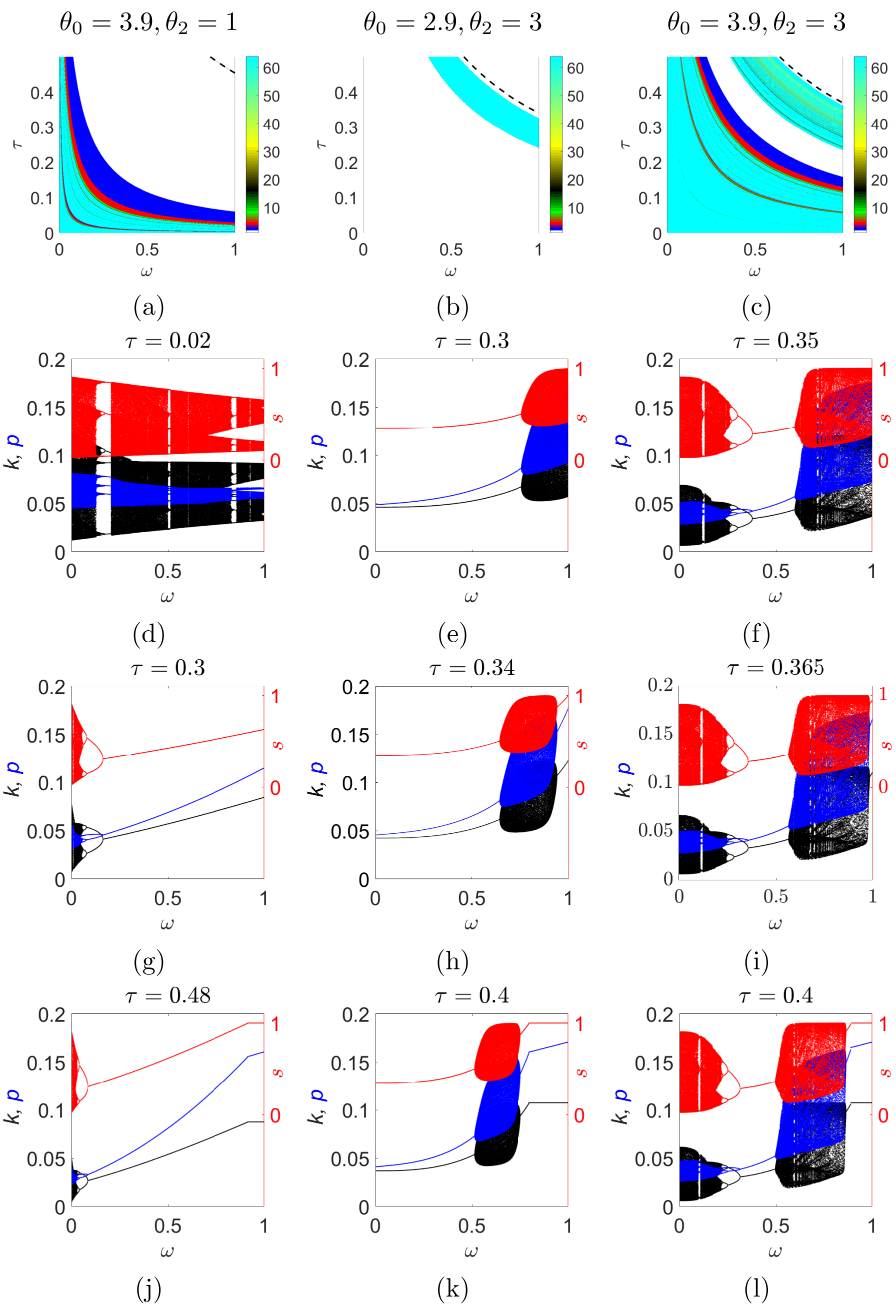

Figure 1 Endogenous θ, exogenous A and β. First row: two dimensional bifurcation diagrams in (ω, τ) plane for different values of θ_0 and θ_2. For different values of τ, the one dimensional bifurcation diagrams, on varying ω, in each column are related to the two dimensional bifurcation diagram in the first row. The black (resp. blue and red) bifurcation diagram is related to variable k (resp. p and s), and refers to the left (resp. left and right) vertical axis of each panel.

Conversely, in panel (b) of Figure 1 we consider a setting for which the first condition in (13) is fulfilled for any $\omega \in [0, 1]$. Note that θ_0 is small, so the endemic steady state is stable for the classic SIS model with contact rate θ_0, which is the rate obtained in the present model with $\tau = 0$. For small taxation rates, we have that $\boldsymbol{\xi}^*$ is stable for any $\omega \in [0, 1]$, while as τ increases, $\boldsymbol{\xi}^*$ becomes firstly unstable for large ω (panel (e)), then it is unstable for intermediate ω, with stability that is recovered for suitably large values of ω (panel (h)). Also in this case the disease-free steady state can become stable

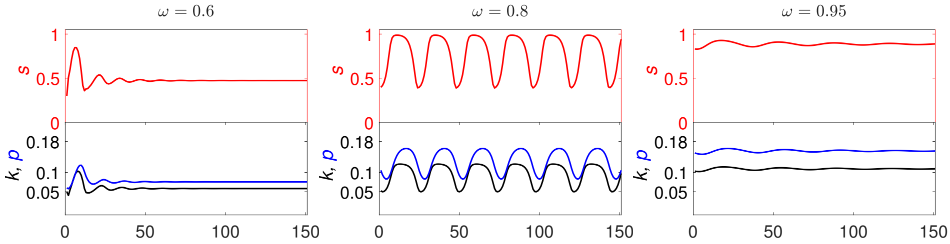

Figure 2 Time series related to the simulation reported in Figure 1 (h) for different values of ω.

if τ and ω are sufficiently large (panel (k)). In all these cases, $\boldsymbol{\xi}^*$ loses/recovers stability through Neimark-Sacker bifurcations. We remark that the SIS equation is not a source of instability, as the contact rate is small. The three domains, when separately considered, would just provide stable dynamics. Conversely, when coupled, we can observe the emergence of quasi-periodic trajectories.

The rationale can be explained by observing the quasi-periodic time series reported in Figure 2, obtained for different values of ω corresponding to values respectively before, belonging to and after the instability interval. In all the reported simulations the epidemic is initially at its maximum spread, and this reflects on a depressed capital level (the number of workers is low) and a consequent reduced pollution level (production is low). If ω is small, the government expenditures for healthcare are able to initially counteract the epidemic diffusion, and the number of healthy agents starts increasing, reviving the economic course and, consequently, deteriorating the environmental situation. However, the raise of pollution has a negative effect on the contact rate, and this is not counterbalanced by the effect of investments on healthcare, and the epidemic proliferates again, even if reaching a smaller diffusion. The sequence of effects on s_t, k_t and p_t repeats, but giving rise to increasingly less extreme situations, which leads to damped oscillations and convergence toward the endemic steady state characterized by a relevant number of infected agents, even if the pollution level is small. In this setting, the negative effect of pollution on the epidemiological situation is stronger than the positive outcome of healthcare expenditure. If we increase ω, we obtain a scenario in which these two effects, on average, balance out, so we have persistent, large oscillations. When the epidemiological and economic situations are good, the large pollution level drives the scenario toward the opposite state. The epidemiological and economic domains reach the bottom situation, from which they come out thanks to the healthcare expenditures and the improved environmental situation. If ω is further increased, the healthcare policy is suitably effective to counterbalance the larger persistent pollution level, so oscillations again dampen, now settling to an endemic steady state characterized by a small fraction of infected people but a large pollution level.

In panel (c) of Figure 1 we consider a setting for which both conditions in (13) can be violated, and what we observe can be thought of as the superposition of what happens in panels (a) and (c). Since θ_0 is large, the endemic steady state is unstable for the SIS model if the contact rate is θ_0. So $\boldsymbol{\xi}^*$ is unstable when $\tau = 0$ and for small taxation rates for any $\omega \in [0, 1]$, while as τ mildly increases, $\boldsymbol{\xi}^*$ can recover stability for large values of ω. If τ is further increased, we have the superimposition of a couple of destabilizing/stabilizing Neimark-Sacker bifurcations like those observed in the middle column of Figure 1. In this case $\boldsymbol{\xi}^*$ can have two (panel (f)) or three (panel (i)) stability changes (in addition to the final transcritical bifurcation), with different kinds of bifurcations occurring. If τ and ω are large enough, the disease-free steady state can recover stability (panel (l)).

We remark that, differently from the results in Cavalli et al. (2024c), taxation can have a destabilizing effect.

4.2 Endogenous θ and A

Now we take into account the effect of the epidemic spread on productivity, leaving β as the only exogenous term. In this case, it is immediate to see that conditions (13) simplify as follows.

Corollary 8 *If $\beta(s) \equiv \beta$ and $\gamma < \theta(\eta^*)$ conditions (13) become*

$$\begin{cases} E_\theta(\eta^*)\left(1 + \frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta^*)}\right)\right) < \frac{(a+1)(2-\delta)[2-(\theta(\eta^*)-\gamma)]}{2(1-a)(\theta(\eta^*)-\gamma)} \\ E_\theta(\eta^*)\left(1 + \frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta^*)}\right)\right) > -\frac{(1-a+a\delta)[1-a+a(\theta(\eta^*)-\gamma)][\delta+(1-\delta)(\theta(\eta^*)-\gamma)]}{(\theta(\eta^*)-\gamma)(1-a)[1-a(1-\delta)(1-(\theta(\eta^*)-\gamma))]} \\ E_\theta(\eta^*)\left(1 + \frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta^*)}\right)\right) > \frac{(1+a)(1-\delta)+a-3-(1+a-\delta)(\theta(\eta^*)-\gamma)}{(1-a)(\theta(\eta^*)-\gamma)} \end{cases} \tag{17}$$

Also in this case the first condition in (17) is fulfilled when $\theta(\eta^*) - \gamma < 2$, and the comments after Corollary 5 are still valid. Since A has positive elasticity, ceteris paribus, it has a stabilizing effect. This is predictable and in line with the outcomes in Cavalli et al. (2024c), as it points out a more reactive effect of a decrease of the fraction of infected agents on the productivity. The opposite occurs for the second condition in (17), with the elasticity of A having a destabilizing effect. If the productivity more quickly reacts to a reduction of the infection spread, this corresponds to an increase of pollution, and recalling the discussion in the previous section, this has a destabilizing effect. Now we provide sufficient conditions on E_A to obtain a result similar to that in Proposition 6.

In what follows, we denote by $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'$ and $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)''$ respectively the first and the second derivative of $E_A\left(\frac{\gamma}{\theta(\eta)}\right)$ with respect to η, i.e. $\frac{dE_A(\gamma/\theta(\eta))}{d\eta}$ and $\frac{d^2E_A(\gamma/\theta(\eta))}{d\eta^2}$.

Proposition 9 *If $\theta'(\eta) \neq 0$, $E'_\theta(\eta) < 0$, $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' > 0$ and*

$$E''_\theta(\eta) > \left(\frac{\theta''(\eta)}{\theta'(\eta)} - \frac{2\theta'(\eta)}{\theta(\eta)-\gamma}\right)E'_\theta(\eta) - 2E'_\theta(\eta)\frac{\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'}{E_A\left(\frac{\gamma}{\theta(\eta)}\right)} \tag{18}$$

and

$$\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'' < \left(\frac{-2\theta'(\eta)}{\theta(\eta)-\gamma} + \frac{\theta''(\eta)}{\theta'(\eta)}\right)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' \tag{19}$$

We then have that the possible bifurcations are the same as those in Proposition 6.

For the numerical simulations we introduce function

$$A(s) = \max\left\{\min\left\{A_m + \frac{1-A_m}{A_1-A_0}(s-A_0), 1\right\}, A_m\right\} \tag{20}$$

with $0 \leq A_0 < A_1 \leq 1$ and $A_m \in [0,1)$. With function (20) the total factor productivity equals the smallest possible A_m for $s \leq A_0$, is linearly increasing on (A_0, A_1) and attains its maximum value 1 for $s \geq A_1$. We study under what conditions on the parameters of function θ defined by (16) and on the restriction to (A_0, A_1) of function A the previous proposition can be applied.

Corollary 10 *Let θ be defined by (16) and A by (20). For $\eta \in \left(0, \min\left\{\frac{\delta\tau}{\alpha}, \theta^{-1}\{\gamma\}\right\}\right) \cap \left(\theta^{-1}\left(\frac{\gamma}{A_0}\right), \theta^{-1}\left(\frac{\gamma}{A_1}\right)\right)$ and if*

$$\begin{cases} A_1A_m - A_0 > 0 \\ A_0 > \frac{1}{3} \end{cases} \tag{21}$$

conditions of Proposition 9 hold true and so the possible bifurcations are the same as those in Proposition 6.

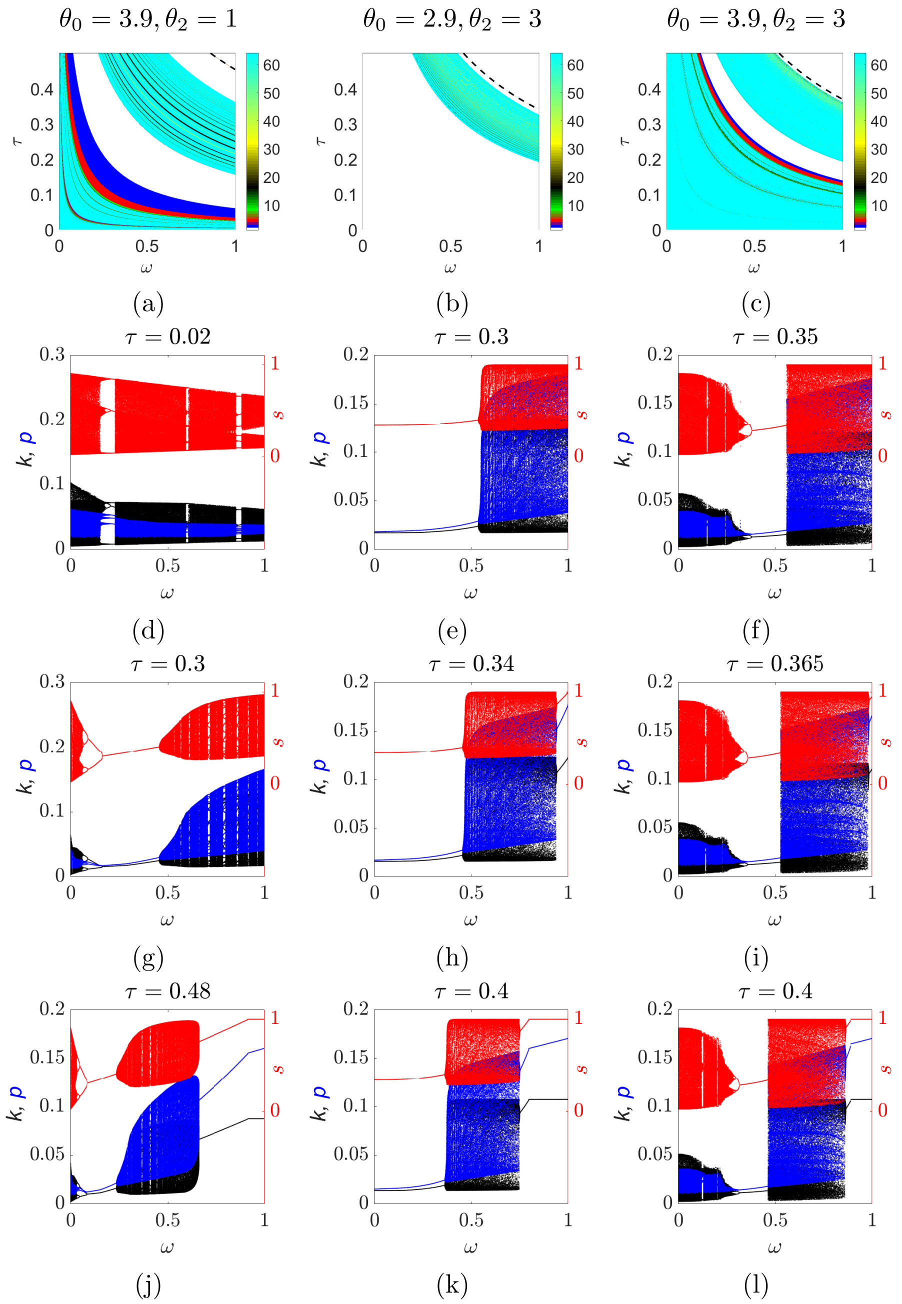

Figure 3 Endogenous θ, and A, exogenous β. First row: two dimensional bifurcation diagrams in (ω, τ) plane for different values of θ_0 and θ_2. For different values of τ, the one dimensional bifurcation diagrams, on varying ω, in each column are related to the two dimensional bifurcation diagram in the first row. The black (resp. blue and red) bifurcation diagram is related to variable k (resp. p and s), and refers to the left (resp. left and right) vertical axis of each panel.

In Figure 3 we report the results obtained by considering the same parameter settings used for Figure 1, now with the endogenous total factor productivity function (20), for which we set $A_m = 0.5, A_0 = 0.35, A_1 = 0.75$. Comparing the two dimensional bifurcation diagrams, the most evident difference is the presence of an additional region of instability in Figure 3 (a) with respect to Figure 1 (a), which highlights one (Figure 3 (g)) or two (Figure 3 (j)) Neimark-Sacker bifurcations. As noted after Corollary 8, endogenizing total factor productivity can lead to a violation of the second stability condition in (17), which gives rise to the Neimark-Sacker bifurcation. Also in the other bifurcation diagrams of Figure 3 we can see that the instability interval due to the Neimark-Sacker bifurcation is larger than in those corresponding to Figure 1, with wider oscillations.

4.3 Endogenous θ, A and β

Now take into account all the possible sources of interaction, by also considering endogenous β. For the lack of analytical tractability, we avoid to provide sufficient conditions under which the scenarios of Propositions 6 and 9 are guaranteed. We note that the probability to survive alters stability conditions through the elasticity of the saving propensity, which appears as a multiplicative factor of the elasticity of the contact rate. However, the extensive numerical investigations we performed seem to point out that its effect on stability is the weakest one when compared to those of θ and A. To show this, we consider

$$\beta(s) = s^{\alpha}, \tag{22}$$

in which $\alpha \in (0, 1)$ allows regulating the concavity of the probability to survive [13]. In Figure 4 we report the simulations obtained by using function (22) with $\alpha = 1/2$ and the same setting adopted for the scenarios reported in Figure 3. As we can see, corresponding panels are almost identical.

5 Model with public debt and subsidies

In this section we study the robustness of the results related to model (6) by taking into account subsidies and the possibility for the government to issue a debt. In Sections 3 and 4 we studied the limit situation in which adult agents, if infected, are not able to work and hence do not have any kind of income. In this section, we show that the results we have found are reliable for the realistic setting, in which the government allocates a suitable amount of resources for the subsidies of non working agents. In what follows, we take into account this issue and, to this end, and to support environmental and healthcare expenditures, in addition to the resources collected from taxation, we assume that the government can issue a debt. In Figure 5 we report the simulation obtained introducing subsidies in the setting considered for Figure 4. The additional parameters are $b = 0.02, \tau^{ex} = 0.018$ and $\tau^{end} = 0.05\tau$, which means that the endogenous amount of resources allocated to subsidies corresponds to the five percent of the overall taxation. As in Cavalli et al. (2024c), we remark that the values for these additional parameters are appropriately selected with respect to k_t. As we can see, the results reported in the corresponding panels of Figures 4 and 5 are very similar, highlighting a mild overall effect on dynamics of the introduction of subsidies. The results confirm the trustworthiness of the analysis related to the limit case without subsidies. Hence, the discussions and comments in the previous sections can be applied also to the case with subsidies.

6 Conclusions

The study of interacting economic-epidemiological-environmental domains allowed for some interesting insights. Firstly, comparative statics show that in some settings it may be not possible to avoid trade-offs in the distribution of resources for different purposes, and the regulator must be careful in order to balance them and avoid penalizing too

[13] The analytical expression for β is actually not differentiable at $s = 0$, but this is not a problem since $s^* > 0$, so the analytical results for stability are still valid.

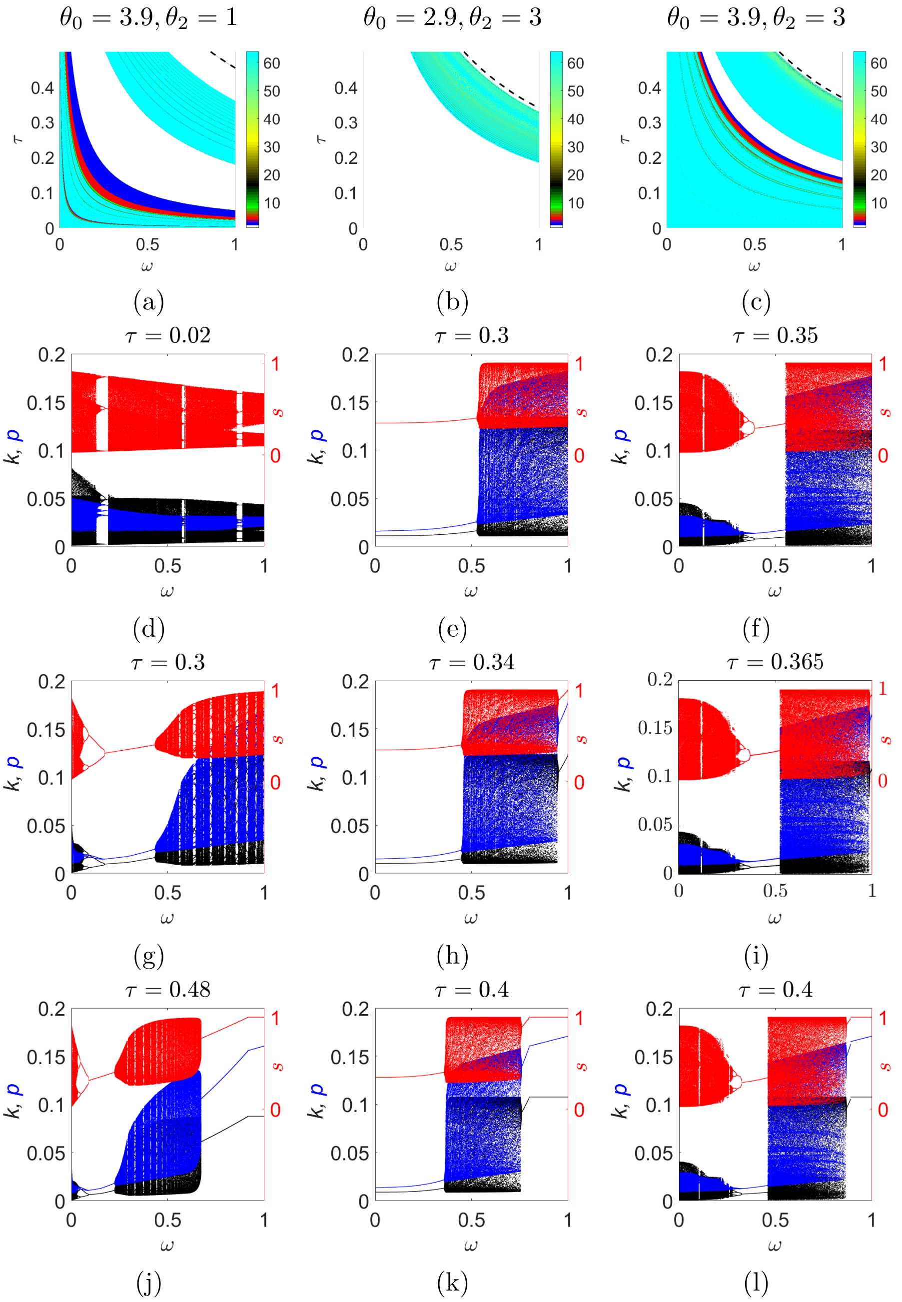

Figure 4 Endogenous θ, A and β. First row: two dimensional bifurcation diagrams in (ω, τ) plane for different values of θ_0 and θ_2. For different values of τ, the one dimensional bifurcation diagrams, on varying ω, in each column are related to the two dimensional bifurcation diagram in the first row. The black (resp. blue and red) bifurcation diagram is related to variable k (resp. p and s), and refers to the left (resp. left and right) vertical axis of each panel.

much one aspect. In addition to this, even if, for any reason, it is needed to devote more resources to either healthcare or environment, it should be clear that this can be a source of instability. The exhibited level of complexity highlights the need of a precise dynamical investigation of the problem. In particular, a too simplified modelling for a single domain may cause misleading results in view of effective policy interventions. Moreover, dynamical complexity can arise even if each domain, considered on its own,

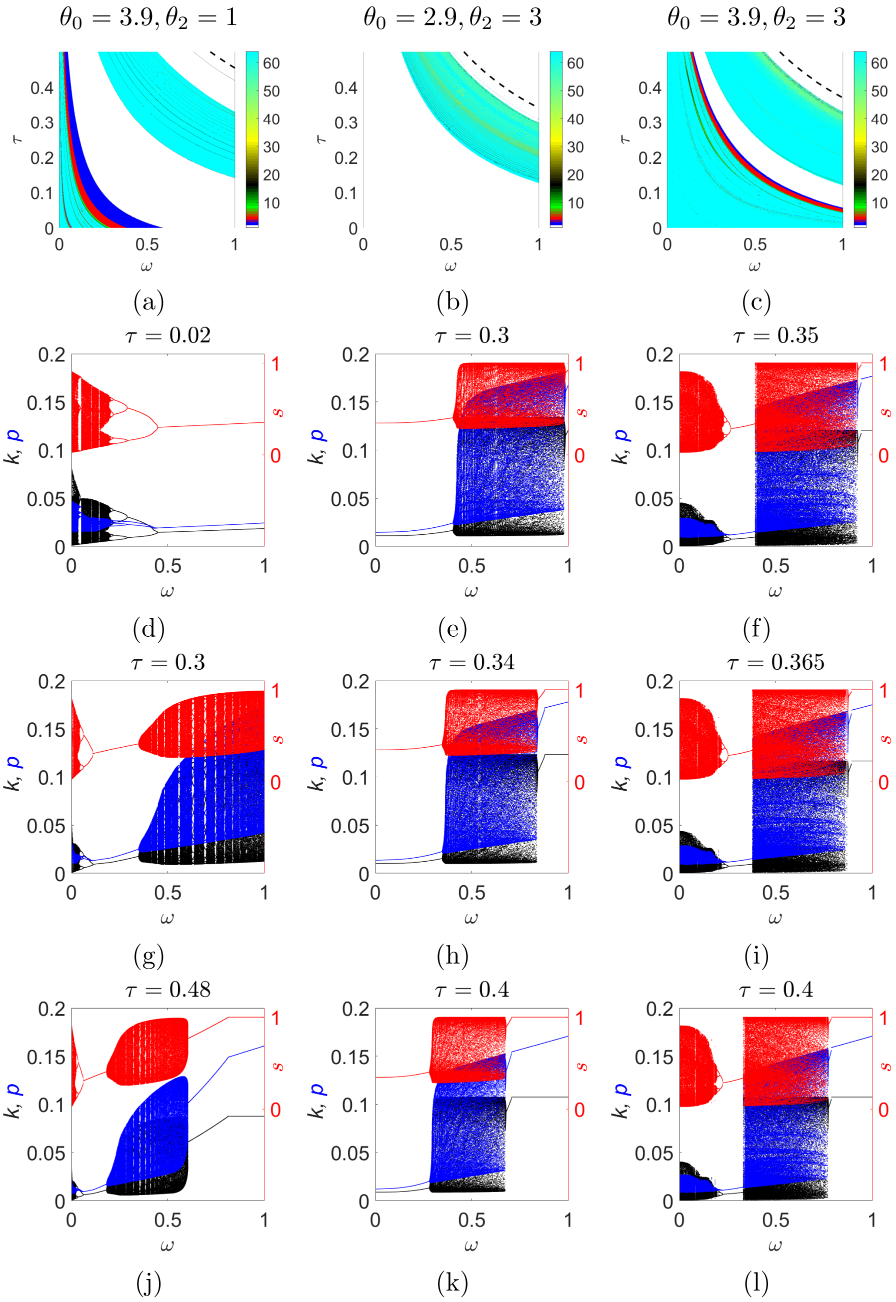

Figure 5 Endogenous θ, A and β and subsidies. First row: two dimensional bifurcation diagrams in (ω, τ) plane for different values of θ_0 and θ_2. For different values of τ, the one dimensional bifurcation diagrams, on varying ω, in each column are related to the two dimensional bifurcation diagram in the first row. The black (resp. blue and red) bifurcation diagram is related to variable k (resp. p and s), and refers to the left (resp. left and right) vertical axis of each panel.

would not be a source of instabilities, and this points out the relevance of an approach based on integrated domains.

For this reason, in the future research we aim at generalizing the interdependence between the domains, in particular by introducing a direct effect of the environmental quality on the productivity. Other possible extensions include the possibility to develop and study dynamical policy interventions that are able to endogenously adapt to the given contexts. Moreover, we aim to address the issue of firm uncertainty regarding the estimation of (marginal) total factor productivity, with the goal of proposing and studying suitable approximation methods.

Declarations

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- Competing and financial interests: the authors declare no competing interests, and have no relevant financial or non-financial interests to disclose

Appendix A Proofs

Prop. 1 Setting $s_t = s_{t+1} = s$, $k_t = k_{t+1} = k$ and $p_t = p_{t+1} = p$ in (6) we find

$$\begin{cases} (1-s)\left(\gamma - \theta\left(\frac{\omega g}{p}\right)s\right) = 0, \\ k = (1-\tau)(1-a)\frac{\beta(s)}{1+\beta(s)}A(s)k^a s^{1-a}, \\ p = \frac{\alpha-\lambda(1-\omega)\tau}{\delta}A(s)s^{1-a}k^a. \end{cases} \tag{A1}$$

where $g = \tau A(s)s^{1-a}k^a$. Solving the second equation with respect to k (which is assumed to be non null), we find

$$k = \left(\frac{\beta(s)}{1+\beta(s)}(1-\tau)(1-a)A(s)\right)^{\frac{1}{1-a}} s \tag{A2}$$

Using the third equation in (A1), we obtain

$$\frac{\omega g}{p} = \frac{\delta\omega\tau}{\alpha - \lambda(1-\omega)\tau}$$

which inserted in the first equation in (A1) allows finding the two solutions $s^*_{df} = 1$ and $s^* = \frac{\gamma}{\theta\left(\frac{\delta\omega\tau}{\alpha-\lambda(1-\omega)\tau}\right)}$. This latter solution is admissible if $s^* < 1$, i.e. $\theta\left(\frac{\delta\omega\tau}{\alpha-\lambda(1-\omega)\tau}\right) > \gamma$, otherwise no endemic steady state is possible. Replacing s with either $s^*_{df} = 1$ or s^* in (A2) and third equation in (A1) allows concluding. □

Prop. 2 Computing the partial derivatives of each component of $\boldsymbol{\xi}^*$ with respect to ω we find

$$\begin{aligned}
\frac{\partial s^*}{\partial \omega} &= -\frac{\gamma(\alpha-\lambda\tau)}{\omega[\alpha-\lambda(1-\omega)\tau]\theta(\eta^*)}E_\theta(\eta^*) \\
\frac{\partial k^*}{\partial \omega} &= \frac{k^*}{s^*}\left[1 + \frac{1}{1-a}\left(E_{\frac{\beta}{1+\beta}}(s^*) + E_A(s^*)\right)\right]\frac{\partial s^*}{\partial \omega} \\
\frac{\partial p^*}{\partial \omega} &= \frac{A(s^*)}{\delta}(s^*)^{1-a}(k^*)^a\left[\lambda\tau - \frac{\alpha-\lambda\tau}{\omega}E_\theta(\eta^*)\,(1\right. \\
&\quad \left. + \frac{a}{1-a}E_{\frac{\beta}{1+\beta}}(s^*) + \frac{1}{1-a}E_A(s^*)\Big)\right] \\
&= \frac{A(s^*)}{\delta}(s^*)^{1-a}(k^*)^a\left[\lambda\tau - \frac{\alpha-\lambda\tau}{\omega}E_\theta(\eta^*)\,E_{g^*}(s^*)\right]
\end{aligned}$$

Recalling that $E_\theta(g) < 0$ for $g > 0$ and both $E_{\frac{\beta}{1+\beta}}$ and E_A are strictly positive, we can conclude. □

Prop. 3 and 4 We begin with the study of stability for the endemic steady state. We compute each element of the Jacobian matrix of M evaluated at $\boldsymbol{\xi}^*$. We have

$$J_{11} = \frac{\partial M_1}{\partial s}(s,k,p)$$

$$= -\theta(\eta)(1-s) - \gamma + s\left[\theta(\eta) - \frac{\omega\tau k^a s^{1-a}}{p}\theta'(\eta)\left(A'(s) + \frac{A(s)}{s}(1-a)\right)(1-s)\right] + 1$$

from which

$$J_{11} = -\theta(\eta)(1-s) - \gamma + s\theta(\eta) - \eta\theta'(\eta)\left(s\frac{A'(s)}{A(s)} + 1 - a\right)(1-s) + 1$$

Evaluating it at $\boldsymbol{\xi}^*$, at which we have $s^* = \gamma/\theta(\eta^*)$, we find

$$J_{11}^* = (\gamma - \theta(\eta^*))\left[1 + E_\theta(\eta^*)\left(1 - a + E_A(s^*)\right)\right] + 1$$

We have

$$J_{12} = \frac{\partial M_1}{\partial k}(s,k,p) = -\frac{a\omega\tau A(s)}{p}s(1-s)\theta'(\eta)\left(\frac{s}{k}\right)^{1-a} = -\frac{a\eta\theta'(\eta)s(1-s)}{k}$$

so

$$J_{12}^* = \frac{as^*E_\theta(\eta^*)(\gamma - \theta(\eta^*))}{k^*}$$

We have

$$J_{13} = \frac{\partial M_1}{\partial p}(s,k,p) = s(1-s)\theta'(\eta)\frac{\omega\tau A(s)s^{1-a}k^a}{p^2} = \frac{\eta\theta'(\eta)s(1-s)}{p}$$

so

$$J_{13}^* = -\frac{E_\theta(\eta^*)s^*(\gamma - \theta(\eta^*))}{p^*}$$

Let us consider the second equation of (6). We have

$$J_{21} = \frac{\partial M_2(s,k,p)}{\partial s} = \frac{M_2}{s}E_{M_2}(s)$$

where

$$E_{M_2}(s) = E_{\beta/(1+\beta)}(s) + E_A(s) + 1 - a$$

$$J_{21} = \frac{\partial M_2(s,k,p)}{\partial s} = \frac{M_2(s,k,p)}{s}\left(E_{\frac{\beta}{1+\beta}}(s) + E_A(s) + 1 - a\right)$$

so

$$J_{21}^* = \frac{k^*}{s^*}\left(E_{\frac{\beta}{1+\beta}}(s^*) + E_A(s^*) + 1 - a\right)$$

We have

$$J_{22} = \frac{\partial M_2(s,k,p)}{\partial k} = a(1-\tau)(1-a)\frac{\beta(s)}{1+\beta(s)}A(s)k^{a-1}s^{1-a} = \frac{aM_2(s,k,p)}{k}$$

and hence $J_{22}^* = a$. Moreover, $J_{23} = J_{23}^* = 0$. Noting that

$$(\alpha - \lambda(1-\omega)\tau)A(s)k^a s^{-a} = \frac{M_3(s,k,p) - (1-\delta)p}{s}$$

we have

$$J_{31} = \frac{\partial M_3(s,k,p)}{\partial s} = (\alpha - \lambda(1-\omega)\tau)A(s)\left(\frac{k}{s}\right)^a\left[s\frac{A'(s)}{A(s)} + 1 - a\right] = \left(\frac{M_3 - (1-\delta)p}{s}\left[s\frac{A'(s)}{A(s)} + 1 - a\right]\right)$$

from which

$$J_{31}^* = \frac{\delta p^*}{s^*}(E_A(s^*) + 1 - a)$$

Finally, we have

$$J_{32} = \frac{\partial M_3(s,k,p)}{\partial k} = a(\alpha - \lambda(1-\omega)\tau)A(s)k^{a-1}s^{1-a} = a\frac{M_3 - (1-\delta)p}{k}$$

so

$$J_{32}^* = a\delta\frac{p^*}{k^*}$$

and

$$J_{33} = J_{33}^* = \frac{\partial M_3}{\partial p}(\boldsymbol{\xi}^*) = 1 - \delta$$

The resulting Jacobian matrix is then

$$J^* = \begin{pmatrix} (\gamma - \theta(\eta^*))[1 + E_\theta(\eta^*)(E_A(s^*) + 1 - a)] + 1 & \frac{as^*E_\theta(\eta^*)(\gamma-\theta(\eta^*))}{k^*} & -\frac{E_\theta(\eta^*)s^*(\gamma-\theta(\eta^*))}{p^*} \\ \frac{k^*}{s^*}\left[E_{\frac{\beta}{1+\beta}}(s^*) + E_A(s^*) + 1 - a\right] & a & 0 \\ \frac{\delta p^*}{s^*}(E_A(s^*) + 1 - a) & a\delta\frac{p^*}{k^*} & 1 - \delta \end{pmatrix}$$

As reported in Lines et al. (2020), stability is guaranteed by

$$\begin{cases} 1+m(J^*)+\operatorname{tr}(J^*)+\det(J^*)>0 \\ 1+m(J^*)-\operatorname{tr}(J^*)-\det(J^*)>0 \\ 1-(\det(J^*))^2-m(J^*)+\operatorname{tr}(J^*)\det(J^*)>0 \\ 3-m(J^*)>0 \end{cases}$$

where $m(J^*)$ denotes the sum of principal minors of order two of the Jacobian. We stress that the first and the third conditions are respectively related to the possible emergence of a flip and Neimark-Sacker bifurcation. Since

$$\begin{aligned}
\operatorname{tr}(J^*) =& a-\delta+2+(\gamma-\theta(\eta^*))[1+E_\theta(\eta^*)(E_A(s^*)+1-a)] \\
\det(J^*) =& a\left[1-\delta+(\gamma-\theta(\eta^*))\left(1-\delta-E_\theta(\eta^*)E_{\frac{\beta}{1+\beta}}(s^*)\right)\right] \\
m(J^*) =& (1+a)(1-\delta)+a+(\gamma-\theta(\eta^*))\left[1+a-\delta+E_\theta(\eta^*)\left(-aE_{\frac{\beta}{1+\beta}}(s^*)\right.\right. \\
&\left.\left. +E_A(s^*)+1-a\right)\right]
\end{aligned}$$

we find that the second condition becomes $\delta(\theta(\eta^*)-\gamma)(1-a)>0$ and hence it is always fulfilled.

Using the expression of E_{g^*} defined by (12) and rearranging terms in the first, third and fourth stability conditions, we find the three conditions in (13).

We conclude with the study of stability for the disease free steady state. Recalling the expressions of J_{ij} and evaluating them at $\boldsymbol{\xi}^*_{df}=(1,p^*_{df},k^*_{df})$ we find that J^*_{df} is a lower triangular matrix in which the diagonal elements, providing its eigenvalues, are

$$J^*_{11}=\theta(\eta^*)-\gamma+1, \quad J^*_{22}=a\in(0,1), \quad J^*_{33}=1-\delta\in(0,1).$$

Since $\theta(\eta^*)-\gamma<0$, $J^*_{11}<1$ and since, thanks to $\theta(\eta^*)-\gamma>-1$, we have $J^*_{11}>0$, and this concludes the proof. □

Proof of Proposition 6 Since we focus on possible bifurcations, we take into account only the first two conditions in (14). We recall that $\eta^*(\omega)$ is increasing and $\alpha>\lambda\geq\lambda\tau$ thanks to Assumption 1. Moreover, we recall that the existence of the endemic steady state requires $\gamma/\theta(\eta^*)<1$. If this is true for any $\omega\in[0,1]$, we can choose $\omega_{tr}=1$, otherwise, ω_{tr} corresponds to the value of ω at which $\gamma/\theta(\eta^*)=1$. In both cases, stability of the endemic steady state must be studied on $[0,\omega_{tr})$.

The remainder of the proof proceeds as follows: we take into account the two stability conditions in (14) and we find the set on which they are not fulfilled, whose ending points identify possible bifurcation values.

Since

$$\left(\frac{2-(\theta(\eta)-\gamma)}{(\theta(\eta)-\gamma)}\right)'=-\frac{2\theta'(\eta)}{(\theta(\eta)-\gamma)^2}>0 \tag{A3}$$

the right hand side of the first condition in (14) is strictly increasing while, thanks to $E'_\theta(\eta)<0$, its left hand side is strictly decreasing, and hence the inequality is either always fulfilled or there exists a unique η_f such that it is true for $\eta>\eta_f$. Consequently, if the inequality is always fulfilled, we do not have any value of ω to remove from the stability set, and we can choose $\omega_f<0$ so that $[0,\omega_f]$ is empty. Conversely, solving $\frac{\delta\omega\tau}{\alpha-\lambda(1-\omega)\tau}>\eta_f$, we obtain the inequality

$$\tau(\delta-\lambda\eta_f)\omega>(\alpha-\lambda\tau)\eta_f.$$

If $\delta-\lambda\eta_f>0$, we obtain the stability interval $\omega>\omega_f=\frac{\eta_f(\alpha-\lambda\tau)}{\tau(\delta-\lambda\eta_f)}$, and the first condition in (14) is false on $[0,\omega_f]$. If instead $\delta-\lambda\eta_f\leq 0$, then the inequality is not fulfilled and we may pose $\omega_f=1$. This provides the conclusions about the flip bifurcation threshold.

Now we focus on the second condition in (14). The right hand side can be rewritten as

$$\frac{\delta}{1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)}-\frac{\delta}{\theta(\eta)-\gamma}-\frac{1-a+a\delta}{1-a}$$

Let us introduce $f:\left[0,\min\left\{\frac{\delta\tau}{\alpha},\theta^{-1}\{\gamma\}\right\}\right]\to\mathbb{R},\eta\mapsto f(\eta)$ defined by

$$f(\eta)=E_\theta(\eta)-\left(\frac{\delta}{1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)}-\frac{\delta}{\theta(\eta)-\gamma}-\frac{1-a+a\delta}{1-a}\right)$$

whose domain takes into account all the possible and feasible values of η^* for $\omega \in [0,1]$.

We have

$$\frac{(\theta(\eta)-\gamma)^2 f'(\eta)}{\theta'(\eta)} = (\theta(\eta)-\gamma)^2 \frac{E'_\theta(\eta)}{\theta'(\eta)} - \delta + \frac{a\delta(1-\delta)(\theta(\eta)-\gamma)^2}{[1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)]^2} \tag{A4}$$

Note that

$$\left(\frac{(\theta(\eta)-\gamma)^2}{[1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)]^2}\right)' = \frac{2(1-a+a\delta)(\theta(\eta)-\gamma)}{[1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)]^3}\theta'(\eta) < 0 \tag{A5}$$

and

$$\left((\theta(\eta)-\gamma)^2 \frac{E'_\theta(\eta)}{\theta'(\eta)}\right)' = 2(\theta(\eta)-\gamma)E'_\theta(\eta) + (\theta(\eta)-\gamma)^2 \frac{E''_\theta(\eta)\theta'(\eta)-E'_\theta(\eta)\theta''(\eta)}{(\theta'(\eta))^2} \tag{A6}$$

is negative thanks to (15). This allows concluding that function $(\theta(\eta)-\gamma)^2 f'(\eta)/\theta'(\eta)$ is strictly decreasing and can have at most one zero. Consequently, also f can have at most one critical point, which guarantees that $f(\eta)=0$ can have at most two solutions.

Different situations can occur, but they must be characterized by a positive f on a right neighborhood of $\eta=0$, since, noting that $E_\theta(0)=0$, we have

$$f(0) = \frac{(1-a+a\delta)[1-a+a(\theta(0)-\gamma)][\delta+(1-\delta)(\theta(0)-\gamma)]}{(\theta(0)-\gamma)(1-a)[1-a(1-\delta)(1-(\theta(0)-\gamma))]} > 0$$

A first situation is that in which we may have $f(\eta)>0$ for any η, in which the second condition in (14) is always fulfilled and we can choose $\omega_{1,ns} > \omega_{2,ns}$ so that $[\omega_{1,ns}, \omega_{2,ns}]$ is empty.

In a second situation, there is just a unique solution to $f(\eta)=0$ and $f(\eta)$ is positive on $0<\eta<\eta_{1,ns}$. In this case, setting $\omega_{1,ns}$ as the value for which $\frac{\delta\omega\tau}{\alpha-\lambda\tau+\lambda\omega\tau} = \eta_{1,ns}$ and $\omega_{2,ns} = \omega_{tr}$, we have that the second condition in (14) is not fulfilled for $\omega \in [\omega_{1,ns}, \omega_{tr}]$. In the last possible situation we have two solutions to $f(\eta)=0$, and $f(\eta)$ is positive for $\eta < \eta_{1,ns}$ and $\eta > \eta_{2,ns}$ and negative otherwise. Setting $\omega_{1,ns}$ and $\omega_{2,ns}$ as the values for which $\frac{\delta\omega\tau}{\alpha-\lambda\tau+\lambda\omega\tau}$ is equal to $\eta_{1,ns}$ and $\eta_{2,ns}$, respectively, we have that the second condition in (14) is not fulfilled for $\omega \in [\omega_{1,ns}, \omega_{2,ns}] \cap [0, \omega_{tr}]$. This provides the conclusions about the Neimark-Sacker bifurcation thresholds. □

Proof of Corollary 7 We have $E_\theta(\eta) = -\theta_1\theta_2\eta^{\theta_2}$ and $E'_\theta(\eta) = -\theta_1\theta_2^2\eta^{\theta_2-1} < 0$. Setting $z = \theta_0 e^{-\theta_1\eta^{\theta_2}}$, inequality (15) corresponds to

$$\theta_1\theta_2^2\eta^{\theta_2-2}(1-\theta_2) > -\theta_1\theta_2^2\eta^{\theta_2-1}\frac{(\theta_2-1)(z-\gamma)+\theta_1\theta_2\eta^{\theta_2}z+\gamma\theta_1\theta_2\eta^{\theta_2}}{\eta(z-\gamma)} \tag{A7}$$

Noting that for $\eta \in \left(0, \min\left\{\frac{\delta\tau}{\alpha}, \theta^{-1}\{\gamma\}\right\}\right)$ we have $z-\gamma>0$, the previous inequality is equivalent to

$$\theta_1\theta_2\eta^{\theta_2}(z+\gamma) > 0 \tag{A8}$$

and hence it is true. □

Proof of Proposition 9 We stress once more that since we study bifurcations, we take into account only the first two conditions in (17). We start noting that the right hand sides in both conditions in (14) and (17) are the same, so we focus on the left hand sides.

Since $E_\theta(\eta)<0$, $E_A\left(\frac{\gamma}{\theta(\eta)}\right)>0$, $E'_\theta(\eta)<0$ and $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'>0$, we have

$$\left(E_\theta(\eta)\left(1+\frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)\right)' = E'_\theta(\eta)\left(1+\frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)+\frac{E_\theta(\eta)}{1-a}\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' < 0.$$

The left hand side of the first condition in (17) is then decreasing and, recalling (A3), the same conclusions of Proposition 6 related to the first stability condition hold true also in this situation.

Let us introduce $\tilde{f} : \left[0, \min\left\{\frac{\delta\tau}{\alpha}, \theta^{-1}\{\gamma\}\right\}\right] \to \mathbb{R}, \eta \mapsto \tilde{f}(\eta)$ defined by

$$\tilde{f}(\eta) = E_\theta(\eta)\left(1+\frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right) - \left(\frac{\delta}{1-a+a\delta+a(1-\delta)(\theta(\eta)-\gamma)} - \frac{\delta}{\theta(\eta)-\gamma} - \frac{1-a+a\delta}{1-a}\right)$$

whose domain takes into account all the possible and feasible values of η^* for $\omega \in [0,1]$. Following the proof of Proposition 6, we want to show that $\left(\frac{(\theta(\eta)-\gamma)^2 \tilde{f}'(\eta)}{\theta'(\eta)}\right)'$ is negative, in order to have the same conclusions about the second stability condition.

Recalling (A4) and (A5), this is guaranteed if

$$\left(\frac{(\theta(\eta)-\gamma)^2 \left(E_\theta(\eta)\left(1+\frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)\right)'}{\theta'(\eta)}\right)' < 0$$

We have

$$\begin{aligned}
\left(\frac{(\theta(\eta)-\gamma)^2 \left(E_\theta(\eta)\left(1+\frac{1}{1-a}E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)\right)'}{\theta'(\eta)}\right)' &= \left(\frac{(\theta(\eta)-\gamma)^2 (E_\theta(\eta))'}{\theta'(\eta)}\right)' \\
&\quad + \frac{1}{1-a}\left(\frac{(\theta(\eta)-\gamma)^2 \left(E_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'}{\theta'(\eta)}\right)'
\end{aligned}$$

in which, recalling (A6), the former addend is negative under condition (15), which is implied by condition (18) since $E'_\theta(\eta) < 0$, $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' > 0$ and $E_A\left(\frac{\gamma}{\theta(\eta)}\right) > 0$. Concerning the latter addend we have

$$\begin{aligned}
\left(\frac{(\theta(\eta)-\gamma)^2 \left(E_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'}{\theta'(\eta)}\right)' &= \left(\frac{(\theta(\eta)-\gamma)^2 \left(E'_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right)+E_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'\right)}{\theta'(\eta)}\right)' \\
&= \left(\frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)}\right)' E'_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right) + \left(\frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)}\right)' E_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' \\
&\quad + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)} E_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'' + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)} E'_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' \\
&\quad + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)} E''_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right) + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)} E'_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'
\end{aligned}$$

Considering the second and third addends in the previous expression (i.e. those having factor $E_\theta(\eta)$) we have

$$\left(\frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)}\right)' E_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)} E_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'' < 0$$

thanks to the bound on $\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)''$ in (19). Considering all the remaining addends we have

$$\left(\frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)}\right)' E'_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right) + \frac{(\theta(\eta)-\gamma)^2}{\theta'(\eta)}\left[2E'_\theta(\eta)\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' + E''_\theta(\eta)E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right] < 0$$

thanks to the lower bound on $E''_\theta(\eta)$ in (18). This allows concluding the proof as in Proposition 6. □

Proof of Corollary 10 We have already shown in the proof of Corollary 7 that $E_\theta(\eta) < 0$. We have

$$E_A(s) = \frac{(1-A_m)s}{(1-A_m)s + A_1A_m - A_0}$$

and hence, setting $z = \theta_0 e^{-\theta_1\eta^{\theta_2}}$,

$$\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)' = \frac{\eta^{\theta_2-1}\gamma\theta_1\theta_2 z(A_1A_m - A_0)(1-A_m)}{[\gamma(1-A_m)+(A_1A_m-A_0)z]^2}$$

which is positive thanks to the first condition in (21).

Let us focus on condition (18). Recalling (A7) and using

$$-2E'_\theta(\eta)\frac{\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'}{E_A\left(\frac{\gamma}{\theta(\eta)}\right)} = \frac{2\eta^{2\theta_2-2}\theta_1^2\theta_2^3 z(A_1A_m-A_0)}{\gamma(1-A_m)+z(A_1A_m-A_0)}$$

condition (18) can be rephrased into

$$\frac{-(A_1A_m - A_0)z^2 + (3A_1A_m - 3A_0 + 1 - A_m)\gamma z + (1 - A_m)\gamma^2}{(z - \gamma)[\gamma(1 - A_m) + (A_1A_m - A_0)z]} > 0$$

Since $z > \gamma$, the sign of the left hand side depends on that of $-(A_1A_m - A_0)z^2 + (3A_1A_m - 3A_0 + 1 - A_m)z\gamma + (1 - A_m)\gamma^2$. We study its positivity for $z \in (\gamma, \gamma/A_0)$, which guarantees that condition (18) is fulfilled for $\eta \in \left(0, \min\left\{\frac{\delta\tau}{\alpha}, \theta^{-1}\{\gamma\}\right\}\right) \cap \left(\theta^{-1}\left(\frac{\gamma}{A_0}\right), \theta^{-1}\left(\frac{\gamma}{A_1}\right)\right)$. The concave polynomial $-(A_1A_m - A_0)z^2 + (3A_1A_m - 3A_0 + 1 - A_m)\gamma z + (1 - A_m)\gamma^2$ of degree two in z is positive for $z = \gamma$, while at $z = \gamma/A_0$ positivity requires

$$-A_0^2(A_m + 2) + A_0(2 - A_m + 3A_1A_m) - A_1A_m > 0. \tag{A9}$$

The previous condition is fulfilled for $A_0 \in \left(\frac{1}{3}, A_1\right)$. In fact, the left hand side is a concave parabola in variable A_0. Since for $A_0 = \frac{1}{3}$ we have that

$$-\frac{1}{9}(A_m + 2) + \frac{1}{3}(2 - A_m + 3A_1A_m) - A_1A_m = -\frac{4}{9}A_m + \frac{4}{9} \geq 0$$

since $A_m \leq 1$ and for $A_0 = A_1$ we have that

$$(1 - A_m)(A_1 - A_1^2) \geq 0$$

this gives (A9).

This guarantees that condition (18) holds true.

Let us focus on condition (19). We have

$$\begin{aligned}\left(E_A\left(\frac{\gamma}{\theta(\eta)}\right)\right)'' =& \frac{(A_1A_m - A_0)(1 - A_m)\gamma\theta_1\theta_2\eta^{\theta_2 - 2}z}{[\gamma(1 - A_m) + (A_1A_m - A_0)z]^3}[\gamma(1 - A_m)(\theta_2 - 1 - \theta_1\theta_2\eta^{\theta_2}) \\ &+ (A_1A_m - A_0)z(\theta_2 - 1 + \theta_1\theta_2\eta^{\theta_2})]\end{aligned}$$

so condition (19) can be rephrased as

$$-\frac{2\eta^{2\theta_2}\gamma^2\theta_1^2\theta_2^2z^2(A_1A_m - A_0)(1 - A_m)(A_1A_m - A_0 + 1 - A_m)}{\eta^2(z - \gamma)[\gamma(1 - A_m) + z(A_1A_m - A_0)]^3} < 0$$

which is true thanks to the first condition in (21). □

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