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Symmetry-in-Asymmetry: On the Existence and Structure of Asymmetric Pure-Strategy Equilibria in Symmetric Tullock Two-Group Contests

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Abstract

When information is complete, symmetric group-contest games *à la* Tullock are known to display within-group-symmetric (WGS) pure-strategy equilibria. Far less is known about the possible existence and structure of their asymmetric counterparts—pure-strategy equilibria where, symmetry notwithstanding, members of the same group exert different effort levels. This note fills the gap by proving that within-group-asymmetric (WGA) pure-strategy equilibria indeed exist, but only for a specific subset of incentivisation schemes. All such equilibria display a peculiar 'group-splitting' structure: a subgroup of members free-ride exerting null effort, whereas the other members provide the same positive effort level. Because of this symmetry-in-asymmetry, the game displays the same aggregative structure identified in the context of WGS equilibria. Accordingly, an explicit characterisation is feasible also for their WGA counterparts.

JEL classification: C72, D71, D74

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1 Introduction

Tullock group-contest games provide a compact yet rich analytical framework for the study of strategic environments where (*i*) competition is stratified by a partitioning of the player set into groups, and (*ii*) randomness affects the equilibrium outcomes jointly with collective effort provision. At the macro level, groups compete for the appropriation of a desirable prize. At the micro level, single group members strikes a balance between free-riding (individually profitable but undesirable for the group) and cooperation (desirable for the group but individually costly).

Within-group symmetry. Since its inception with Nitzan (1991b,a), the by now vast literature on Tullock group-contest games relied heavily on the notion of *within-group-symmetric equilibrium* (WGS equilibrium, henceforth),[1] which mandates that all members of the same group exert, in equilibrium, identical effort levels.[2] The mathematical convenience of the WGS assumption is apparent: as Bosco and Gilli (2024) formally prove in the context of a symmetric Tullock group contests with arbitrarily many groups, the WGS assumption implies aggregativity at the group level, and the resulting block-recursive structure allows for an explicit characterisation of the equilibrium aggregates by 'compounding' the individual FOCs.[3] Also the intuitive rationale for the WGS assumption is clear: if all members of the same group have the same payoff function, the game is symmetric within the group—even if this is not the case among groups. Therefore, it must have a WGS equilibrium.[4]

Issues with symmetry. In more recent contributions, Ueda (2002) and Balart et al. (2016) not only rely on WGS equilibria, but also *suggest* that, when the game's primitives entail perfect symmetry among players, such equilibria *may* also be unique.[5] Although no formal claims in this direction are explicitly made by authors, their *implicit* use of the WGS assumption, together with the standing of the WGS assumption in the literature, add to a general 'gut feeling' that, for symmetric Tullock group contests, pure-strategy within-group-asymmetric (WGA) equilibria may be unlikely to exist.

However, the possibility of asymmetric pure-strategy equilibria in symmetric games is well known, and understanding the origins of behavioural diversity in symmetric strategic environments is increas-

[1] To ease the exposition, in the remainder of this note the acronym WGS will be used to abbreviate both the adjective within-group-symmetric (e.g. within-group-symmetric equilibrium) and the noun within-group symmetry (e.g. within-group symmetry assumption).

[2] Among others, see Nitzan (1991b,a), Baik and Shogren (1995), Lee (1995), Davis and Reilly (1999) and Noh (1999). For a recent survey on contest games, see Beviá and Corchón (2024).

[3] In the context of n-player Tullock contests, Ewerhart (2026) builds on similar arguments.

[4] For general results on this issue, see Reny (1999) and Ewerhart and Reny (2022).

[5] Ueda (2002) is of particular importance, since it is the first contribution that provides sharp result for group-contest games with more than two groups.

ingly perceived as a central goal of research in different areas of economics and social sciences.[6] In view of the diversity of economic research areas invoking symmetry breaking, different approaches have been developed. One common approach consider games with strategic complementarities and possibly multiple Nash equilibria. An alternative approach, pioneered by Matusyama, builds on a peculiar model specification with two a priori identical players, whose actions are pairwise strategic complements but substitutes for the single players due to a fixed total resource constraint, so that multiple asymmetric and symmetric equilibria arise.[7] Finally, more recently Amir et al. (2010) considers two-player symmetric game with two key properties: strategic substitute actions and player's payoffs that, though continuous, admit a robust concavity-destroying kink along the diagonal in action space.

In the context of Tullock group-contest games, the additive separability of individual effort provision in the determination of group effort levels implies a high degree of intra-group substitutability. In turn, this points to the fact that asymmetric pure-strategy equilibria may indeed exist where, within the same group, a (possibly large) cohort of inactive members free-rides on the effort provided by a (possibly small) subgroup of active contributors.

Research question and results. In the context of a two-group specification, this note proves that symmetric Tullock group-contest games indeed admit WGA equilibria at specific profiles of incentivisation schemes. When they exist, such WGA equilibria display a peculiar 'group-splitting structure': in every group that displays asymmetry, a subgroup of members (the *free riders*) exerts null effort, whereas the other members (the *contributors*) all provide the same positive effort level. Therefore, WGA equilibria display *symmetry-in-asymmetry.* This property is crucial from a mathematical point of view, for it preserves the group-level aggregativity identified by Bosco and Gilli (2024) in the case of WGS equilibria. In turn, this allows for an *explicit* equilibrium characterisation even with an arbitrary number of competing groups via the same algorithmic procedure outlined in Bosco et al. (2026) for the characterisation of WGS equilibria. Second, the subset of incentivisation schemes that sustains the WGA equilibria has measure zero with respect to the entire set of feasible schemes. This provides a sound mathematical rationale for the traditional focus of the literature on WGS equilibria. The approach we follow is constructive—we proceed by guess and solve. In this way, we propose a general methodology for generating inter-agent differences out of a symmetric conflict game with fully rational and completely informed players.

[6] See e.g. Cooper and John (1988), Cooper (1999), Besanko and Doraszelski (2004), Matsuyama (2002) and Amir et al. (2010).

[7] See Matsuyama (2000, 2002, 2004).

Structure. This note is structured as follows. Section 2 outlines the Tullock two group model and recaps the standard results on WGS equilibria. Section 3, the core of the note, proves constructively the generic non-uniqueness of WGS equilibria by showing that WGA equilibria indeed exist. Moreover, it highlights the peculiar group-splitting structure of such equilibria. Section 4 concludes the note.

2 Model and WGS Equilibria

In this section we outline the canonical complete-information Tullock group-contest game in its two-group specification, and we briefly recap its equilibrium characterisation under the WGS assumption.

2.1 Primitives

We consider a strategic environment with two groups N_1 and N_2, with set cardinalities $n_1 \geq 2$ and $n_2 \geq 2$, respectively. The generic group is labelled $i \in \{1, 2\}$, so that $-i$ indicates the competing group. The generic group member is indexed by $j \in \{1, \ldots, n_i\}$. Hence, the total set of players, denoted by $N := N_1 \cup N_2$, has cardinality $n = n_1 + n_2 \geq 4$. Groups compete in effort for the allocation of a public good (the *prize*) of common value $v > 0$.

Each member of each group exerts an individual effort level $x_{ij} \geq 0$, with $(x_{i1}, \ldots, x_{in_i}) =: \mathbf{x}_i \in \mathbb{R}_+^{n_i}$ the generic effort profile of group i and $(x_{11}, \ldots, x_{1n_1}, x_{21}, \ldots, x_{2n_2}) =: \mathbf{x} \in \mathbb{R}_+^N$ the generic effort profile across both groups. The cost of effort is linear and identical for all players. Normalizing to one the marginal cost of effort without significant loss of generality, the common cost function can be written as $C_{ij}(x_{ij}) = C(x_{ij}) = x_{ij}$. Individual effort provision maps into its group counterpart $X_i \geq 0$ via the linear *impact function* $X_i = \sum_{j=1}^{n_i} x_{ij}$. Upon winning, the share $q_{ij}(\mathbf{x}_i)$ of the prize allocated to the j-th member of group i is determined by the group-specific *sharing rule*

$$q_{ij}(\mathbf{x}_i) = \begin{cases} (1-\alpha_i)\,\dfrac{x_{ij}}{X_i} + \dfrac{\alpha_i}{n_i} & \text{if } X_i > 0 \\ 1/n_i & \text{otherwise} \end{cases}, \tag{1}$$

whose elements $1 - \alpha_i$ and α_i[8] constitute, respectively, the *incentivising* and *equalising* part.[9] The interpretation is straightforward. The share $1 - \alpha_i$ is effort-contingent and strictly increases in the relative effort provision of member ij; accordingly, it drives competition among members of the same

[8] We focus on two scenarios: $\alpha_i \in [0, 1]$ (restricted case) and $\alpha_i \in \mathbb{R}$ (unrestricted case)—$i = 1, 2$. As our analysis will clearly show, the choice of the state-space of sharing rules has a significant bearing on the equilibrium play.

[9] Note that expression (1) entails a (possible) discontinuity in players' expected payoffs at the zero-effort profile $\mathbf{x}_i = \mathbf{0}$. While it is customary in (some of) the literature to overlook this problematic case, we explicitly address it by assigning to the members of the winning group a purely egalitarian share $1/n_i$. The rationale is the following: at a zero-effort profile the prize is allocated randomly, hence no member of the winning group has privileged claims over it.

group. Conversely, the share α_i is effort-independent, and reflects e.g. group-specific social/moral customs; accordingly, it invites free-riding. Therefore, the equalising part α_i parameterises the level of *egalitarianism* within group i—the larger α_i, the more egalitarian group i is, and *vice versa.*

The allocation of the prize is stochastic and effort-contingent. In particular, the *ex ante* probability of winning faced by group i, $p_i(\mathbf{x})$, is determined by the Tullock *contest success function*

$$p_i(\mathbf{x}) = \begin{cases} \frac{X_i}{X_i+X_{-i}} & \text{if } X_i + X_{-i} > 0 \\ 1/2 & \text{otherwise} \end{cases}.$$

Therefore, the *ex ante* expected payoff $\pi_{ij}(x_{ij}; \mathrm{x}_{-ij}) = p_i(\mathbf{x})\,[v\, q_{ij}(\mathbf{x}_i)] - x_{ij}$ of the generic group member ij can be written as

$$\pi_{ij}(x_{ij}; \mathbf{x}_{-ij}) = \begin{cases} \frac{X_i}{X_i+X_{-i}} \left\{ v \left[(1-\alpha_i)\frac{x_{ij}}{X_i} + \frac{\alpha_i}{n_i} \right] \right\} - x_{ij} & \text{if } X_i + X_{-i} > 0 \\ v/2n_i & \text{otherwise.} \end{cases}, \qquad (2)$$

where $\mathbf{x}_{-ij} \in \mathbb{R}_+^{N-1}$ denotes the profile of individual effort levels across groups excluding that of player ij. Finally, let $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)$ denote the generic profile of group-specific incentivisation schemes. To ease the exposition, throughout the note we generically call Tullock Group Contest (TGC) the static game outlined in this section, that we formally define as follows.

DEFINITION 1 (Tullock Group Contest)**.**

We call Tullock Group Contest (TGC) the game

$$TGC = \langle \{N_1, N_2\}, N, \{\mathcal{X}_s\}_{s=1}^n, \{\pi_s\}_{s=1}^n \rangle$$

with:

1. $\{N_1, N_2\}$ *the group set;*
2. $N := N_1 \cup N_2$ *the player set;*
3. *the strategy sets* $\mathcal{X}_s \equiv \mathbb{R}_+$ *for each player* $s \in \{1, 2, \ldots, n\}$*;*
4. *the ex ante expected payoff functions* π_s *defined by (2)—for each player* $s \in \{1, 2, \ldots, n\}$*.* ■

2.2 WGS Equilibria

In this subsection we briefly recap the WGS equilibrium characterisation outlined in Bosco et al. (2026). For the sake of rigour, and for later use, we begin by formally stating what the WGS assumption implies.

DEFINITION 2 (WGS assumption)**.**

We say that the WGS assumption holds if the equilibrium profiles of both groups $i = 1, 2$ *are required a priori to satisfy the condition* $x_{i1}^* = x_{i2}^* = \ldots = xi^* \geq 0$*.* ■

In words: the WGS assumption mandates that, in equilibrium, all members of the same group provide the same level of effort $x_i^* \geq 0$.

Building on the model primitives outlined by Subsection 2.1, and imposing the WGS assumption outlined by Definition 2, the set of pure-strategy Nash equilibria can be easily characterised in explicit form. The analysis presented in the remainder of this subsection is a brief recap of Bosco et al. (2026). The following lemma begins the characterisation with the (sub)set of zero-effort equilibria.

LEMMA 1 (WGS case: zero-effort equilibria)**.**

The strategy profile $\mathbf{x}^* = (0, ..., 0)$ *is a Nash equilibrium of the TGC if and only if*

$$\alpha_i \geq \frac{2n_i - 1}{2(n_i - 1)} \ . \tag{3}$$

Proof. See the mathematical proof in Bosco et al. (2026). □

Now turn to the positive-effort WGS equilibria. The Lagrangian function for the generic j-th member of group i is

$$L_{ij} = \frac{(1-\alpha_i)\, v x_{ij}}{X_1 + X_2} + \frac{\alpha_i v X_i}{n_i (X_1 + X_2)} - x_{ij} + \lambda_{ij} x_{ij} \ ,$$

where $x_{ij} \geq 0$ and $\lambda_i \geq 0$. The Karush-Kuhn-Tucker (KKT) FOC for member j of group i is

$$\frac{\partial L_{ij}}{\partial x_{ij}} = (1-\alpha_i)\, v \left[\frac{X_1 + X_2 - x_{ij}}{(X_1 + X_2)^2}\right] + \frac{\alpha_i v}{n_i}\left[\frac{X_1 + X_2 - X_i}{(X_1 + X_2)^2}\right] - 1 + \lambda_{ij} = 0$$

with the complementary slackness condition $x_{ij} \geq 0$, $\lambda_{ij} \geq 0$ and $\lambda_{ij} x_{ij} = 0$. From these conditions, we can prove the following lemma.

LEMMA 2 (Magnitude of the Lagrange multipliers)**.**

For all members of all groups, $\lambda_{ij} \in [0, 1)$ *must necessaily hold.*

Proof. See Appendix A.2. □

The above individual FOC can be rewritten as

$$\frac{\partial L}{\partial x_{ij}} = 0 \iff \frac{v}{n_i (1 - \lambda_{ij})}\Big[(1-\alpha_i)\, n_i (X_1 + X_2 - x_{ij}) + \alpha_i X_{-i}\Big] = (X_1 + X_2)^2 \ ,$$

which implies

$$X_1 + X_2 = \sqrt{\frac{v}{n_i (1-\lambda_{ij})}} \sqrt{\Big[(1-\alpha_i)\, n_i (X_1 + X_2 - x_{ij}) + \alpha_i X_{-i}\Big]} \ ,$$

that finally yields player ij's explicit best-response

$$
\begin{aligned}
x_{ij}^{BR}\left(\mathbf{x}_{-ij}\right) \;=\; & -\sum_{k\neq j} x_{ik} - \sum_{j} x_{-ij} + \\
& + \sqrt{\frac{v}{n_i\left(1-\lambda_{ij}\right)}} \sqrt{\left(1-\alpha_i\right) n_i \left(\sum_{k\neq j} x_{ik} + \sum_{j} x_{-ij}\right) + \alpha_i \sum_{j} x_{-ij}}
\end{aligned}
\quad . \tag{4}
$$

Two observations ensue from the inspection of expression (4). First, within groups there is global strategic substitutability in individual effort provision. Second, across groups both complementarity and substitutability can be present, depending on the aggregate level of effort across groups. The following proposition wraps up the characterisation of WGS equilibria as derived in Bosco et al. (2026).

PROPOSITION 1 (WGS equilibria: summary of the characterisation)**.**

Let the WGS assumption outlined by Definition 2 hold. Then, the TGC under analysis has:

1. *no pure-strategy WGS equilibria for every $\boldsymbol{\alpha} \in A^{\nexists}$;*
2. *a pure-strategy WGS equilibria, unique at every $\boldsymbol{\alpha} \notin A^{\nexists}$, with*

$$
x_{1j}^{U*}\left(\boldsymbol{\alpha}\right) = \begin{cases}
\dfrac{v}{n_1^2}\left(1-\alpha_1\right)\left(n_1-1\right) & \text{if } \boldsymbol{\alpha} \in A^{+0} \\
\frac{v}{n_1} K^*\left(\boldsymbol{\alpha}\right)\left[n_1 - \alpha_1 n_2\left(n_1-1\right) + \alpha_2 n_1\left(n_2-1\right)\right] & \text{if } \boldsymbol{\alpha} \in A^{++} \\
0 & \text{if } \boldsymbol{\alpha} \in A^{0+} \\
0 & \text{if } \boldsymbol{\alpha} \in A^{00}
\end{cases} ,
$$

$$
x_{2j}^{U*}\left(\boldsymbol{\alpha}\right) = \begin{cases}
\dfrac{v}{n_2^2}\left(1-\alpha_2\right)\left(n_2-1\right) & \text{if } \boldsymbol{\alpha} \in A^{0+} \\
\frac{v}{n_2} K^*\left(\boldsymbol{\alpha}\right)\left[n_2 + \alpha_1 n_2\left(n_1-1\right) - \alpha_2 n_1\left(n_2-1\right)\right] & \text{if } \boldsymbol{\alpha} \in A^{++} \\
0 & \text{if } \boldsymbol{\alpha} \in A^{+0} \\
0 & \text{if } \boldsymbol{\alpha} \in A^{00}
\end{cases} ,
$$

with

$$
K^*(\boldsymbol{\alpha}) := \frac{n_1 + n_2 - 1 - \alpha_1(n_1-1) - \alpha_2(n_2-2)}{(n_1+n_2)^2} \;>\; 0 \; ,
$$

and where the subspaces A^{00}, A^{0+}, A^{+0}, A^{++} and $A^{\nexists}$ are formally defined in Appendix A.1.

Proof. The characterisation is formally derived in Bosco et al. (2026). □

Since $A^{\nexists} = \mathbb{R}^2 \setminus \left\{A^{00} \cup A^{0+} \cup A^{+0} \cup A^{++} \cup\right\} \neq \emptyset$ holds, for later use denote with $A^{\exists} := \mathbb{R}^2 \setminus A^{\nexists} \subsetneq \mathbb{R}^2$ the set of profiles (α_1, α_2) of incentivisation schemes such that a WGS equilibrium exists for the TGC.

3 Asymmetric Equilibria

We proceed by proving the existence of WGA pure-strategy equilibria and characterising necessary and sufficient conditions for their existence. For later use, we provide the following definition.

DEFINITION 3 (Active and inactive subgroups)**.**
Consider the generic group $i = 1, 2$. We call active subgroup *of group i, and denote it with $\mathscr{A}_i^+$, the (possibly empty) subset of members j of that group with $x_{ij} > 0$. Formally*

$$\mathscr{A}_i^+ := \Big\{ j \in N_i \,:\, x_{ij} > 0 \,,\, i = 1, 2 \Big\} .$$

Moreover, we call inactive subgroup *of group i the complement of $\mathscr{A}_i^+$ in i, and denote it with $\mathscr{A}_i^-$, with $x_{ij} = 0$ for all $j \in \mathscr{A}_i^-$. Moreover, we call* contributor *and* free rider *any member of group i included, respectively, into the active and inactive subgroup.* ■

In words: the active subgroup of group i is the subset of its members that provide positive effort (hence the labelling 'contributors'), whereas the inactive subgroup is the subset of its members that, in equilibrium, free-ride on the other members' effort (hence the labelling 'free riders'). By definition, for every profile $\boldsymbol{\alpha} := (\alpha_1, \alpha_2)$ of sharing rules such that $\boldsymbol{\alpha} \in A^{\exists}$, in any group at least one subgroup *must* be nonempty, whereas one of the two subgroups *can* be empty. The equilibrium characterisation we propose is logically structured as follows.

1. We first prove that, in any group $i = 1, 2$, all the contributors (if any) must provide, in equilibrium, the same amount of effort. This implies that the structure of any WGA equilibrium must display symmetry-in-asymmetry: any group whose equilibrium distribution of effort provision is uneven across members must be split into two subgroups, those of contributors and free riders, in which effort provision is symmetric across members.

2. We then prove that a nonempty set of profiles of incentivisation schemes always exists such that, in any group where both the active and inactive subgroups are nonempty, no free rider has an incentive to become a contributor. In turn, this identifies a straightforward necessary condition for the existence of WGA equilibria.

3. We then show that the necessary condition for existence implies that, in any WGA equilibrium, both groups must actively provide some effort.

4. Finally, we identify additional necessary and sufficient conditions for the aforementioned WGA profiles to be consistent with equilibrium.

3.1 Structure of the WGA Equilibria

Focus on the generic group $i = 1, 2$, and assume that the the cardinality of the active subgroup $\mathscr{A}_i^+$ is $m_i > 0$—i.e. that the active subgroup of group i is nonempty and includes m_i members. Then, $X_i = \sum_{\ell=1}^{n_i} x_{i\ell} = \sum_{\ell=1}^{m_i} x_{i\ell} > 0$ must hold by definition. Denote with X_i^{-j} the effort provided in the aggregate by group i excluding that of its member j—formally, $X_i^{-j} = \sum_{\ell \neq j} x_{i\ell} = X_i - x_{ij}$. Then, via expression (4), for the generic member $j \in \mathscr{A}_i^+$ it must hold that

$$x_{ij}^{BR}(\mathbf{x}_{-ij}) = -X_i^{-j} - X_{-i} + \sqrt{\frac{v}{n_i}\left[n_i(1-\alpha_i)\left(X_i^{-j} + X_i\right) + \alpha_i X_{-i}\right]} \, . \tag{5}$$

It is here important to stress that the FOC underlying expression (5) pins down the optimal effort provision *of contributors only*: the null effort provision of free riders, if any, is sustained by the fact that no deviation to positive effort level exists, given the aggregates, such that the expected payoff differential is strictly positive.

Direct inspection of (5) reveals that the generic individual best-response displays intra-group global strategic substitutability: effort-provision decisions by single active members $j \in \mathscr{A}_i^+$ are perfect substitutes, since they contribute symmetrically to the determination of the group effort level X_i. This strong form of intra-group substitutability suggests that asymmetric equilibria may indeed exist where, within the same group, some members actively provide effort while the other members free-ride. The following lemma proves an important intermediate result: if the active subgroup of group i is nonempty, then all its m_i members must provide, in equilibrium, the same effort level.

LEMMA 3 (Within-group symmetry among contributors)**.**
In equilibrium, for every $j', j'' \in \mathscr{A}_i^+$ *it holds that* $x_{ij'}^* = x_{ij''}^*$.

Proof. See Appendix A.3. □

Lemma 3 proves an important symmetry-in-asymmetry result: since, in any equilibrium, all active members of group i must provide the same effort levels, *if* an asymmetric equilibrium indeed exists it must be such that effort provision is strictly *symmetric* within the active subgroup—all active members exert the same effort $x_{ij}^* > 0$. Symmetry among free riders comes by definition. Therefore, because of their peculiar structure, we refer to such potential equilibria as "group-splitting" equilibria.

DEFINITION 4 (Group-splitting equilibrium and profile)**.**
We define group-splitting equilibrium *any equilibrium of the TGC under analysis whose strategy profile displays the intra-group clustered structure*

$$\mathbf{x^*} = (\mathbf{x_1^*}, \mathbf{x_2^*}) = \Big((x_{11}^*, \ldots, x_{1n_1}^*), (x_{21}^*, \ldots, x_{2n_2}^*)\Big) = \Bigg(\Big(\underbrace{x_1^*, x_1^*, \ldots}_{m_1}, \underbrace{0, 0, \ldots, 0}_{n_1 - m_1}\Big), \Big(\underbrace{x_2^*, x_2^*, \ldots}_{m_2}, \underbrace{0, 0, .., 0}_{n_2 - m_2}\Big)\Bigg),$$

with $m_i \in \{0, 1, \ldots, n_i\}$, and where $x_i^* > 0$ is the common effort level provided by the active members $i \in \mathscr{A}_i^+$ of group $i \in \{1, 2\}$. Accordingly, we refer to any profile that implies a group-splitting structure as a group-splitting profile. ■

It is immediate to check that Definition 4 subsumes the canonical WGS equilibria as special (degenerate) cases. Note indeed that a zero-effort equilibrium is a degenerate group-splitting equilibrium where $m_i = 0$ holds for both groups, whereas any positive-effort WGS equilibrium is a degenerate group-splitting equilibrium where $m_i = n_i$ holds for all active groups. For the sake of clarity, the following remark restates Lemma 3 in terms of the group-splitting equilibrium notion outlined by Definition 4.

REMARK 1 (All asymmetric equilibria are group-splitting).
If an asymmetric pure-strategy equilibrium exists for the TGC under analysis, then in must be in group-splitting form. Therefore, since WGS equilibria are group-splitting, too, all pure-strategy equilibria of a symmetric Tullock two-group contest are group-splitting equilibria. ■

To ease the exposition, we henceforth refer to any group to which a non-degenerate group-splitting profile is assigned as a 'split group'.

3.2 Necessary Condition for Existence

As already stressed in the previous subsection, in any WGA equilibrium the FOC (5) only constrains the behaviour of effort contributors. In any split group the compatibility of the generic free rider's null effort provision with the equilibrium play does not stem from optimality *a priori*, but rather from the absence of profitable unilateral deviations to some positive effort level *given group-specific incentivisation and the behaviour of that group's contributors*. Building on the analysis of unilateral incentives to deviate, the following lemma identifies a necessary existence condition.

LEMMA 4 (Necessary condition for the existence of across-group-symmetric WGA equilibria).
Let $X_i > 0$ hold for both groups $i = 1, 2$. Moreover, let each group i be split, so that $x_{ij} > 0$ holds for $m_i \in \{1, \ldots, n_i - 1\}$ group members and $x_{ij} = 0$ hold for all other $n_i - m_i > 0$ members. Then,

$$\alpha_i > 1 \qquad \text{for both groups } i = 1, 2$$

is a necessary condition for null effort provision by the free riders to be compatible with equilibrium.

Proof. See Appendix A.5. □

The underlying intuition is straightforward: $\alpha_i > 1$ implies that the incentivisation scheme of group i discourages effort provision by taxing its contributors to cross-subsidise its free riders. When this is the case, any increase in effort provision entails, from the perspective of the individual free rider, *two* additional costs: (i) the direct cost that stems from switching to null to positive effort provision; (ii) the indirect cost that stems from the fact that, upon switching to costly effort provision, the former free rider gets his/her fair share of the incentivising part $1 - \alpha_i$ that, with $\alpha_i > 1$, amounts *de facto* to a distorsionary tax.

Direct implication of Lemma 4 is that any WGA equilibrium must entail positive effort provision by both groups. Direct inspection of Proposition 1 reveals the underlying intuition: across-group-asymmetric equilibria in which group i is the only active group can only be sustained by incentivisation schemes with $\alpha_i < 1$—i.e. by incentivisation schemes that either redistribute the prize or tax the free riders. Given the necessary condition identified by Lemma 4, therefore, inverse cross-subsidisation from the contributor to the free riders is incompatible with the existence of WGA equilibria with a single active group. The following lemma formally states and proves the result.

LEMMA 5 (Both groups must be active in any WGA equilibrium)**.**
No WGA equilibrium exists such that $X_i^* > 0$ *and* $X_{-i}^* = 0$, $i = 1, 2$.

Proof. See Appendix A.4. □

3.3 Characterisation

Building on the result of Lemma 5, we proceed with the characterisation by focusing on candidate WGA equilibria where both groups are active. We characterise first the effort levels x_1^+ and x_2^+ that maximize $\pi_{ij}\left(x_i^+;\mathbf{x^*}\right)$ for the generic contributors $j \in \mathscr{A}_i^+$. Building on the symmetry result of Lemma 3, we can write the expected payoff of the generic contributor ij as

$$\begin{aligned}\pi_{ij}\left(x_i^+;\mathbf{x^*}\right) &= v\left(\frac{m_i x_i^+}{m_i x_i^+ + m_{-i}x_{-i}^+}\right)\left[(1-\alpha_i)\frac{x_i^+}{m_i x_i^+} + \alpha_i \frac{1}{n_i}\right] - x_i^+ = \\ &= \frac{v\,(1-\alpha_i)\,x_{ij}^+}{m_i x_i^* + m_{-i}x_{-i}^+} + v\frac{\alpha_i}{n_i}\left(\frac{m_i x_i^+}{m_i x_i^+ + m_{-i}x_{-i}^+}\right) - x_i^+ \,.\end{aligned} \qquad . \tag{6}$$

The expected payoff (6) is strictly concave in x_i^+. Accordingly, the FOC

$$\frac{\partial}{\partial x_i^+}\pi_{ij} = 0 \iff v\left[(1-\alpha_i)\left(\frac{m_i x_i^+ + m_{-i}x_{-i}^+ - x_i^+}{\left(m_i x_i^+ + m_{-i}x_{-i}^+\right)^2}\right) + \frac{\alpha_i}{n_i}\left(\frac{m_i x_i^+ + m_{-i}x_{-i}^+ - m_i x_i^+}{\left(m_i x_i^+ + m_{-i}x_{-i}^+\right)^2}\right)\right] = 1 \,,$$

which implies

$$v\left\{(1-\alpha_i)\left[(m_i-1)\,x_i^+ + m_{-i}x_{-i}^+\right] + \alpha_i \frac{m_{-i}}{n_i} x_{-i}^+\right\} \;=\; \left(m_i x_i^+ + m_{-i}x_{-i}^+\right)^2 \,,$$

is sufficient for the identification of a (global) maximum. Then, the positive effort levels x_1^+ and x_2^+ are the unique solution of the 2×2 determinate system of equations

$$\begin{cases} v\left\{(1-\alpha_1)\left[(m_1-1)\,x_1^+ + m_2 x_2^+\right] + \alpha_1 \frac{m_2}{n_1} x_2^+\right\} \;=\; \left(m_1 x_1^+ + m_2 x_2^+\right)^2 \\ v\left\{(1-\alpha_2)\left[(m_2-1)\,x_2^+ + m_1 x_1^+\right] + \alpha_2 \frac{m_1}{n_2} x_1^+\right\} \;=\; \left(m_1 x_1^+ + m_2 x_2^+\right)^2 \end{cases} . \tag{7}$$

To ease calculations, we henceforth focus on the convenient parameterisation $n_1 = n_2 = n \geq 2$ and $m_1 = m_2 = m \in \{1, \dots, n-1\}$.

ASSUMPTION 1 (Super-symmetric paramterisation)**.**

Henceforth, $n_1 = n_2 = n \geq 2$, $m_1 = m_2 = m \in \{1, \dots, n-1\}$ and $\alpha_1 = \alpha_2 = \alpha$ are assumed to hold. ■

In words: on top of symmetry in group sizes, we are imposing that, in any WGA equilibrium, the number of contributors in both group be identical. It is immediate to check from (7) that Assumption 1 implies $x_1^+ = x_2^+$, i.e. that effort provision is symmetric across groups. Then, the following characterisation can be easily derived.

PROPOSITION 2 (WGA equilibria: characterisation for the super-symmetric parameterisation)**.**

Let Assumption 1 hold. Moreover, let $m \in \{1, \dots, n\}$. Then, the condition

$$\alpha \;\in\; \left(1\,,\, \frac{n(2m-1)}{n(2m-1)-m}\right) \neq \emptyset \tag{8}$$

is necessary and sufficient for the TGC under analysis to have a WGA equilibrium such that, in each group, exactly $m > 0$ members each provide strictly positive effort

$$x_1^* \;=\; x_2^* \;=\; x^+ \;=\; \frac{v}{4nm^2}\left[\,n(2m-1)(1-\alpha) + m\alpha\,\right] \;>\; 0 \,, \tag{9}$$

whereas all the other $n-m > 0$ members free-ride by providing $x^* = 0$.

Proof. See Appendix A.6. □

The characterisation outlined by Proposition 2 implies that the free riders enjoy, in expected terms, a higher payoff that the contributors. Note indeed that expression (9) entails a payoff differential

$$\varDelta\pi_i \;:=\; \pi_{ij}(x_{ij}^* = 0; \mathbf{x}^*) - \pi_{ij}(x_{ij}^* = x^+; \mathbf{x}^*) \;=\; \frac{v}{4nm^2}\left[\,\alpha(m+n) - 1\,\right] \;>\; 0 \,, \tag{10}$$

that strictly decreases both in m and n.[10] It is finally worth stressing that the necessary and sufficient condition identified by Proposition 2 is consistent with the WGS characterisation summarised by Lemma 1 and Proposition 1. Note indeed that, for $n = m$, the upper bound of the open interval in (8) becomes $(2n-1)/[2(n-1)]$, that is exactly the critical value (3) of Lemma 1.

4 Conclusion

In the context of a two-group specification for a symmetric, complete-information Tullock group contest, this note proved that within-group-asymmetric equilibria exist where both groups must be active. Such equilibria necessarily come in group-splitting form: in at least one group, effort provision is partitioned so that a subgroup of members free-rides on the positive effort provided by all other members. This implies that, albeit intuitive and mathematically convenient, the widespread use of the WGS assumption in the literature, is effective but restrictive.

A promising avenue for future research uncovered by our analysis is the characterisation of endogenous incentivisation schemes when downstream effort provision is consistent with WGA equilibria. Such a characterisation would settle a question yet unaddressed by our analysis: can the symmetric incentivisation schemes consistent with WGA equilibria be reached within a subgame-perfect equilibrium where incentivisation is *endogenously* determined? It may well be possible *in principle* that the incentivisation schemes that sustain the WGA equilibria are 'salient' in some sense when sequential rationality is taken into account. Finally, the aggregative structure at the group level displayed by the WGA equilibria calls for a generalisation of the analysis presented in this note to a many-group specification via the algorithmic procedure developed by Bosco and Gilli (2024).

[10]Note indeed that $\partial \Delta\pi_i/\partial m \propto -[\alpha(m+2n)+2] < 0$ and $\partial \Delta\pi_i/\partial n \propto -(\alpha m+1) < 0$.

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A Proofs

A.1 Definitions for the Subspaces A^{00}, A^{0+}, A^{+0}, A^{++} and $A^{\nexists}$

The subspaces A^{00}, A^{0+}, A^{+0} and A^{++} mentioned in Proposition 1 are formally defined as follows:

$$
\begin{aligned}
A^{00} &= \left\{(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R} : \alpha_2 \geq -\alpha_1\frac{n_1-1}{n_2-1} + \frac{n_1+n_2-1}{n_2-1}\right\}, \\
A^{+0} &= \left\{(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R} : \left\{\begin{array}{c} \alpha_2 \leq -\alpha_1\frac{n_1-1}{n_2-1} + \frac{n_1+n_2-1}{n_2-1} \\ \alpha_2 \geq \alpha_1\frac{(n_1-1)n_2}{n_1(n_2-1)} + \frac{n_2}{n_1(n_2-1)} \end{array}\right\}\right\}, \\
A^{++} &= \left\{(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R} : \left\{\begin{array}{c} \alpha_2 \leq -\alpha_1\frac{n_1-1}{n_2-1} + \frac{n_1+n_2-1}{n_2-1} \\ \alpha_2 \geq \alpha_1\frac{(n_1-1)n_2}{n_1(n_2-1)} - \frac{1}{n_2-1} \\ \alpha_2 \leq \alpha_1\frac{(n_1-1)n_2}{n_1(n_2-1)} + \frac{n_2}{n_1(n_2-1)} \end{array}\right\}\right\}, \\
A^{0+} &= \left\{(\alpha_1,\alpha_2)\in\mathbb{R}\times\mathbb{R} : \left\{\begin{array}{c} \alpha_2 \leq -\alpha_1\frac{n_1-1}{n_2-1} + \frac{n_1+n_2-1}{n_2-1} \\ \alpha_2 \leq \alpha_1\frac{(n_1-1)n_2}{n_1(n_2-1)} - \frac{1}{n_2-1} \end{array}\right\}\right\}.
\end{aligned}
$$

Finally, the non-existence area $A^{\nexists}$ is defined as the 'residual' with respect to the domain $\mathbb{R}^2$, i.e

$$A^{\nexists} = \mathbb{R}^2 \setminus \left\{A^{00} \cup A^{0+} \cup A^{+0} \cup A^{++}\right\}.$$

A.2 Proof of Lemma 2

The proof is straightforward. The Lagrange multipliers are nonnegative because of the Complementary-Slackness Condition (CSC). We now prove by contradiction that they are strictly smaller than 1 using the individual FOCs. Assume that $\lambda_{ij} \geq 1$ holds. Then, $(1-\alpha_i)\, v\left[\frac{X_1+X_2-x_{ij}}{(X_1+X_2)^2}\right] + \frac{\alpha_i v}{n_i}\left[\frac{X_1+X_2-X_i}{(X_1+X_2)^2}\right] = 1-\lambda_i < 0$ must hold, which is impossible $\forall\,(X_1+X_2) > 0$. Therefore, $\lambda_{ij} \in [0,1)$ must hold for all players ij, which proves the lemma. QED

A.3 Proof of Lemma 3

Focus on group i, and assume that its active subgroup $\mathscr{A}_i^+$ be nonempty—with cardinality $m_i \leq n_i$. Assume further that individual players in each group i are ordered so that the first m_i members are the effort providers. Then, the system of m_i FOCs can be written as follows:

$$
\begin{aligned}
x_{i1}^+ &= \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^* + X_i^{*-1}\right) + \frac{\alpha_i}{n_i}X_{-i}^*\right]} - \left(X_{-i}^* + X_i^{*-1}\right), \\
x_{i2}^+ &= \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^* + X_i^{*-2}\right) + \frac{\alpha_i}{n_i}X_{-i}^*\right]} - \left(X_{-i}^* + X_i^{*-2}\right), \\
\vdots \quad & \quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots
\end{aligned}
$$

$$
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
x_{im_i}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-m_i}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-\left(X_{-i}^{*}+X_i^{*-2}\right),
\end{array}
$$

where the notation x_{ij}^{+} stresses the fact that the FOC constrains only the (optimal) behaviour of the contributors of group i. We can rewrite the above list of m_i FOCs as

$$
\begin{array}{ccc}
x_{i1}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-1}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i2}^{+}-x_{i3}^{+}-...-x_{im_i}^{+}, \\
x_{i2}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-2}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i1}^{+}-x_{i3}^{+}-...-x_{im_i}^{+}, \\
\vdots & \vdots & \vdots \\
x_{im_i}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-m_i}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i1}^{+}-x_{i2}^{+}-...-x_{im_i-1}^{+}.
\end{array}
$$

Swapping the LHS of each FOC with the term x_{i1}^{+} at the RHS yields the system of m_i FOCs

$$
\begin{array}{ccc}
x_{i1}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-1}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i2}^{+}-x_{i3}^{+}-...-x_{im_i}^{+}, \\
x_{i1}^{+} & = & \sqrt{v\left[(1-\alpha_i)\,,\left(X_{-i}^{*}+X_i^{*-2}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i2}^{+}-x_{i3}^{+}-...-x_{im_i}^{+}, \\
\vdots & \vdots & \vdots \\
x_{i1}^{+} & = & \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-m_i}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-X_{-i}^{*}-x_{i1}^{+}-x_{i2}^{+}-...-x_{im_i}^{+},
\end{array}
\tag{A.1}
$$

that can be concatenated as

$$
X_i^{*-1} = X_i^{*-2} = \ldots = X_i^{*-m_i}. \tag{A.2}
$$

Via mathematical induction note that: (i) $X_i^{*-1} = X_i^{-2}$ implies $x_{i1}^{*} = x_{i2}^{*}$; (ii) $X_i^{*-j} = X_i^{*-(j+1)}$ implies $x_{i(j+1)}^{+} = x_{ij}^{+}$ for every $j \in \mathscr{A}_i^{+}$. Therefore, expression (A.2) yields $x_{i1}^{+} = x_{i2}^{+} = \cdots = x_{im_i}^{+} := x_i^{+}$. QED

A.4 Proof of Lemma 5

Focus on group i, and assume that its active subgroup $\mathscr{A}_i^{+}$ be nonempty—with cardinality $m_i \leq n_i$. Assume further that group i is the sole active group, whereby $X_{-i} = 0$ holds. Via expression (5) in the main text, fhe FOC of the generic contributor ij becomes

$$
\left(X_i^{*}\right)^2 = v(1-\alpha_i)\left(X_i^{*}-x_{ij}^{+}\right) \tag{A.3}
$$

Via Lemma 3 we know that, in any equilibrium, effort provision must be symmetric across contributors.

Therefore, by substituting for $X_i^* = m_i x_i^+$ expression (A.3) yields

$$x^+ \;=\; \frac{v}{m_i^2}\,(1-\alpha_i)\,(m_i-1)\;, \tag{A.4}$$

that is indeed strictly positive if and only if $\alpha < 1$ and $m_i > 1$. Assume that $m_i > 1$ holds, so that the effort (A.4) is compatible with equilibrium for the generic contributor. Consider now the generic free rider ij. His/her expected payoff is simply $v\alpha_i/n_i > 0$. Upon unilaterally deviating to a generic effort level x'_{ij} that free rider gets

$$\pi_{ij}\left(x'_{ij};\mathbf{x}_{-ij}\right) \;=\; v\left[(1-\alpha_i)\frac{x'_{ij}}{m_i x_i^+ + x'_{ij}} + \frac{\alpha_i}{n_i}\right] - x'_{ij}\;.$$

Since the equalising part $v\alpha_i/n_i$ is identical across both payoffs, the unilateral deviation is strictly profitable if and only if

$$x'_{ij} \;<\; v(1-\alpha_i) - m x_i^+ \tag{A.5}$$

that, substituting for the definition (A.4) of x_i^+, yields the condition $x'_{ij} < v(1-\alpha_i)$. Recall that $\alpha_i < 1$ is a necessary condition for the group-splitting profile under analysis to be a WGA equilibrium. However, via (A.5) it is immediate to check that, for every $\alpha < 1$, the generic free rider finds it profitable to unilaterally deviate to some effort level $x'_{ij} \in (0, v(1-\alpha_i)) \neq \emptyset$. Therefore, no WGA equilibrium can exist where group i is the sole active group, which proves the result. QED

A.5 Proof of Lemma 4

Consider the generic unilateral deviation $x'_{ij} > 0$ by the generic free rider in group i. Upon deviating, he/she is faced with the expected payoff

$$\begin{aligned}\pi_{ij}(x'_{ij};\mathbf{x}_{-\boldsymbol{ij}}) \;&=\; \frac{v\left(X_i + x'_{ij}\right)}{X_i + X_{-i} + x'_{ij}}\left[(1-\alpha_i)\frac{x'_{ij}}{X_i + x'_{ij}} + \frac{\alpha_i}{n_i}\right] - x'_{ij}\\ &=\; \frac{v(1-\alpha_i)x'_{ij}}{X_i + X_{-i} + x'_{ij}} + \frac{v\alpha_i\left(X_i + x'_{ij}\right)}{n_i\left(\sum_{\ell=1}^m X_\ell + x'_{ij}\right)} - x'_{ij}\;.\end{aligned} \tag{A.6}$$

Via expression (A.6) it is immediate to check that, in equilibrium $\pi_{ij}(x_{ij}\!=\!x_{ij}^{dev};\mathbf{x}^*_{-\boldsymbol{ij}}) > \pi_{ij}(x_{ij}\!=\!0;\mathbf{x}^*_{-\boldsymbol{ij}})$ if and only if

$$x'_{ij} \;<\; v\left[(1-\alpha_i) + \frac{\alpha_i}{n_i}\left(\frac{X^*_{-i}}{X_i^* + X^*_{-i}}\right)\right] - (X_i^* + X^*_{-i}) \;=:\; \bar{x}^{dev}\;. \tag{A.7}$$

Condition (A.7) reveals that profitable deviations to positive effort provision are *not* available to the generic free rider if and only if $\bar{x}^{dev} \le 0$ holds, which implies

$$(X_i^* + X^*_{-i}) \;\ge\; v\left[(1-\alpha_i) + \frac{\alpha_i}{n_i}\left(\frac{X^*_{-i}}{X_i^* + X^*_{-i}}\right)\right]$$

that, multiplying both sides by $(X_i^* + X^*_{-i})$, yields the condition

$$\bar{x}^{dev} \le 0 \quad\Longleftrightarrow\quad (X_i^* + X^*_{-i})^2 \;\ge\; v\left[(1-\alpha_i)\,(X_i^* + X^*_{-i}) + \frac{\alpha_i}{n_i}X^*_{-i}\right]\;. \tag{A.8}$$

Recall from Lemma 3 that the FOC for the generic contributor j is

$$x_{ij}^{+} \;=\; \sqrt{v\left[(1-\alpha_i)\left(X_{-i}^{*}+X_i^{*-j}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]}-\left(X_{-i}^{*}+X_i^{*-j}\right)\;. \tag{A.9}$$

Recall further that $x_{ij}^{*} = x_{ij}^{+} > 0$ holds for the generic contributor. Recall finally that $X_i^{*-j} = \sum_{\ell\neq j} x_{i\ell}^{*} = X_i^{*} - x_{ij}^{*} = X_i^{*} - x_{ij}^{+}$ holds by definition. Therefore, the generic FOC (A.9) can be rewritten as

$$\left(X_i^{*}+X_{-i}^{*}\right)^2 \;=\; v\left[(1-\alpha_i)\left(X_i^{*}+X_{-i}^{*}-x_{ij}^{+}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right].$$

Bringing the element $x_{ij}\,[v(1-\alpha_i)]$ to the LHS we finally obtain

$$x_{ij}^{+}\,[v(1-\alpha_i)] \;=\; v\left[(1-\alpha_i)\left(X_i^{*}+X_{-i}^{*}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]-\left(X_i^{*}+X_{-i}^{*}\right)^2\;. \tag{A.10}$$

Note that the LHS of expression (A.10) is strictly negative for every $\alpha_i > 1$, since $x_{ij}^{+} > 0$ holds by definition. Therefore, via (A.10) the condition $\alpha_i > 1$ implies

$$\left(X_i^{*}+X_{-i}^{*}\right)^2 \;>\; v\left[(1-\alpha_i)\left(X_i^{*}+X_{-i}^{*}\right)+\frac{\alpha_i}{n_i}X_{-i}^{*}\right]\;,$$

which in turn implies $\bar{x}^{dev} < 0$ via condition (A.8). Therefore, the condition $\alpha_i > 1$ is necessary to guarantee that deviating to some $x'_{ij} > 0$ is never profitable for the generic free rider in group i, which proves the result. QED

A.6 Proof of Proposition 2

Note that Assumption 1 implies that the 2×2 system (7) is over-determinate, since $x_1^{+} = x_2^{+} = x^{+}$ must hold. Substituting for $n_1 = n_2 = n$ and $m_1 = m_2 = n$ into the representative FOC yields

$$x^{+} \;=\; \frac{v}{4nm^2}\left[n(2m-1)(1-\alpha)+m\alpha\right], \tag{A.11}$$

that is strictly positive if and only if

$$\alpha \;<\; \bar{\alpha}_{++} \;:=\; \frac{n(2m-1)}{n(2m-1)-m}\,. \tag{A.12}$$

Direct inspection of the definition for $\bar{\alpha}_{++}$ in (A.12) immediately reveals that $\bar{\alpha}_{++} > 1$ holds for every $(n,m) \in \mathbb{N}^2 \setminus \{(1,m)\}$. Recall from Lemma 4 that $\alpha > 1$ is necessary for across-group-symmetric WGA equilibria to exist. Recall further that across-group-asymmetric WGA equilibria are unique in their areas of existence. Therefore, $\alpha \in (1,\bar{\alpha}_{++}) \neq \emptyset$ is necessary and sufficient for (A.11) to be compatible with equilibrium for all the effort contributors. Recall finally that $\alpha > 1$ guarantees that no free rider has a unilateral incentive to deviate to positive effort provision. Therefore, at every $\alpha \in (1,\bar{\alpha}_{++})$ unique across-group-symmetric WGA equilibrium exists such that all contributors provide effort (A.11) and all free riders provide null effort. Note finally that the equilibrium payoff of the generic free rider is

$$\pi_{ij}(x_{ij}^{*}\!=\!0;\mathbf{x}^{*}) \;=\; \frac{v}{2}\frac{\alpha}{n} \;>\; 0\;,$$

whereas the equilibrium payoff of the generic contributor is

$$\pi_{ij}(x^*_{ij}=x^+;\mathbf{x}^*) \;=\; \frac{v}{4nm^2}\Big[\,n(1-\alpha)+m(2m-1)\alpha\,\Big], \tag{A.13}$$

that is strictly positive for $\alpha < \bar{\bar{\alpha}}_{++} := n/[n-m(2m-1)]$. It is immediate to check that $\bar{\alpha}_{++} \leq \bar{\bar{\alpha}}_{++}$ for every $m \geq 1$. Therefore, the expected payoff (A.13) is always positive, too, which completes the proof. QED