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Abstract

Alternative market allocations are compared in a linear setting with product differentiation. With two commodities, assuming that both Ramsey and Cournot quantities are positive, the Cournotian quantities are shown to be not larger than their Bertrand counterparts, regardless of product substitutability. It also turns out that the case of a Bertrand price exceeding its Cournotian counterpart (which requires product complementarity) and the existence of a “subordinate” commodity receiving a negative price-cost margin *à la* Ramsey are equivalent conditions.

1 Introduction

It has been known since Singh and Vives (1984) that there is no general ranking between Cournot and Bertrand outcomes when products are differentiated, even when they are just two² and demand and cost functions are linear. For example, even if products are symmetric the Bertrand profits are smaller than the Cournotian profits under product substitutability, but the reverse applies to the case of product complementarity.³ In spite of that, it is a widespread view in the literature that price competition leads to lower prices (see e.g. Belleflamme and Peitz, 2015: p. 66, Lesson 3.11), the reason being that own demand elasticity *tends* to be locally higher in a Bertrand setting.⁴ Clearly, this can be

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²The results of Häckner (2000) and Amir and Jin (2001) suggest forcefully that no general ranking is available when there are more than two products.

³Under complementarity the lower Bertrand price is in fact closer to the price that would be used by a two-product monopolist. Indeed, this ranking reversal is perhaps not a surprise since with independent commodities (meaning $\frac{\partial x_i}{\partial p_j} = 0$, $i \neq j$) oligopoly outcomes would be equal to the monopoly ones. Moreover, as shown by Zanchettin (2006), Bertrand profits can be larger than Cournotian profits even under substitutability if products are asymmetric.

⁴Own demand elasticity in a Bertrand setting is locally higher than in its Cournotian counterpart if $\text{sign} \left\{ \frac{\partial x_i}{\partial p_j} \right\} = -\text{sign} \left\{ \frac{\partial p_i}{\partial q_j} \right\}$ for all i and j , $i \neq j$, a condition which always holds with two goods: see e.g. Vives (1999: chapter 6).

misleading if elasticities are not constant (as it happens with linear demands) and products are asymmetric.

Indeed, as far as we know, a general price (quantity) ranking can only be achieved when the underlying price (quantity) game is also supermodular (Vives, 1999: p. 156). When applied to the linear case these results say that if products are (all) substitutes⁵ the Cournotian equilibrium prices are higher and if products are (all) complements⁶ the Cournotian equilibrium quantities are smaller than their Bertrand counterparts.⁷ Accordingly, these results admit the possibility (discussed by Amir and Jin, 2001) that even in the linear case under “overall” (see Section 2) product complementarity some Bertrand price might be higher and under “overall” product substitutability some Bertrand quantity might be smaller than their Cournotian counterparts.

In this note we explore these issues focusing on the case of two commodities in a linear setting. It turns out that assuming that the Ramsey and Cournotian quantities are positive the latter are both smaller than their Bertrand counterparts, regardless of product substitutability. In addition, we show that the Bertrand price of one commodity can actually be higher than its Cournotian counterpart if and only if that commodity is “subordinate” to the other, meaning that its Ramsey price would be inferior to its marginal cost. This requires the products to be complements and it also implies that the Bertrand price of the other commodity is lower than the price that a multiproduct monopolist would charge for it.

To the best of our knowledge, Bertoletti (2024) provides the first example of the price ranking characterized in this note. The other paper most closely related to this is Amir and Jin (2001), whose setting is very much the same as ours. Focusing on the case of $n \geq 3$ products they answer in the negative the question about the possibility of ranking quantities, and assuming that the marginal willingness to pay when *all* quantities are zero exceeds the corresponding marginal cost for *each* commodity they do not identify the connection between the reversal of the usual price Bertrand-Cournot ranking and the existence of a subordinate commodity.

2 Ramsey pricing

Suppose that the representative consumer has quasi-linear preferences represented by the following utility function defined over the consumption of n commodities:

$$u(\mathbf{x}) = \mathbf{a}'\mathbf{x} - \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{2},$$

⁵ Meaning $\text{sign} \left\{ \frac{\partial x_i}{\partial p_j} \right\} > 0$ for all i and j , $i \neq j$.

⁶ Meaning $\text{sign} \left\{ \frac{\partial p_i}{\partial q_j} \right\} > 0$ for all i and j , $i \neq j$.

⁷ We are not aware of general results covering cases in which some products are substitutes and others are complements according to the signs of $\frac{\partial x_i}{\partial p_j}$ or $\frac{\partial p_i}{\partial x_j}$: see Amir and Jin (2001).

where: $\mathbf{B}$ is a symmetric, positive definite matrix (whose (i, j) element we denote b_{ij}), $\mathbf{x}$ the non-negative quantity vector, and $\mathbf{a}$ the non-negative vector of the marginal willingnesses to pay for products when consumption is null.

The corresponding linear, inverse demand system is given by

$$\mathbf{p}(\mathbf{x}) = Du(\mathbf{x}) = \mathbf{a} - \mathbf{B}\mathbf{x},$$

where $\mathbf{p}$ is the non-negative price vector of the consumption goods. Consumer surplus is then given by

$$s(\mathbf{x}) = \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{2} \text{ with } Ds(\mathbf{x}) = \mathbf{B}\mathbf{x}:$$

note that $s(\mathbf{x})$ is homothetic, and then preferences belong to the class characterized by Armstrong and Vickers (2016).

The demand system is provided by

$$\mathbf{x}(\mathbf{p}) = \tilde{\mathbf{a}} - \mathbf{D}\mathbf{p},$$

where matrix $\mathbf{D} = \mathbf{B}^{-1}$ is also symmetric and definite positive, and $\tilde{\mathbf{a}} = \mathbf{D}\mathbf{a}$. It is standard to say that products i and j are substitutes if $d_{ij} < 0$ and complements if $d_{ij} > 0$. One can also classify as substitutes products i and j if $b_{ij} > 0$ and as complements if $b_{ij} < 0$, but these classifications are possibly inconsistent when $n > 2$. However, they are consistent whenever the extra-diagonal elements of $\mathbf{D}$ or $\mathbf{B}$ are all non-positive, implying respectively that products are overall (weakly) substitutes (since also $\mathbf{B} \geq \mathbf{0}$) or overall (weakly) complements (since also $\mathbf{D} \geq \mathbf{0}$): see Takayama (1985: p. 392).

Assume that the overall cost function is $c(\mathbf{x}) = \mathbf{c}'\mathbf{x}$, where $\mathbf{c}$ is the non-negative vector of marginal costs. In this case the profit function of a multi-product monopolist and the corresponding Ramsey welfare expression would be given by

$$\pi(\mathbf{x}) = (\mathbf{a} - \mathbf{c})'\mathbf{x} - \mathbf{x}'\mathbf{B}\mathbf{x}, \quad W^\alpha(\mathbf{x}) = \pi(\mathbf{x}) + \alpha s(\mathbf{x}) = (\mathbf{a} - \mathbf{c})'\mathbf{x} - \frac{2 - \alpha}{2}\mathbf{x}'\mathbf{B}\mathbf{x},$$

where $0 \leq \alpha \leq 1$. Notice that both $\pi(\mathbf{x})$ and $W^\alpha(\mathbf{x})$ are strictly concave (for positive quantities): Ramsey pricing implies (in an interior solution)

$$\mathbf{p}(\mathbf{x}) - \mathbf{c} = (1 - \alpha) Ds(\mathbf{x}) = (1 - \alpha) \mathbf{B}\mathbf{x}, \quad (1)$$

leading to

$$\begin{aligned} \mathbf{x}^R(\alpha) &= \frac{\mathbf{D}(\mathbf{a} - \mathbf{c})}{2 - \alpha}, \quad \mathbf{p}^R(\alpha) = \frac{1 - \alpha}{2 - \alpha} \mathbf{a} + \frac{1}{2 - \alpha} \mathbf{c}, \\ \mathbf{p}^R(\alpha) - \mathbf{c} &= \frac{1 - \alpha}{2 - \alpha} (\mathbf{a} - \mathbf{c}). \end{aligned} \quad (2)$$

These formula deliver the monopoly and perfect competition allocations respectively for $\alpha = 0$ and $\alpha = 1$, and generalize the expressions familiar from the case of a single commodity. Notice that Ramsey quantities $\mathbf{x}^R$ and price-cost margins $\mathbf{p}^R - \mathbf{c}$ are proportional, as discussed by Armstrong and Vickers (2016).

The natural case of positive Ramsey quantities does not require that $\mathbf{a} > \mathbf{c}$: however, from (1) this must be the case if commodities are overall substitutes. Since from (2)

$$\mathbf{a} - \mathbf{c} = (2 - \alpha) \mathbf{B} \mathbf{x}^R(\alpha),$$

it must also be the case that $a_i > c_i$, and thus $p_i^R(\alpha) > c_i$ for $\alpha < 1$, at least for some i , which we assume. Conversely, $p_i^R(\alpha) < c_i$ for some i and $\alpha < 1$ if and only if $a_i < c_i$: this requires that $\mathbf{b}_i' \mathbf{x}^R(\alpha) < 0$, where $\mathbf{b}_i$ is the i -th line of $\mathbf{B}$, implying that at least one of the elements of $\mathbf{b}_{-i}' = [b_{i1}, \dots, b_{i,i-1}, b_{i,i+1}, \dots, b_{in}]$ is negative, meaning that commodity i has at least one complement.⁸ Intuitively, it is the consumption of this (these) complementary good(s) which sustains the consumption of such a “subordinate” commodity (it might well be the case that $a_i = 0$ but $x_i^R > 0$: see Bertoletti, 2024 for an example).

3 Oligopoly pricing

In a Cournotian market in which each firm produces a single commodity the profit of firm i is given by:

$$\pi_i(\mathbf{x}) = [p_i(\mathbf{x}) - c_i] x_i,$$

and condition

$$a_i - \mathbf{b}_i \mathbf{x} - c_i - b_{ii} x_i = 0$$

characterizes its reaction curve $x_i(\mathbf{x}_{-i})$ when this is positive. Notice that quantities are *strategic* substitutes if $\mathbf{b}_{-i} > \mathbf{0}$ for all i , and *strategic* complements if $\mathbf{b}_{-i} < \mathbf{0}$ for all i (meaning that products are overall complements).

Accordingly, in the unique interior Nash equilibrium it would hold that

$$\mathbf{a} - \mathbf{c} = [\mathbf{B} + \widehat{\mathbf{b}}_{ii}] \mathbf{x}^C,$$

where $\widehat{\mathbf{b}}_{ii}$ is the diagonal matrix which shares the same diagonal as $\mathbf{B}$, i.e.,

$$\mathbf{x}^C = \mathbf{B}_C^{-1} (\mathbf{a} - \mathbf{c}), \quad \mathbf{p}^C = \widehat{\mathbf{b}}_{ii} \mathbf{B}_C^{-1} \mathbf{a} + \mathbf{B} \mathbf{B}_C^{-1} \mathbf{c}, \quad (3)$$

$$\mathbf{p}^C - \mathbf{c} = \widehat{\mathbf{b}}_{ii} \mathbf{B}_C^{-1} (\mathbf{a} - \mathbf{c}),$$

where also $\mathbf{B}_C = \mathbf{B} + \widehat{\mathbf{b}}_{ii}$ is a symmetric, positive definite matrix.

Under Bertrand competition with each firm producing a single commodity the profit of firm i is given by:

$$\pi_i(\mathbf{p}) = [p_i - c_i] x_i(\mathbf{p}),$$

and condition

$$\widetilde{a}_i - \mathbf{d}_i' \mathbf{p} - [p_i - c_i] d_{ii} = 0,$$

⁸That is, $\frac{\partial p_i}{\partial x_j} < 0$ for some $j \neq i$.

where $\mathbf{d}_i$ is the i -th line of $\mathbf{D}$, characterizes its reaction curve $p_i(\mathbf{p}_{-i})$ when this is higher than c_i . Notice that prices are *strategic* substitutes if $\mathbf{d}'_{-i} = [d_{i1}, \dots, d_{i,i-1}, d_{i,i+1}, \dots, d_{in}] > \mathbf{0}$ for all i , and *strategic* complements if $\mathbf{d}_{-i} < \mathbf{0}$ for all i (and thus products are overall substitutes).

Accordingly, at the unique interior Nash equilibrium it would hold that

$$\tilde{\mathbf{a}} + \hat{\mathbf{d}}_{ii}\mathbf{c} = [\mathbf{D} + \hat{\mathbf{d}}_{ii}] \mathbf{p}^B,$$

where $\hat{\mathbf{d}}_{ii}$ is the diagonal matrix which shares the same diagonal as $\mathbf{D}$, i.e.,

$$\mathbf{p}^B = \mathbf{D}_B^{-1} [\mathbf{D}\mathbf{a} + \hat{\mathbf{d}}_{ii}\mathbf{c}], \quad \mathbf{x}^B = \hat{\mathbf{d}}_{ii}\mathbf{D}_B^{-1}\mathbf{D}(\mathbf{a} - \mathbf{c}), \quad (4)$$

$$\mathbf{p}^B - \mathbf{c} = \mathbf{D}_B^{-1}\mathbf{D}(\mathbf{a} - \mathbf{c}),$$

where $\mathbf{D}_B = \mathbf{D} + \hat{\mathbf{d}}_{ii}$ is again a symmetric definite positive matrix.

However, since at the oligopoly equilibrium *each* price must be (weakly) higher than the corresponding marginal cost, an interior equilibrium need not to exist even when the Ramsey quantities are positive.⁹ It appears also clear that the previous expressions do not allow a general comparison between the Bertrand and Cournot prices and quantities: see Amir and Jin (2001).

3.1 Bertrand versus Cournot with two commodities

When both oligopoly equilibria are interior, from (3)-(4) we get, in the case of two commodities:

$$\mathbf{x}^B - \mathbf{x}^C = [\hat{\mathbf{d}}_{ii}\mathbf{D}_B^{-1}\mathbf{D} - \mathbf{B}_C^{-1}] (\mathbf{a} - \mathbf{c}) = k\mathbf{D}(\mathbf{a} - \mathbf{c}) = k(2 - \alpha)\mathbf{x}^R(\alpha),$$

$$\mathbf{p}^C - \mathbf{p}^B = [\hat{\mathbf{b}}_{ii}\mathbf{B}_C^{-1} - \mathbf{D}_B^{-1}\mathbf{D}] (\mathbf{a} - \mathbf{c}) = k(\mathbf{a} - \mathbf{c}) = k\frac{2 - \alpha}{1 - \alpha} [\mathbf{p}^R(\alpha) - \mathbf{c}]$$

where $k = \frac{b_{12}^2}{4b_{11}b_{22} - b_{12}^2} > 0$ if $b_{12} \neq 0$. Accordingly, in a linear setting with two positive Ramsey and Cournot quantities the latter cannot be larger than their Bertrand counterparts (regardless of product substitutability), suggesting that consumer surplus is larger in a Bertrand equilibrium. In addition, $p_i^R(\alpha) < c_i$ ($\alpha < 1$) and $p_i^B > p_i^C$ are *equivalent* conditions which require $a_i < c_i$ and $a_j > c_j$ with $b_{12} < 0$ and thus $d_{12} > 0$ (commodities are complements, prices are *strategic* substitutes and quantities are *strategic* complements), and imply $p_j^R(\alpha) > c_j$ and $p_j^B < p_j^C$ ($i \neq j$, $i, j = 1, 2$).

One could also compare the oligopolistic and the monopolistic prices: for example, it turns out that if i is subordinate then $p_j^m > p_j^B$. We summarize our main results in the following Proposition.

⁹ A counterexample with two commodities arises with $a_2 = 0$, $\mathbf{B} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ and $c_1 + 2c_2 < a_1 < c_1 + 4c_2$ (in such a case the Cournotian quantity of the subordinate commodity 2 would be null): see Bertolotti (2024).

Proposition 1. *In a linear setting with two commodities and positive Ramsey and Cournot quantities: i) Cournotian quantities are never larger than their Bertrand counterparts; ii) a Bertrand price is higher than its Cournotian counterpart if and only if it is the price of a subordinate commodity.*

4 Conclusions

In this note we have revisited the Bertrand-Cournot price and quantity ranking in the case of two differentiated products under linear demand and cost functions. Assuming that the Ramsey and Cournotian quantities are positive, we have shown that, contrary to a widespread view in the literature, one Bertrand price can actually be higher than its Cournotian counterpart, but if and only if it is the price of a subordinate commodity (meaning that it receives a negative Ramsey price-cost margin). The existence of such a price requires product complementarity: on the contrary, we have also shown that the Cournotian quantities cannot be larger than their Bertrand counterparts regardless of product substitutability.

References

- Armstrong, Mark and John Vickers, 2018, Multiproduct pricing made simple, *Journal of Political Economy*, 126, 1444-71.
- Amir, Rabah and Jim Y. Jin, 2001, Cournot and Bertrand equilibria compared: substitutability, complementarity and concavity, *International Journal of Industrial Organization*, 19, 303-17.
- Belleflamme, Paul and Martin Peitz, 2015, *Industrial Organization. Markets and Strategies*, 2nd Edition, Cambridge (UK): Cambridge University Press.
- Bertoletti, Paolo, 2024, Ramsey pricing: A simple example of a subordinate commodity, *Portuguese Economic Journal*, 23, 241-7.
- Häckner, Jonas, 2000, A note on price and quantity competition in differentiated oligopolies, *Journal of Economic Theory*, 93, 233-9.
- Singh, Nirvikar and Xavier Vives, 1985, Price and quantity competition in a differentiated duopoly, *Rand Journal of Economics*, 15, 546-54.
- Takayama, Akyra, 1985, *Mathematical Economics*, CUP, Cambridge (UK).
- Vives, Xavier, 1999, *Oligopoly Pricing. Old Ideas and New Tools*, MIT Press, Cambridge (Mass).
- Zanchettin, Piercarlo, 2006, Differentiated duopoly with asymmetric costs, *Journal of Economics & Management Strategy*, 15, 999-1015.