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a Dynamic Model**

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Monopolistic Pricing in the Banking Industry: a Dynamic Model*

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Abstract

This work develops a portfolio model of the banking firm where both the size and composition of the portfolio are jointly determined. The model provides a micro-foundation of the credit channel of transmission of monetary policy. It allows to analyse the pricing policies of the banking firm, and shows how interest rate shocks and credit quality shocks (the real shocks that change expected default costs) affect the equilibrium level of loans and deposits. Besides it shows the factors affecting the provision of insurance services by means of the smoothing of shocks.

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1 Introduction

The theory of the credit market on which this work is founded highlights the particular features of credit contracts due in particular to the relevance of uncertainty, limited information and transaction costs. According to this conception, because of their particular institutional features, banks can reduce the influence of market imperfections. Banking institutions can be considered to be market solutions to the problems caused by the limited availability of information, and the high cost that is necessary to undertake in order to obtain, select, and process the relevant information. The peculiar institutional framework of contemporary banks, based on the joint provision of depository and lending services, can be explained viewing the bank as an institution specialized in the provision of liquidity, to both households and firms. Banks have been increasingly studied as firms specialized in the analysis of a set of information, internal information, that is not available to the market. The specific expertise that the intermediary develops in the analysis of information provides the rationale for the actual existence of banking intermediaries.

The bank described in the model is a profit maximizing firm, that faces an infinite horizon problem. The banking firm is assumed to be risk-neutral, so that the conclusions do not depend on the assumption of risk-aversion on part of the bank. The need for an intertemporal framework comes from the assumption of the existence of long term relationships between the bank and its customers, either depositors or borrowers. The bank has to consider the effect that decisions made in every point in time have on future period balance sheets. Short-run policy choices taken in different periods are not independent, as it is normally assumed describing other markets. In a credit transaction the object itself of the exchange contract is a promise, and time is explicitly taken into account in every contract.

The model adopted is a dynamic model. Different market imperfections might determine the need for a dynamic model. Most dynamic models of banking¹ have simply assumed the presence of quadratic adjustment cost for deposits, loans or both. Here the dynamic is driven by the consideration that loans and deposits are not independent; loans feedback in deposits of future periods.² The feedback mechanism is quite complex and works through the interactions of the entire banking system. Anyway it can be reasonable to formulate some drastically simplifying assumptions, in order to take in consideration its effect. As it will be shown, considering the effect of the feedback, the maximisation problem becomes dynamic even *without* explicitly formulating quadratic adjustment costs for deposits or loans. If the bank takes into account the feedback, a standard quadratic cost function on loans produces implicitly a quadratic adjustment cost on deposits. The intuition behind this result is that the decision to extend any loan facility of the bank implies a variation in the stock of deposits.

Three important limitations of the model must be spelled out.

We assume price and cost flexibility and neutrality, so that inflation has no effect. The only market imperfections we want to consider are in fact linked to the limited and costly availability of information. The limited availability of information produces both market power and the peculiar structure of the cost functions that we introduce in the model.

In second order, we choose not to deal with liquidity problems, assuming that they are adequately managed by means of the compulsory reserve requirement and the deposit insurance. Liquidity costs could easily be introduced in the model, but they would complicate the results without increasing the understanding of the problems that we want to study.

Finally, we disregard the influence of net worth that we introduce in the model in a peculiar way. We discuss to some extent how the result would change partially relaxing our assumption, but a general limitation remains: we do not introduce equity markets in the analysis. This simplification is almost standard in the microeconomic theory of banking, but nevertheless this assumption is a relevant omission. Our model in fact, in line with most of the recent literature, focus on an explanation of the role of banking intermediaries based on the limited availability of information. But when information is not perfect the Modigliani-Miller theorem does not hold, so that the composition of the liabilities of the firm matters. In this work though we limit our analysis of the liability of the bank to debt, deposits in particular. The explicit introduction of equity markets would be a fundamental extension of this line of research.

¹Such as Elyasiani, Kopecky and Van Hoose [8] and Cosimano [3] and [4].

²This mechanism provides the dynamic constraint.

2 A model with contemporaneous feedback

The model is in discrete time, and has the following time structure. At the beginning of every period, households and firms dispose of a certain amount of funds that are the bequest of previous periods. Households take decisions regarding their portfolio allocation and their consumption plans for the period. Firms plan their investments for the period and evaluate their finance needs. Deposits are necessary for households in order to carry out transactions, since there is no currency. At the end of every period households and firms dispose of an amount of funds that reflects the evolution of the value of their assets, the income of the period and their consumption choices. Firms obtain the liquidity that is necessary to carry out their transactions by means of loans. The feedback process of loans on deposits that we describe can be understood as the result of the liquidity creation at the firm level: at the end of the period part of the liquidity generated by means of loans is distributed to households. This assumption fits well with Ramey's [20] findings of cointegration between M1 and business M1.

The bank can invest its deposits in loans or other assets, that we implicitly assume to be bonds. Besides it is compelled to hold a fraction of its deposits as reserves that can eventually provide a return.³ Deposits are immediately invested in loans or assets by the bank, except the share that is kept as reserves. At the end of the period loans are paid back and depositors are reimbursed. With the beginning of the new period there is a new inflow of deposits and so on. Loans feedback in deposits because firms invest the sums received, creating deposits in the system proportionally to the amount of the loans. The circulation of money allows banks to provide payment services with the same funds they have loaned, just keeping a fraction of deposits as reserves.

2.0.1 The budget constraint

The budget constraint is the following:

$$L_t + F_t + R_t = D_t + NW_t. \quad (1)$$

The bank can buy securities or lend as loans only the part of deposits that it does not keep as reserve. Defining with q the legal reserve coefficient, so that $R_t = qD_t$, the equation becomes:

$$L_t + F_t = (1 - q)D_t + NW_t. \quad (2)$$

The value of F_t represents the amount of assets that are invested on assets, such as bonds. The value of L_t represents the amount of loans issued by the bank. NW_t is the net worth of the bank, and we assume that it remains constant over time: $NW_{t+1} = NW_t = NW$. The bank cannot get access to the capital markets to increase its capital. Since we assume the existence of a monopolistic framework, profits are not pushed down to the normal rate by competition, but we assume that there is a one hundred per cent dividend payout, so that all profits are distributed to shareholders in every period.

2.0.2 Cost functions

The analysis of the problem of the banking firm in its most general form is impossible without specifying a simplified cost structure that allows an analytical treatment. A solution often adopted is to specify the cost structure in terms of the different components of the portfolio, such as bonds loans and deposits. And a further simplification used is to assume a cost function that is separable in the arguments. Formally:

$$C(K, L) = C(D(K, L)) + C(L(K, L)) + C(B(K, L)).$$

Using this formulation the existence of a separate production function for each class of assets and for deposits is implicitly assumed. The last simplification is not a big problem as long as the eventual economies of scope between assets and liabilities or among assets are not crucial for the problem studied. The empirical evidence regarding the relevance of the economies of scope among

³The model could be structured in order to allow the possibility for the bank to issue other liabilities, and the results would not change radically.

different component of the portfolio is not uncontroversial. This is not surprising, though, because complementarities and economies of scope do not arise between the provision of deposit services and loans. They arise between the two separate economic functions that banks fulfil: the provision of payment services and financial intermediation. The empirical analysis is complicated by the fact that revenues of one service are often confused with revenues or costs of the other and vice-versa.

We choose to describe the cost of servicing deposits and loans as linear in the quantity. We assume that the bookkeeping transaction function implies a deterministic industrial cost that the bank has to incur in order to provide bookkeeping services to depositors and borrowers. The cost of checks clearing and other desk operation is in fact linked to the number of transactions made by the customers, and we can for simplicity assume that they are proportional to the amount of deposits and loans.⁴ In general, on there are no obvious reasons for the industrial costs to be convex. On the contrary, they might be concave, because of the presence of some fixed costs. But since we choose an infinite horizon problem, we can disregard the eventual relevance of fixed costs, which in the banking sector, anyway, are not overwhelming. The large empirical literature regarding the existence of scale economies in the banking system (economies that besides could be linked to portfolio management function) has not reached undisputed conclusions.⁵ The simultaneous survival of banks of different size in almost every country shows that economies of scale certainly are not overwhelming.

The assumptions regarding the structure of costs linked with the provision of financial intermediation services are crucial. Banks are normally assumed to face two kinds of costs, default costs and liquidity costs. Both liquidity and default costs are to be assumed as stochastic. They are in fact essentially due to the uncertainty regarding shocks that may hit borrowers or depositors. The first might cause the impossibility of borrowers to refund loans, the second may cause a bank run. We choose to disregard the importance of liquidity costs, focusing just on default costs, because their introduction would not change in a relevant way the analysis we want to develop.

The fundamental assumption of the model is that the default cost function is quadratic. This assumption implies that the returns on the investment in information are decreasing. Banks cannot increase direct lending at will without reducing the efficiency of their monitoring and screening processes. Increasing direct lending indefinitely sooner or later they would finance investment projects of decreasing quality, taking a higher risk without a proportional increase in the return. They would end up not pricing risk properly. We assume that the default cost affects loans only. This seems counterfactual since banks can hold corporate bonds as well as gilts. This does not imply that there are no defaults on bonds, but that the market is efficient and prices risk correctly. Any agent can take as much market risk as he desires at the market price for risk, as assumed by the CAPM model. So banks can buy bonds without incurring in non-linear default costs because they just buy market risk, and we assume that no bank is large enough to affect returns. The decreasing returns are just with the banks own activity, the pricing of uncertain investments, whose information is not common knowledge and has not been disclosed to the market. Investments whose risk is virtually unknown so that the market can not price it.

It might be reasonable to assume a non linear cost function only if assets holding are allowed to be negative, so that the bank can borrow issuing bonds in order to lend more. In this case the non-linearity would imply that the cost of borrowing has to be increasing with the quantity. This case can be studied with the same framework, but the results would not be radically different. In order to simplify the notation we do not specify neither a linear default cost nor the transaction cost of bonds. With no loss of generality we can define the returns on bonds as net of default and transaction costs.

Formally:

$$\frac{\partial C(D_t)}{\partial D_t} > 0 \quad \frac{\partial^2 C(D_t)}{\partial D_t^2} = 0; \quad \frac{\partial C(L_t)}{\partial L_t} > 0 \quad \frac{\partial^2 C(L_t)}{\partial L_t^2} = 0. \quad (3)$$

⁴A detailed study of the industrial costs of deposit is provided by Osborne [19], and our assumptions are compatible with it.

⁵The most recent empirical evidence regarding the return to scale of banks is in Weelock and Wilson [29]. They showed that after 1985 there is evidence of increasing returns to scale for small and medium size banks, while the restriction of constant returns to scale could not be rejected for large banks. The finding of relevant return to scale is probably due to the progressive deregulation of the banking sector.

We can express the last functions simply as:

$$C(D_t) = uD_t \quad C(L_t) = zL_t, \quad (4)$$

where u and z are positive real numbers.

A stochastic default cost must be added to the industrial costs:

$$D(L_t) = \frac{1}{2}vL_t^2, \quad (5)$$

where

$$\frac{\partial D(L_t)}{\partial L_t} > 0 \quad \frac{\partial^2 D(L_t)}{\partial L_t^2} < 0, \quad (6)$$

and with

$$v = v_d + \epsilon^d \quad \text{with} \quad E[\epsilon_d] = 0 \quad E[\epsilon_d^2] = \sigma_d^2. \quad (7)$$

The cost functions we have assumed are constant over time, because with time-varying coefficients no closed-form solution can be obtained. Anyway the model could be extended to study this more general case. For the purposes of this study this further complication was unnecessary. The only important problem that could be obscured by our assumption regards the effect of expected inflation on the problem of the bank. But if prices are not sticky and expectations are rational, expected inflation is fully incorporated in all interest rates. And under these assumptions the price of factors of production adjusts instantaneously to the change of other prices. This implies that cost functions are homogeneous of degree one with respect to inflation. With our assumptions, marginal costs coefficients and interest rates are proportionally shifted by variations of the price level.

2.0.3 The demand for deposits and the dynamic constraint

Deposits are demanded not just as a financial asset for portfolio allocation, but mainly because banks provide depositors with transaction services. The market for payment services has always been highly competitive, since commercial banks were competing with issuing banks (only later state-owned central banks) that provide those services by means of bank-notes. In order to get remunerated for the payment services that they provide by means of checks, bookkeeping entries and credit cards, banks charge fees on the transactions undertaken. On the contrary, transactions by means of banknotes, whose technology is much simpler and cheaper, do not need the payment of fees. As a consequence, commercial banks have to attract depositors offering an interest rate that banknotes do not pay. The technological developments of the 20th century have reduced the competitive pressure from banknotes, whose role has become smaller. But new competitors have come out. At the beginning of the twentieth century savings institutions, which were developed initially exclusively to provide financial intermediation services, have been allowed to provide payment services by means of the giro. Only later they have been allowed to issue loans, becoming in all respect analogous to commercial banks. More recent technological developments have allowed money market mutual funds and other financial intermediaries to provide many of the payment services that banks provide at a low cost. As a consequence the need to pay interest rates has increased.

On the other hand the relevance of transaction costs, (search costs in particular) in the market for deposits has been shown by Flannery [11] and Hess [15], and is quite uncontroversial. Deposits are increasingly described as quasi-fixed inputs. Since search costs allow the firm to charge non-competitive prices,⁶ in presence of search costs monopolistic competition becomes the normal market structure. This suggests that each bank does not suffer a strong competitive pressure from other banks and can price deposits monopolistically.

We can conclude that the need to pay an interest rate on deposits comes from the competition of intermediaries different from banks. These new competitors can in fact offer interest rates no too far from the rates that bonds pay. As a consequence we will assume that each bank can set

⁶See Salop [21] and Salop and Stiglitz [22] and [23].

monopolistically the price of deposits, while the demand for deposits is negatively affected by the market interest rate on bonds that intermediaries pay when providing payment services.

In order to obtain a demand schedule for deposits services, we study separately the demand of households and firms. The demand of both classes of agents is assumed to depend on two different interest rates, the own rate on deposits and the rate on bonds, that is an opportunity cost. For simplicity, we assume that transaction fees do not affect the demand for deposits. This assumption can be justified, following Fama [10], considering that the market for payment services is competitive, marginal costs are constant and the supply of these services is normally infinitely elastic.

Interest rates on bonds are assumed to follow a pure random walk process:

$$r_{t+1}^B = r_t^B + \epsilon_{t+1}^B \quad \text{with} \quad E[\epsilon_t^B] = 0 \quad E[\epsilon_{t+i}^B \epsilon_{t+j}^B] = \sigma_B^2 \quad i = j, \quad E[\epsilon_{t+i}^B \epsilon_{t+j}^B] = 0 \quad i \neq j. \quad (8)$$

Interest rates on deposits are the result of the equilibrium condition of the market, and are set by the banking system in function of the rate on bonds. So they are assumed to follow another random walk process, correlated with the process of bonds:

$$r_{t+1}^D = r_t^D + \epsilon_{t+1}^D \quad \text{with} \quad E[\epsilon_t^D] = 0 \quad E[\epsilon_{t+i}^D \epsilon_{t+j}^D] = \sigma_D^2 \quad i = j, \quad E[\epsilon_{t+i}^D \epsilon_{t+j}^D] = 0 \quad i \neq j \text{ and} \\ E[\epsilon_{t+i}^D \epsilon_{t+j}^B] = Cov(DB) \quad i = j, \quad E[\epsilon_{t+i}^D \epsilon_{t+j}^B] = 0 \quad i \neq j. \quad (9)$$

Household's deposits are assumed to depend (positively) on nominal income,⁷ the own interest rate on deposits and (negatively) on the interest rate on bonds. The coefficients are assumed to be constant over time.

$$D_t^h = f_1 Y_t + f_2 r_t^D - f_3 r_t^B. \quad (10)$$

The first term can be considered to capture mainly the behaviour of demand deposits while the second two of time deposits. It has in fact been shown that both demand deposits and $M1$ are not very sensitive to interest rates.⁸ We assume that nominal income is an AR(1) process, whose trend and error coefficients depend on the growth of both real income and prices:

$$Y_{t+1} = \gamma_Y Y_t + \epsilon_{t+1}^Y = \gamma_{Y_r} \gamma_P Y_t + \epsilon_{t+1}^{Y_r} + \epsilon_{t+1}^P. \quad (11)$$

The expected value of deposits of the following period is:

$$E[D_{t+1}^h] = f_1 \gamma_Y Y_t + f_2 E[r_{t+1}^D] - f_3 E[r_{t+1}^B]. \quad (12)$$

this can be rewritten as:

$$E[D_{t+1}^h] = \gamma_Y D_t^h - (\gamma_Y - 1) f_2 E[r_{t+1}^D] + (\gamma_Y - 1) f_3 E[r_{t+1}^B]. \quad (13)$$

Firm's deposits are assumed to depend on both rates as before, and on the quantity of loans issued by the bank. This dependence is due to the liquidity creation that loans allow because of the convertibility on demand of deposits.⁹ Besides, banks compel firms to deposit a fraction of the loans they issue. In this way they manage the payments of the borrower, earning fees, as was suggested by Sprenkle,¹⁰ and they can monitor his liquidity. We assume that deposits depend on the amount of loans of the current period.¹¹

$$D_t^f = \kappa L_t + f_4 r_t^D - f_5 r_t^B. \quad (14)$$

The coefficient κ synthesizes the effect of the feedback of loans on deposits. Loans have a direct impact on deposits through the effect of firms' deposits, indirectly through firms' expenditure. The effect of the increase of loans is assumed to be analogous among different banks of the system,

⁷Since their transaction demand is assumed to be a function of income.

⁸See Hess [14] and [15] and Moosa [17].

⁹See Diamond and Rajan [7].

¹⁰See Sprenkle [26] and [27].

¹¹Alternatively the dependence can be assumed to be lagged, and deposits of the current period depend on loans of the previous one. It can be shown that the results do not change in a relevant way.

assuming that banks operate in regions that are homogeneous. For simplicity loans are assumed to affect deposits for the following period only.¹²

Summing deposits of firms and households we can obtain the expected level of deposits of the bank as:

$$E[D_{t+1}] = \gamma_Y D_t + g_3 E[r_{t+1}^D] - g_4 E[r_{t+1}^B] + \kappa E[L_{t+1}] \quad (15)$$

$$\text{where } g_3 = [f_4 - (\gamma_Y - 1)f_2] \quad \text{and} \quad g_4 = [f_5 - (\gamma_Y - 1)f_3].$$

Interest rates on deposits are assumed to affect equilibrium levels, but not the dynamic behaviour of deposits. The assumption of a unit root implicitly excludes the existence of a trend, and it seems reasonable because of the infinite time period of the maximization problem. In the long run the interest rate should in fact ultimately depend on the productivity of capital and the average time-preference coefficient, both of which are unlikely to follow a trend, neither deterministic nor stochastic. The interest rate has increasingly been model as a mean-reverting stochastic process, such as the Uhlenberg-Ulbeck in continuous time.

In conclusion, the demand for deposits services is assumed to have three components. One component is completely exogenous and cannot be influenced by the bank in any way, and it is the behaviour of income. The second depends on the portfolio choices of the bank, since a fixed proportion of loans has to be kept or feedbacks in deposits. The interest rate on deposits represents just the third component. If banks are willing to increase deposits further than they could achieve with just the normal flow plus the component dependent on the loans issued, they have to pay an interest rate. But since they cannot separate different components of the demand banks can't price discriminate and have to pay the interest rate on all their deposits

2.0.4 The demand for loans

The costly availability of information generates monopoly power in the market for loans. Relationship lending allows the bank to price monopolistically and the higher return due to the market power makes the higher risks of the project worth. Sharpe [25] has shown that establishing long-term relationships with its customers, a bank learns more than others about the business and the capability of the borrower. This information asymmetry generates a rent that allows banks to finance risky projects whose information is very opaque, which cannot be financed in the market. Establishing the relationship and developing their knowledge, banks provide a valuable service, they create the knowledge necessary to price the risk. The price that firms pay for this service is the monopolistic rent that they pay on loans.

The empirical tests for the presence of market power were traditionally performed studying the behaviour of the rate on loans, which has been found to be stickier than the rate on bonds, in different estimates conducted in different periods of time and different countries. This evidence though was not uncontroversial, since the stickyness of the rate can be explained as well as the outcome of credit rationing, or the result of implicit contracts for the smoothing of interest rate shocks. An important recent result has been provided by Cosimano and Mc Donald [5], which, analysing a large panel of bank loans, have shown that banks in the US exploit significant market power in the market for loans.

2.0.5 Revenues

The main stream of profits of the bank stems from the difference between the interests rate r_{Lt} , that the bank charges on loans, and the interest rate r_{Dt} , that it pays to depositors. For simplicity we assume that banks do not buy shares, and that the only available alternative to the issuance of loans is the purchase of bonds. The alternative source of revenues is given by the spread between the interest rate on bonds r_{Bt} and the rate on deposits:

$$v_t = r_t^B - r_t^D. \quad (16)$$

¹²This assumption is necessary in order to make the model tractable. But it can be justified considering that the lag in the operation of the feedback should not be too long: firms keep part of their loans as deposits, and in general most of the portfolio of retail banks is made of short-term loans.

The rate on bonds is assumed to be set exogenously, and the bank is assumed to be price taker. Reserves are assumed to provide (eventually) a return equal to r_t^R . Since banks are normally compelled to hold reserves in form of cash, or non-interest bearing deposits at the central bank, the net return $\rho_t = r_t^R - r_t^D$ on reserves is usually negative, since the return is zero, but the bank has to bear the costs of the proportional share of deposits.

2.0.6 Monopolistic pricing

The profit function that results from the previous assumptions is the following:

$$\Pi = \sum_{t=0}^{\infty} \beta^t \left[(r_t^L - r_t^D) L_t + v_t F_t + \rho_t R_t - \frac{1}{2} v L_t^2 - u_t D_t - z L_t \right]. \quad (17)$$

Its logical structure is very simple: revenues come from the interest rates spreads, the costs that must be detracted are the cost functions previously defined. The importance of search costs in the market for deposits and the relevance of relationship lending in the market for loans create the need of a monopolistic model. As a consequence, the bank takes into account the demand schedule for deposits and loans in its maximization problem. The demand for loans is introduced in the model in a standard way, obtaining an inverse demand function and substituting its value for the value of the interest rate on loans.

$$r_t^L = -\frac{1}{b} L_t + \frac{a(Y_t)}{b} + \frac{d}{b} r_t^B + \epsilon_t^L \quad (18)$$

The peculiarity of the model lays in the way the deposit demand schedule is introduced in the model. Because of the presence of the quantity of deposits in two different periods of time, the demand schedule introduces another unknown in the problem, which becomes a dynamic problem. The information provided by this equation cannot be used to eliminate the interest rate on deposits, because it must be used to solve for the two quantities. The bank has to choose: it can either get rid of the rate on deposits and solve for the quantity of deposits in just one period, or solve for both quantities and treat the dynamic of the interest rate as exogenous. The correct solution is the second. Solving for the quantity of loans, from:

$$D_t = \gamma_Y D_{t-1} + g_3 r_t^D - g_4 r_t^B + \kappa L_t, \quad (19)$$

the following can be obtained:

$$L_t = \frac{1}{\kappa} \left\{ D_t - \gamma_Y D_{t-1} - g_3 r_t^D + g_4 r_t^B \right\}. \quad (20)$$

Substituting this function for L_t in the profit function, we can observe that the quadratic cost on loans works as a quadratic adjustment cost on deposits. Our model becomes formally identical to a standard dynamic model, but its structure is much simpler than the structure of any other dynamic models of banking.

Rather than solving the model for L_t , it is convenient to use the budget constraint differently, forming a Lagrangian. It becomes possible in this way to solve the model for the other two variables, obtaining as a the solution, both the optimal size of the portfolio, and the optimal composition of the portfolio of assets. Making a further assumption, it is possible to obtain the rate on deposits from the solution on the quantities, as it happens in the perfectly competitive models.

2.1 Intertemporal maximization

The firm maximizes its expected profits over an infinite horizon period. The budget constraint is implemented by substitution. The problem of the banking firm, for every pair of positive real numbers (v, u) , can be expressed as:

$$\begin{aligned} & \text{Max} \\ & \{F_t, D_t\}_0^{\infty} \\ \Pi = & \sum_{t=0}^{\infty} \beta^t \left[(r_t^L - r_t^D) L_t + v_t F_t + \rho_t R_t - \frac{1}{2} v L_t^2 - u_t D_t - z L_t \right], \end{aligned} \quad (21)$$

s.t.

$$L_t + F_t + R_t = D_t + NW, \quad (22)$$

$$R_t = qD_t, \quad (23)$$

$$L_t = a - br_t^L + dr_t^B + \eta_t, \quad (24)$$

and

$$D_t = \gamma_Y D_{t-1} + \kappa L_t + g_3 r_t^D - g_4 r_t^B. \quad (25)$$

The discount factor β^t represents the temporal perspective of the bank: $\beta^t = \frac{1}{(1+r^E)^t}$, where we have assumed that r^E , the return on equities, is the discount rate that the bank applies.¹³ Under our assumption deposits are the state variable of the problem, while F_t is the control variable of the bank.

From the Euler equations of the problem, defining $\left(\frac{bv+2}{b}\right) = \alpha$, the following difference equation can be obtained:

$$\begin{aligned} E[F_{t+1}] = & \frac{1 - (1-q)\kappa}{\gamma_Y \beta} F_t + \frac{(1-q)\{\gamma_Y^2 \beta - [1 - (1-q)\kappa]\}}{\gamma_Y \beta} D_t + \\ & + \frac{\beta \gamma_Y + (1-q)\kappa - 1}{\gamma_Y \beta} NW + (1-q)X_t + \frac{1 - (1-q)\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right], \end{aligned} \quad (26)$$

where:¹⁴

$$\begin{aligned} Z_{t+1} = & \left[\left(1 - \frac{d}{b}\right)(\gamma_Y \beta - L) + (1-q)\kappa L\right] r_{t+1}^B - (\gamma_Y \beta - L) \epsilon_{t+1}^L \\ & + \kappa [r_t^R q - r_t^D - u] + (\beta \gamma_Y - 1) \left(z - \frac{a}{b}\right), \end{aligned} \quad (27)$$

$$X_t = g_3 r_t^D - g_4 r_t^B. \quad (28)$$

Equation (26) together with the original dynamic constraint, which we rewrite, form a system of difference equations:

$$D_t = \frac{\gamma_Y}{1 - \kappa(1-q)} D_{t-1} - \frac{\kappa}{1 - \kappa(1-q)} F_t + \frac{\kappa}{1 - \kappa(1-q)} NW + \frac{1}{1 - \kappa(1-q)} X_t. \quad (29)$$

Stability conditions

It can be shown that the eigenvalues of the system are:

$$\lambda_1 = \frac{1}{\beta \gamma_Y} \quad \text{and} \quad \lambda_2 = \gamma_Y. \quad (30)$$

Necessary and sufficient condition for the two eigenvalues to be one smaller the other larger than one are the following:

$$\begin{cases} \gamma_Y > 1 & \text{and} & \beta \gamma_Y < 1 & \text{or} & \frac{1+g_Y}{1+r_t^E} < 1 & \text{or} & g_Y < r_t^E \\ \gamma_Y < 1 & \text{and} & \beta \gamma_Y > 1 & \text{or} & \frac{1-g_Y}{1+r_t^E} > 1 & \text{or} & r_t^E < -g_Y \end{cases}$$

This condition is necessary, but not sufficient, in order to assure the saddle path stability of the system. Here g_Y is the rate of growth of nominal GDP, and we have made the strong assumption of risk neutrality for the investors.

The meaning of this condition is that the stability of the banking system depends on the relationship between the rate of growth of the demand for deposits and the interest rate that the bank uses as a discount factor. If the discount factor is smaller than the rate of growth, and the second is positive, the dimensions of the bank tend to increase indefinitely. If the nominal rate of

¹³Assuming investors to be risk neutral.

¹⁴Defining $\epsilon_t^L = \frac{1}{b} \eta_t$.

growth is negative, the bank does not tend to disappear only if the expected return on equities is negative too and larger in absolute value than the rate of growth. The rate of growth of income is the fundamental variable for the problem of the bank as long as the demand for deposits is a function of nominal income. For simplicity we have assumed that deposits change as income, but the conclusions would hold anyway.

If markets are efficient and shareholders are for simplicity considered to be risk-neutral, the standard assumption is that the bank uses the expected return on equities as a discount rate. As long as wages and rents do not adjust instantaneously this condition should normally be guaranteed. In this case in fact profits are more volatile than income. And as a consequence the expected return on equities should always be in absolute terms larger than the expected rate of growth of income.

2.1.1 General solution

The Rational Expectations Equilibrium of the system can be obtained substituting one of the equations in the other. Substituting the first in the second we obtain an equation for the stock of bonds:¹⁵

$$E[F_{t+1}] - \left[\frac{1}{\gamma_Y \beta} + \gamma_Y \right] F_t + \frac{1}{\beta} F_{t-1} = \frac{[1 - \gamma_Y][\beta \gamma_Y - 1]}{\gamma_Y \beta} NW + \\ -E \left[\frac{1}{\beta \alpha} \right] Z_t + E \left[\frac{1 - (1 - q)\kappa}{\gamma_Y \beta \alpha} Z_{t+1} \right] + \frac{(1 - q)(\gamma_Y \beta - 1)}{\gamma_Y \beta} X_t + \frac{(1 - q)\gamma_Y}{1 - \kappa(1 - q)} (\epsilon_t^D - \epsilon_t^B). \quad (31)$$

Following the same procedure, we can write the value of D_t as:

$$E[D_{t+1}] - \left[\frac{1}{\gamma_Y \beta} + \gamma_Y \right] D_t + \frac{1}{\beta} D_{t-1} = \frac{\gamma_Y \beta - 1}{\gamma_Y \beta} X_t - \frac{\kappa}{\gamma_Y \beta} E \left[\frac{1}{\alpha} Z_{t+1} \right]. \quad (32)$$

2.1.2 The portfolio of bonds

Using the expectation lag operator H , such that $H^{-j} E_{s-1} x_s = E_{s-1} x_{s+j}$, the left hand side of the equation can be expressed as:

$$E[F_{t+1}] - E \left[\frac{1}{\gamma_Y \beta} + \gamma_Y \right] F_t + \frac{1}{\beta} F_{t-1} = (1 - \lambda_1 H)(1 - \lambda_2 H) E[F_{t+1}]. \quad (33)$$

Where λ_1 and λ_2 are the roots of the system. We already know their values, but they could have easily been obtained realizing that the right hand side can be rewritten as

$$1 - (\lambda_1 + \lambda_2)H + \lambda_1 \lambda_2 H^2, \quad \text{so that:} \\ -(\lambda_1 + \lambda_2) = \frac{1}{\gamma_Y \beta} + \gamma_Y \quad \text{and} \quad \lambda_1 \lambda_2 = \frac{1}{\beta}. \quad (34)$$

$$\lambda_1 = \frac{1}{\beta \gamma_Y} \quad \lambda_2 = \gamma_Y. \quad (35)$$

Equation (31) can be rewritten as:

$$(1 - \lambda_1 H) F_{t+1} = \frac{1}{(1 - \lambda_2 H)} E_t \left\{ \frac{[1 - \gamma_Y][\beta \gamma_Y - 1]}{\gamma_Y \beta} NW + \right. \\ \left. + \frac{1 - (1 - q)\kappa - \gamma_Y L}{\gamma_Y \beta \alpha} Z_{t+1} + \frac{(1 - q)(\gamma_Y \beta - 1)}{\gamma_Y \beta} X_t + \frac{(1 - q)\gamma_Y}{1 - \kappa(1 - q)} (\epsilon_t^D - \epsilon_t^B) \right\}. \quad (36)$$

Remembering that:

$$Z_{t+1} = \left[\left(1 - \frac{d}{b} \right) (\gamma_Y \beta - L) + (1 - q)\kappa H \right] r_{t+1}^B - (\gamma_Y \beta - H) \epsilon_{t+1}^L \\ + \kappa [r_t^R q - r_t^D - u] + (\beta \gamma_Y - 1) \left(z - \frac{a}{b} \right), \quad (37)$$

¹⁵As shown in the appendix.

it is now necessary to isolate the constant terms, dividing the former expression as:

$$Z'_{t+1} = \left[\left(1 - \frac{d}{b}\right)(\gamma_Y \beta - H) + (1 - q)\kappa H \right] r_{t+1}^B - (\gamma_Y \beta - L)\epsilon_{t+1}^L + \kappa(r_t^R q - r_t^D), \quad (38)$$

$$C = (\beta\gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u. \quad (39)$$

We are assuming that $\lambda_1 < 1$ and $\lambda_2 > 1$. The right-hand side can be solved forward, applying the algorithm developed by Sargent.¹⁶ The expression can be rewritten, using the properties that $\frac{1}{(1-\lambda_2 L)} a = \frac{-(\lambda_2 L)^{-1}}{1-(\lambda_2 L)^{-1}} a = \frac{a}{1-\lambda_2}$, and $\frac{1}{(1-\lambda_2 L)} b X_t = b \frac{-(\lambda_2 L)^{-1}}{1-(\lambda_2 L)^{-1}} X_t = -b \sum_{i=1}^{\infty} \left(\frac{1}{\lambda_2}\right)^i X_{t+i+1}$, where a and b are arbitrary constant terms. Applying the transversality condition of the problem (discussed in the Appendix), Equation (31) can be solved as:

$$F_{t+j+1} = \frac{1}{\beta\gamma_Y} F_{t+j} + \frac{\beta\gamma_Y - 1}{\gamma_Y \beta} NW + \frac{1 - (1 - q)\kappa - \gamma_Y}{(1 - \gamma_Y)\gamma_Y \beta \alpha} C - \frac{1 - (1 - q)\kappa}{\beta\gamma_Y} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y}\right)^i E_{t+i} \left[\frac{Z'_{t+j+i+2}}{\alpha} \right] + \frac{\gamma_Y}{\beta\gamma_Y} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y}\right)^i E_{t+i} \left[\frac{Z'_{t+j+i+1}}{\alpha} \right] - \frac{(1 - q)(\gamma_Y \beta - 1)}{\gamma_Y \beta} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y}\right)^i X_{t+j+i+1}. \quad (40)$$

In order to simplify the interpretation of the results, it can be assumed that interest rates follow a random walk process, so that the deterministic component is assumed to remain constant. This assumption is reasonable for market interest rates on bonds, but it is not acceptable for the rates on deposits, since the equilibrium rate of deposits is obtained from the model. Just to simplify the understanding of the results, we will disregard the relevance of the deterministic components, assuming for simplicity that future rates on deposits are expected to behave like a random walk.

If we assume that interest rates are random walks, assuming that the correlation between interest rates and default costs is time-invariant, we can rewrite the expression treating all the terms as constants:

$$F_{t+j+1} = \frac{1}{\beta\gamma_Y} F_{t+j} + \frac{1 - (1 - q)\kappa - \gamma_Y H}{(1 - \gamma_Y)\gamma_Y \beta \alpha} \left\{ \left[\left(1 - \frac{d}{b}\right)(\gamma_Y \beta - H) + (1 - q)\kappa H \right] \cdot \left[r_{t+j+1}^B + COV\left(r^B, \frac{1}{\alpha}\right) \right] + \kappa(qr_{t+j}^R - r_{t+j}^D) \right\} + \frac{(1 - q)(\gamma_Y \beta - 1)}{\beta\gamma_Y(1 - \gamma_Y)} (g_3 r_{t+j}^D - g_4 r_{t+j}^B) + \frac{[\gamma_Y \beta - 1][1 - (1 - q)\kappa - \gamma_Y]}{\beta\gamma_Y(1 - \gamma_Y)\alpha} COV\left(L^D, \frac{1}{\alpha}\right) + \frac{1 - (1 - q)\kappa - \gamma_Y}{(1 - \gamma_Y)\gamma_Y \beta \alpha} C + \frac{\beta\gamma_Y - 1}{\gamma_Y \beta} NW. \quad (41)$$

To understand the results we have to remind our assumption that $\gamma_Y > 1$ and $\left(\frac{bv+2}{b}\right) = \alpha$.

We can observe that interest rates on bonds at time $t + 1$ have a positive sign, so they always increase the quantity of bonds held in the portfolio at time $t + 1$. Rates on bonds at time t have a negative sign, in the first term, showing a positive effect on time t holdings of bonds. We can observe though that under our basic assumption,¹⁷ higher rates produce an increase in the holdings of bonds. The other terms in the interest rate at time t have a positive sign, and this means that there is even a positive dependence on the lagged value of the interest rate. We can conclude that both current and lagged values of interest rates increase the quantity of bonds held in the portfolio by the bank.

Industrial costs and interest rates on deposits have a negative sign, since the profit margin becomes tighter as they increase. On the contrary higher industrial costs on loans have a positive sign, since they increase the relative appeal of bonds.

The covariance between the rate of interest and the reciprocal of the cost function has a positive sign. This implies that a positive correlation between the rate of interest and the default cost has a *negative* effect on the purchase of bonds. the sign of the other covariance is the opposite. This highlights the fact that when the demand for loans is correlated with default costs, the bank issues a proportionally lower quantity of loans, buying more bonds instead.

¹⁶See Sargent [24] p.176.

¹⁷That the nominal rate of growth is positive and larger than the discount rate.

Default costs shrink the size of the forward looking part of the equation, reducing the portfolio of bonds, and the same effect is a function of the elasticity of the demand for loans. The lower the elasticity the smaller the portfolio. This result is due to the fact that both default costs and inelastic demand for loans reduce the optimal quantity of loans and, as a consequence, the size of the whole portfolio.

2.1.3 Deposits

The solution for deposits is given by:

$$D_{t+j+1} = \frac{1}{\gamma_Y \beta} D_{t+j} + \frac{\kappa}{\beta \gamma_Y} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y} \right)^i E_{t+i} \left[\frac{Z'_{t+j+i+2}}{\alpha} \right] - \frac{\kappa}{\beta \gamma_Y \alpha (1 - \gamma_Y)} C +$$

$$- \frac{\gamma_Y \beta - 1}{\gamma_Y \beta} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y} \right)^i E_{t+i} [X_{t+j+i+1}], \quad (42)$$

Under the assumption that interest rates follow a random walk process, we can for simplicity assume that deterministic component of the rates remain constant and treat the values of the rates at time t and $t + 1$ that enter in the solution as constants. In this case the result can be simplified as:

$$D_{t+j+1} = \frac{1}{\gamma_Y \beta} D_{t+j} + \frac{\kappa \left[\left(1 - \frac{d}{b} \right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] + (\gamma_Y \beta - 1) \alpha g_4 H}{\beta \gamma_Y \alpha (\gamma_Y - 1)} r_{t+j+1}^B +$$

$$+ \frac{\kappa}{\beta \gamma_Y \alpha (\gamma_Y - 1)} \kappa r_{t+j}^R q - \frac{\kappa^2 + \alpha (\gamma_Y \beta - 1) g_3}{\beta \gamma_Y \alpha (\gamma_Y - 1)} r_{t+j}^D + \frac{\kappa \left\{ (\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u \right\}}{\beta \gamma_Y \alpha (\gamma_Y - 1)} +$$

$$+ \frac{\kappa}{\beta \gamma_Y \alpha (\gamma_Y - 1)} \left[\left(1 - \frac{d}{b} \right) (\gamma_Y \beta - 1) + (1 - q) \kappa \right] COV \left(r^B, \frac{1}{\alpha} \right) - \frac{(\gamma_Y \beta - 1) \kappa}{\beta \gamma_Y (\gamma_Y - 1)} COV \left(L^D, \frac{1}{\alpha} \right). \quad (43)$$

It can be observed that the intertemporal equilibrium value of deposits depends negatively on the costs of deposits as it should be the case. The variation of the quantity of deposits between two periods is a negative function of the industrial cost of deposits. It is much more surprising that interest rate on deposits has a negative effect. Solving the equation for the rate on deposits, it can easily be realised that interest rates on deposits at time t increase with the level of deposits at time t and vice-versa, as we would expect. The negative relationship is with the level of deposits at time $t + 1$. This result can be understood reading the equation in the opposite way. The current level of deposits has a direct relationship with both the rate on deposits and the level of next period. As a consequence when the rate of interest is high because the demand for deposits is elastic, the higher rates imply a lower level for next period. The other negative component is small since it is of second order and reflects the reduction in profits due to the interest cost. The results regarding the dependence on the rate on deposits, anyway, are not very robust, since they depend on the very strong assumption that their deterministic component is expected by the banker not to change in the future.

The quantity of deposits grows with the rate on bonds. This result is counterintuitive, but it can easily be explained, as the effect of the feedback. It is useful to underline what would the result be if the market for loans were to be competitive, rather than monopolistic. The structure of the model would be the same, the only difference would be that in the final result we would have $-r_{t+1}^L$ instead of $\left(1 - \frac{d}{b} \right) r_{t+1}^B$, and of course the intercept term $\frac{a}{b}$ would disappear and the term in default cost would be v rather than α . The final result would show a negative relationship with the rate on loans, the impact of which depends on the value of the feedback coefficient. In fact lower rates would allow a greater issuance of loans generating more liquidity and vice-versa. Since higher rates on bonds imply lower rates on loans in the monopolistic model, the first part of the result is explained. The second component, $\kappa^2 (1 - q) r_t^B$ is of a smaller order, and is due to the fact that higher rates imply higher returns of the part of the portfolio invested in bonds. This component affects the level of deposits with a one period lag, and is small. The final term $\alpha g_4 r_t^B$ is due to the standard demand effect, and measures the reduction in the demand of deposits services due to the

higher return of alternative assets (that may even provide monetary services, in some cases). The negative impact affects the contemporaneous level of deposits but not the level of the following period. As a consequence, the higher the rate on bonds, the lower the contemporaneous level of deposits, the higher the difference between the levels of the two periods. This result provides a theoretical rational for the empirical evidence provided by Chari, Christiano and Eichenbaum [2], which shows that M1 shows a relevant positive correlation with future values of the interest rate, while the correlation with contemporaneous and past values is negative.

The covariance between the interest rate on bonds and the reciprocal of the default costs increases deposits, so a positive correlation between interest rates and default costs reduces the level of deposits. The opposite is true in the case of a positive correlation between shocks in the demand for loans and default costs. Default costs shrink the size of the forward looking part of the equation, reducing the size of the portfolio. They are the true constraint on the size of the portfolio, putting a limit to the liquidity creation.

Loans

The rational expectation equilibrium quantity of loans can be easily obtained from the budget constraint $L = (1 - q)D - F + NW$:

$$L_{t+j+1} = \frac{1}{\gamma_Y \beta} L_{t+j} + \frac{1}{\gamma_Y \beta} NW - \frac{1}{\gamma_Y \beta \alpha} C + \frac{1 - \gamma_Y H}{\gamma_Y \beta} \sum_{i=0}^{j+1} \left(\frac{1}{\gamma_Y} \right)^i E_{t+i} \left[\frac{Z'_{t+j+i+1}}{\alpha} \right]. \quad (44)$$

And:

$$L_{t+j+1} = \frac{1}{\gamma_Y \beta} L_{t+j} + \frac{1}{\gamma_Y \beta} NW - \frac{\left[\left(1 - \frac{d}{b} \right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] r_{t+j+1}^B + \kappa (r_{t+j}^R q - r_{t+j}^D)}{\gamma_Y \beta \alpha} + \frac{(\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u}{\gamma_Y \beta \alpha} - \frac{\left(1 - \frac{d}{b} \right) (\gamma_Y \beta - 1) + (1 - q) \kappa}{\gamma_Y \beta \alpha} COV \left(r^B, \frac{1}{\alpha} \right) + \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV \left(L^D, \frac{1}{\alpha} \right). \quad (45)$$

This result shows that the contemporaneous rate on bonds has a negative impact on the issuance of loans. Lower interest rates on bonds increase the amount of loans issued and vice-versa. This is not surprising since the rate on loans depends negatively on the rate on bonds and the rate on bonds itself represents the opportunity cost of the issuance of loans.

Loans depend positively on the lagged rate on deposits. At the same time, costs of loans negatively affect the issuance of loans, while the opposite is true for the cost of deposits. In this model the amount of deposits represents a constraint for the issuance of loans since it cannot be adjusted in a costless way. Higher industrial costs (which are linear) and interest rates on deposits have a stronger impact on the alternative investment of the assets portfolio of banks, bonds. Lower margins in the intermediation due to higher industrial costs and interest rates of deposits push the bank to issue proportionally more loans. The reason lies in the fact that there is a crucial difference between the issuance of loans and of the purchase of bonds. Purchasing bonds the size of the bank does not change, while the issuance of loans increases the size. The constraint on the size is caused by the default cost, and is increased by the monopoly power. Default costs are the relevant constraints for the issuance of loans because they work as an adjustment cost on deposits, as shown before.

The covariance between the rate of interest and the reciprocal of the cost function has a negative sign. This implies that a positive correlation between the rate of interest and the default cost has a *positive* effect on the issuance of loans. This condition states that when higher rates on bonds are correlated with higher default costs on loans, the banks, not surprisingly, issues more loans. Any consideration regarding the impact of interest rate shocks must accordingly be qualified.

When the demand for loans is correlated with default costs, the bank issues a proportionally lower quantity of loans, buying more bonds instead. Not surprisingly if a buoyant demand is regarded as implying higher future default cost, the enthusiasm for the issuance of loans is proportionally reduced.

2.1.4 The interest rate on loans

The interest rate on loans can easily be obtained substituting the solution (44) for the quantity of loans in the demand condition. For simplicity we use the solution of (26), but the general result could easily be obtained following the same procedure:

$$r_{t+j}^L = \frac{1}{b} E_{t+j} \left\{ dr_{t+j}^B + \frac{\left[\left(1 - \frac{d}{b}\right)(\gamma_Y \beta - L) + (1 - q)\kappa L \right] r_{t+j+1}^B}{\gamma_Y \beta \alpha} + G \right\}, \quad (46)$$

where

$$G = a + \eta_{t+j} - \frac{1}{\gamma_Y \beta} L_{t+j} - \frac{1}{\gamma_Y \beta} NW + \frac{\kappa(r_{t+j}^R q - r_{t+j}^D)}{\gamma_Y \beta \alpha} + \frac{(\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u}{\gamma_Y \beta \alpha} + \frac{\left(1 - \frac{d}{b}\right)(\gamma_Y \beta - 1) + (1 - q)\kappa}{\gamma_Y \beta \alpha} COV\left(r^B, \frac{1}{\alpha}\right) - \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{\alpha}\right); \quad (47)$$

and L is the lag operator. When the rate on loans is set monopolistically, interest rates on loans depend on contemporaneous and expected rates on bonds. Interest rates on bonds have a strong impact on the rates on loans.

It is useful to show the result when the bank has no market power and the rate on loans is taken as exogenous, in order to separate the effect of market power from the supply side effect due to the presence of bonds in the portfolio. Considering the bank as representative of the sector and aggregating we would obtain:

$$r_{t+j+1}^L = \frac{1 - \gamma_Y \beta v}{\gamma_Y \beta} r_{t+j}^L + \frac{(1 - q)\kappa}{\gamma_Y \beta} r_{t+j}^B + vG', \quad (48)$$

where

$$G' = -\frac{1}{\gamma_Y \beta} L_{t+j} - \frac{1}{\gamma_Y \beta} NW + \frac{\kappa(r_{t+j}^R q - r_{t+j}^D - u)}{\gamma_Y \beta v} + \frac{(1 - q)\kappa}{\gamma_Y \beta v} COV\left(r^B, \frac{1}{v}\right) - \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{v}\right). \quad (49)$$

An interest rate shock would decrease the contemporaneous value of the rate on loans but increase the value of next period. We can see that in general higher interest rates on bonds tend to increase the rate on loans since the difference between the two periods rates on loans is positive. It is important to observe that the impact is in this case quite small, indicating that the rate on loans is much more sluggish than the rate on bonds. The default cost coefficient, v , reduces the dependence on the lagged value of the interest rate, but has no influence on the coefficient of the rate on bonds.

The equilibrium level of the interest rate on deposits can be obtained in the same way, considering the equilibrium rate for which demand and supply are equated. It can easily be realised how the rate on deposits is subject to contrasting demand and supply effects, whose net effect is not obvious on a priori ground.

2.1.5 Cournot competition

Assuming the presence of different banks in the market and that each bank's cost structure is common knowledge, the model developed in the former sections can be structured as a Cournot model. The problem of every individual bank would in this case include the market share as an unknown of the problem, and it would take into account the result of the same optimisation problem performed by the others banks. We would now have n firms facing their n maximization problems, that include the problems of the competitors in the price setting equation. And each individual firm's problem would now include as an unknown the value of the market shares $\psi = \frac{L}{L_j}$ and $\theta = \frac{D}{D_j}$. The n equations would provide the optimal supply functions. The condition of aggregation of the loan and deposits supply schedules provides the two extra equations that allow closing the system:

$$L = \sum_{j=1}^n L_j, \quad D = \sum_{j=1}^n D_j, \quad (50)$$

where n is the number of firms.

3 Discussion

3.1 A positive interest rate shock

When the rate on loans is set monopolistically interest rates on bonds have a strong impact on the rate on loans, in contrast with the perfectly competitive case.¹⁸ Our results show that contemporaneous interest rate shocks are the only that are smoothed.¹⁹ On the contrary shocks to expected future rates need not be smoothed, and can produce a more than proportional increase of the rate on loans. This kind of shocks can be smoothed only if the demand for loans is highly sensitive to the rate on bonds. We can conclude that interest rate shocks are normally smoothed only when the shock is expected to be transitory. If the shock is expected to affect permanently future interest rates, the impact on the rate on loans may even be stronger than on the rate on bonds. Default costs have no influence on the contemporaneous rate coefficient, but affect the coefficient of the expected rate on bonds. The forward looking part becomes less important only when default costs are very large.

An interest rate shock reduces the quantity of loans issued and increases more than proportionally the size of the portfolio of bonds. The main force behind the result is the market power of the bank. The sensitivity of the demand for loans on the rate on bonds is important in this context. When its value is high relative to the coefficient of the own rate, the strength of the result is proportionally reduced. When the competition from the bond market is strong, the impact of the rate on bonds on the issuance of loans is reduced, because the market power of the bank is proportionally reduced. In the case of a perfectly competitive market for loans, and assuming that the rate on loans is perfectly correlated with the rate on bonds, the result is never overturned, but it becomes far less significant.

Another important observation is that the impact of the shock on the two assets is not symmetric, because the size of the portfolio is affected too. Not surprisingly, the effect is much stronger on the bond component of the portfolio, because the negative impact on the issuance of loans is necessarily more limited. We can conclude that the amount of bonds in the portfolio is necessarily much more volatile than the quantity of loans. The reason is that the bond market does not take default costs into account in the way banks do, because, assuming that risk is correctly priced in an efficient market, expected default costs reduce proportionally the expected return in a linear way, so that net returns are not affected. The case of the bank is different because it deals with uncertainty, and the impact of default costs on its profits must be non-linear. It can finally be observed that under our assumptions a higher rate of growth of output increases the rate on loans, reducing the smoothing, even disregarding any effect of income on the demand for loans.²⁰

In the former section we began to discuss the effect of the correlation between interest rates and default costs. We have seen that because of this correlation the bank tends to issue more loans. The assumption that higher rates on bonds increase default costs on loans can be justified considering that large firms need both loans and bonds, and higher market interest rate reduce the cash flow and increase the risk of default on both categories of assets. We did not formally introduce a default cost on bonds, because we did not specify whether the bonds that the bank buys were risk-free or high yield risky bonds. In the second case, assuming that default costs are a linear function of the quantity purchased, these costs would proportionally shrink the net returns of bonds. Even in this case the pattern of our results would not change, only the relevance of the different effects would be different.

The results of our model show that the correlation between the interest rate on bonds and the default costs tends to offset the direct effect of the interest rates on bonds. The correlation is positive because higher interest rates reduce the free cash-flow of the borrowers, so that the risk of default becomes proportionally higher. We have not made any assumption regarding this correlation, but it is likely to be much larger when interest rates are high. The increase of the interest rate implies in fact a reduction of the free cash flow that is proportional to the initial level of the rate. So we can expect the effect of an interest rate shock on the composition of the portfolio

¹⁸Our results for a perfectly competitive framework confirm the finding of Cosimano [4].

¹⁹As it can easily be realised from Equation (46). The own coefficient b is in fact always larger than the cross coefficient d of the demand for loans.

²⁰The derivative of r^L with respect to γ_Y is in fact normally positive.

to be dependent on the initial level of the rates. When the rates are low the direct effect of the interest rate is likely to be predominant, because the opposite effect due to the correlation should not be very large. For higher initial levels of the rate the direct effect of the interest rate is likely to be largely offset by the correlation. When initial interest rates are high, the higher correlation could even imply the dominance of the effect of the covariance. In this final case we would observe the paradoxical result of an increase of the direct lending activity after a positive interest rate shock, notwithstanding the market power of the bank. This could possibly happen in the case of high inflation, when nominal rates are very high and the demand for loans remains strong.

Finally, following the same line of reasoning, in the case of heavy shocks we might expect that the effects are not symmetric in the case of a positive or negative shock. The reduction in the issuance of loans following a large positive shock should be smaller than the increase following a proportional reduction.

3.2 Credit quality shocks

Different types of shocks of real origin affect borrowers, producing variations of current and expected default costs. To keep the analysis simple, we have assumed that the coefficient of the expected default costs are the same as the coefficient of the cost of the current period, and are revised when the current cost changes. As a consequence the only shocks that we consider are those that are regarded as permanent ones by the banker. Shocks that are supposed to affect negatively and permanently the borrowers reduce the average quality of the portfolio of loans, increasing default costs.

Higher default costs reduce the size of the whole portfolio. The higher volatility of the portfolio of bonds implies that the impact on the purchase of bonds is proportionally much stronger than on the issuance of loans. The bank reacts to higher default cost shrinking the whole portfolio, reducing notably the amount of bonds in order to reduce less the issuance of loans. So when an industry is affected by a negative shock and the placement of bonds becomes difficult because spreads get wider, the banking industry does not increase the issuance of loans. The bank cannot lend more because default costs affect the size of the portfolio, and only indirectly the composition. The portfolio shrinks, and the composition changes favouring the issuance of loans just because the reduction of the size affects bonds more. On the contrary when default cost gets smaller the share of bonds in the portfolio grows more than proportionally.

This result shows the usefulness of a dynamic framework, where the size of the portfolio is endogenous. Variations of the size of the whole portfolio are not captured by static portfolio models, and may induce in error. In the case of a negative shock affecting the credit quality a borrower or of an industry, our model predicts that banks reduce the issuance of loans, providing insurance against the shock to a limited extent only. As long as their information allows the formulation of expectations regarding future default cost, banks provide direct lending facilities, but proportionally reducing the amounts involved. On the contrary, after the shock it may become very difficult for the less informed lenders of the market to properly price the risk, so the bond market may easily dry out because of the insurgence of a lemon problem. In the last case borrowers are pushed to rely on banks, the demand for loans surges and bonds are not a substitute for loans any more as a source of finance for risky projects.

A strong enough increase of the demand could in principle push the bank to lend more, since demand grows more than proportionally as default costs rise. But this may be the case only when the initial demand is low, for example when the shock hits an industrial sector whose main source of finance is the bond market. This can easily be realised from Equation (26), which shows that the increase of the demand²¹ change just one of the components of the equilibrium level of loans, while the default costs affects other terms of the sum too. Besides the absence of competition from bonds²² has both a positive and a negative impact in the issuance of loans. The positive effect comes from the higher coefficient of the covariance term, the negative because the coefficient measuring the direct effect of the interest rate on bonds becomes larger. When the level of the rates on bonds affects the demand for loans, if the coefficient is large enough, the higher demand

²¹A variation of the intercept a of the demand for loans schedule, increased by the lower elasticity of the demand, measured by a lower coefficient b .

²²Shown by a value of zero for the coefficient d .

for loans due to an increase of the rate on bonds may offset the negative supply side effect due to the fact that the rate represents the opportunity cost of direct lending. This seems to suggest that banks may provide insurance, increasing the issuance of loans when borrowers are hit by a shock, to firms that have access to the bond market, when it becomes difficult for these firms to get financed in the market. On the contrary, when a negative shock hits small firms or other borrowers that do not get access to the bond market, banks are likely to reduce the issuance of loans.

3.3 Net worth

This model does not allow a correct assessment of the importance of the own capital and of capital requirements, for two reasons. In first instance the volatility of deposits induced by bank runs might be relevant for the problem, but this problem could be solved introducing a stochastic error in the demand for deposits schedule. Second, and much more important, the stock market has not been introduced in the analysis, and it is a relevant omission, because we are explicitly analysing conditions under which the Modigliani-Miller theorem does *not* hold. The explicit introduction of the stock market would be quite complex, since we should model agency problems in condition of opaque information, and is far beyond the scope of this work. Nevertheless it can be useful to study the importance of the level of initial capital, under the very restrictive assumptions described earlier, of a strong equity rationing.

In this case, the relevance of net worth for the problem of the bank does not depend on the lag structure adopted, because the result is exactly the same if adopt a different structure for the lags of the feedback, as in the appendix. Net worth increases the portfolio of assets in a very asymmetric way, because independently of the expected returns of different securities, banks use net worth almost entirely to finance loans, and only a modest fraction is invested in bonds. The share of bonds might become substantial only in the case when the exogenous rate of growth of deposits is much larger than the discount factor. Under our assumptions, this happens when the nominal rate of growth of income is much larger than the real interest rate on equity. The surprising result is the irrelevance of the relative returns of different classes of assets. This result is due to our assumption that the level of net worth remains constant in every period, independently of the dimension of the portfolio, which is not very realistic. This implies that banks are allowed to distribute all profits in every period and there are no legally binding capital requirements. Besides banks do not need a buffer because there is no penalty if profits become negative during a period, for example because of a negative shock that affects the average quality of credit increasing the default cost. In this extreme situation net worth increases the size of the portfolio at no cost, so banks normally use most of the net worth to finance loans because loans allow the portfolio to grow, while bonds do not. As we would intuitively expect net worth in this case has no effect on the other liability, deposits. A larger net worth allows a one-off proportional increase of the portfolio of assets, loans in particular, without increasing the amount of deposits. Yet this result is not trivial, because since a higher capitalisation allows the bank to lend more creating liquidity, it could be supposed that deposits should be positively correlated with net worth. On the contrary, the model shows that a higher net worth increases direct lending *without* creating liquidity.

The model can easily be extended to analyse the impact of the legal requirement of a minimum ratio between capital and loans. If the net worth of the bank has to cover at least a fixed proportion of the loans issued, since in our model the bank would never have an incentive to keep a higher than necessary share of capital, we could assume that:

$$L_t + F_t + R_t = D_t + NW_t, \quad (51)$$

$$R_t = qD_t, \quad (52)$$

$$NW_t = \delta L_t. \quad (53)$$

so that the budget constraint becomes the following:

$$(1 - \delta)L_t = (1 - q)D_t - F_t. \quad (54)$$

It can be easily realised that in this case the results of the model would change in a simple way. The term in net worth obviously disappears, and in the final result both intercept terms are multiplied

for $1 - \delta$. The effect of this legal requirement is to proportionally reduce the forward looking part of the solution. As a consequence in this case the size of the portfolio is proportionally reduced, without changing the composition of the assets. Clearly deposits, the only freely chosen liability are proportionally reduced as well, because deposits positively depend on the main spread Z_t . These constraint virtually acts a reserve coefficients on deposits, but is much more effective. In this case we did not introduce explicitly net worth in the profit function, assuming that capital could not be invested and had no cost. The introduction of those cost and revenues function would affect the optimal composition, that would now take into account this component. But the general result previously discussed would not change. Variations of the capital requirement coefficient would affect only the size of the portfolio, not the composition.

3.4 Growth

An increase in the factor of growth of the economy reduces the importance of the backward looking part of each of the equilibrium equations. The impact on the forward looking part is positive in the case of deposits, negative for loans. This result reflects the fact that the forward looking part is a function of the feedback process and of income growth. Income growth increases the demand of future deposits services exogenously, independently of the feedback process. Since issuing loans the bank increases the level of deposits through the feedback, when income grows the bank obtains more deposits independently on the loans issuance. So the bank has to allocate these new deposits to bonds and reduce proportionally the issuance of loans. This of course assuming that the demand for loans is not affected by the higher income, which is unlikely. A higher demand for loans could naturally offset this portfolio composition effect and reverse the result, but that would not be surprising. The interesting result can be better understood considering the effect of a reduction of the factor of growth of the economy, *ceteris paribus*. In this case the bank reacts shifting the portfolio from bonds to loans, if the demand remains unchanged. This result implies that the banking system shows a tendency to behave anti-cyclically, smoothing macroeconomic shocks.

Until the very recent past, Anglo-saxon economic systems, where stock and bond markets are a relevant source of finance, were considered to be more volatile than continental European systems, where the influence of the banking intermediation is stronger. Our results may help to explain why banking institutions provide insurance. But the stronger impact of the last recession on continental European economies has put in question the overall validity of these considerations. The German banking system in particular has been hardly hit, producing, according to many observers, a credit crunch. The close ties between banks and firms in the German system, where banks hold a large portfolio of shares of their own borrowers, can explain the surprising fragility that the system showed after the bubble in the stock market burst. The main limitation of our work, the absence of an explicit modelling of the stock market, explains the failure of the model to predict such an outcome.

3.5 Monetary policy and the control of monetary aggregates

Studying the composition of the portfolio of a bank, this model provides a micro-foundation of the first step of the credit channel of transmission of monetary policy, showing how banks react to monetary policy shocks. Most of the literature on the credit channel has focused on explaining why the intermediation of banking institutions affects the investment behaviour of firms, and the consumption pattern of households, when interest rates vary. Much less attention has recently been devoted to understand how variations of market interest rates affect the lending decisions of banks. This issue has probably neglected because central banks interventions are increasingly conducted influencing the banking system, for example trough the discount window in the US or the “corridor” of the ECB. Nevertheless in order to evaluate the importance of the credit channel it is necessary to study how both the size and the composition of the portfolio of assets of banks are changed.

The version of the model previously exposed considers interest rates on bonds as a substitute for loans, and no other liabilities different form deposits are introduced in the problem. The interest rate movements that we study are changes in the market equilibrium rates, and because of the timing structure chosen we should consider short term interest rates in particular. As a

consequence monetary policy can be analyzed only through its effect on the bond market and the price of securities in general. The model could easily be changed though to study the discount window or other forms of direct lending from the central bank to the banking system. In this case the formal structure would remain unchanged, but F would change sign and indicate a liability rather than an asset. The only necessary modification would be the need to introduce a non-linear cost on this liability, because the cost of borrowing for any firm is always convex since the risk grows non-linearly with the amount borrowed, and the amount of finance provided by the central bank is always limited. In this case the results would be quite similar, and the coefficient of the cost function would play the same role as the coefficient of the default cost function. The model could similarly be extended to include any number of assets and liabilities. We have not shown these different cases because the solution becomes heavy whenever another liability is added to the problem, since the value of the roots cannot be simplified. On the contrary the pattern of results would remain similar since the formal structure of the model would remain unchanged.

The results of our basic model show that higher interest rates on securities reduce the size of the portfolio and alter the composition in favour of bonds. Banks set the rate on loans in function of the rate on bonds, smoothing transitory interest rates shocks, while in the case of permanent shocks they may even amplify the shocks. Besides we have seen that the change of the portfolio would depend on the initial level of the rates, because the impact of interest rates shocks on the cash-flow of borrowers is likely to be much stronger when the rates are high. If we would have considered direct lending of the central bank, the results would have been very similar, but the forward looking part of the solutions would be smaller in absolute value, because of the presence of a second cost coefficient in the denominator. This implies that the shock would be smoothed a bit more.

These results suggest that the credit channel tends to increase the effectiveness of monetary policy when interest rates are low, while in the case of high interest rates banks might even offset the result of the policy. This explains why central banks need to impose quantitative restrictions on the issuance of loans in order to fight high inflations. The interest rate is a very good instrument as long as interest rates are not very high. In the last case the imposition of ceilings on the issuance on loans can be necessary to control the endogenous growth of money supply.

In order to control the quantity of monetary aggregates the endogenous process of supply of deposits by means of the feedback of loans must be considered. This model provides a useful insight since it considers the incentives that a profit maximising banking firm faces in the process. The model showed that capital requirements are much more effective than reserve coefficients as a constraint on the size of the portfolio, because they affect the scale of the whole dynamic process. Another crucial factor is the value of the parameters of the default cost function, since the size of the portfolio is ultimately constrained by default costs. This implies that regulatory requirements regarding the write-off of bad loans are crucial not just for the efficiency of the banking industry, but for the health of the economic system as a whole. When banks are allowed to roll over loans that should be written off, the constraint on the size of the portfolio is virtually removed, since default costs can be indefinitely postponed. This implies that any investment could in principle be financed, without any selection, producing permanent distortions in the productive structure.

The strength of the results depends on the sensitivity of the demand for loans on the rate on bonds. The development of the bond market tends to reduce the market power of banks, reducing the strength of the previous results. But since they can never be overturned, even in the limit case of perfect competition in the market for loans, the results seem quite robust. It must finally be observed that the results do not depend on the particular feedback process assumed, since it can be shown that using different assumptions regarding the time profile of the feedback process the model provides results that are almost identical.

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Appendix

Appendix: solution of the model with contemporaneous feedback

The Lagrangian of the problem is the following:

$$\begin{aligned} \ell = \sum_{t=0}^{\infty} \beta^t \left\{ \left\{ -\frac{1}{b}[(1-q)D_t - F_t + NW] + \frac{a(Y_t)}{b} + \frac{d}{b}r_t^B - r_t^D + \epsilon_t^L \right\} [(1-q)D_t - F_t + NW] + \right. \\ \left. -\frac{1}{2}v[(1-q)D_t - F_t + NW]^2 - uD_t - z[(1-q)D_t - F_t + NW] + v_tF_t + \rho_tqD_t + \right. \\ \left. -\mu_t \left\{ D_t - \gamma_Y D_{t-1} - \kappa[(1-q)D_t - F_t + NW] - g_3r_t^D + g_4r_t^B \right\} \right\}. \quad (55) \end{aligned}$$

The first order conditions are:

$$\frac{\partial \ell}{\partial F_{t+j}} = \beta^{t+j} \left\{ \left(v + \frac{2}{b} \right) [(1-q)D_{t+j} - F_{t+j} + NW] - \frac{a(Y_{t+j})}{b} - \frac{d}{b}r_{t+j}^B + r_{t+j}^D + z - \epsilon_{t+j}^L + v_{t+j} - \kappa\mu_{t+j} \right\} = 0, \quad (56)$$

$$\begin{aligned} \frac{\partial \ell}{\partial D_{t+j}} = \beta^{t+j} E_{t+j} \left[(1-q) \left[\frac{a(Y_{t+j})}{b} + \frac{d}{b}r_{t+j}^B - r_{t+j}^D + \epsilon_{t+j}^L \right] - \left(v + \frac{2}{b} \right) \left[(1-q)^2 D_{t+j} + \right. \right. \\ \left. \left. - (1-q)F_{t+j} + (1-q)NW \right] - u - (1-q)z + \rho_{t+j}q + \right. \\ \left. - \left[1 - \kappa(1-q) \right] \mu_{t+j} + \beta\gamma_Y E[\mu_{t+j+1}] \right] = 0, \quad (57) \end{aligned}$$

$$\frac{\partial \ell}{\partial \mu_{t+j}} = -D_{t+j} + \gamma_Y D_{t+j-1} + \kappa E \left[(1-q)D_{t+j} - F_{t+j} + NW \right] + g_3r_{t+j}^D - g_4r_{t+j}^B = 0. \quad (58)$$

For $j = T$, Condition (57) implies:

$$\begin{aligned} \beta^{t+T} E \left\{ (1-q) \left[\frac{a(Y_{t+T})}{b} + \frac{d}{b}r_{t+T}^B - r_{t+T}^D + \epsilon_{t+T}^L \right] - \left(v + \frac{2}{b} \right) \left[(1-q)^2 D_{t+T} + \right. \right. \\ \left. \left. - (1-q)F_{t+T} + (1-q)NW \right] - u - (1-q)z + \rho_{t+T}q - \mu_{t+T} \right\} = 0. \quad (59) \end{aligned}$$

The transversality condition of the problem is:

$$\begin{aligned} \lim_{T \rightarrow \infty} \beta^{t+T} E_t \left\{ (1-q)\omega_{t+T} - \alpha \left[(1-q)^2 D_{t+T} - (1-q)F_{t+T} + (1-q)NW \right] - u - (1-q)z + \right. \\ \left. + \rho_{t+T}q \right\} - \frac{1}{\kappa} \left\{ -\omega_{t+T} + v \left[(1-q)D_{t+T} - F_{t+T} + NW \right] + v_{t+T} \right\} = 0. \quad (60) \end{aligned}$$

To keep the notation simpler, we omit from now on the index j , to show it only when it will be necessary, in the final solutions. Besides we have used the definitions $\left(\frac{bv+2}{b} \right) = \alpha$ and $\frac{a(Y_t)}{b} + \frac{d}{b}r_t^B - r_t^D + \epsilon_t^L = \omega_t$. From Equation (56) we can obtain an equation that can be solved in order to get a solution for the multiplier μ_t .

$$\mu_t = \frac{1}{\kappa} \left\{ \alpha \left[(1-q)D_t - F_t + NW \right] - \omega_t + z + v_t \right\}. \quad (61)$$

This value can be substituted in the other first order condition, shown in equation (57), that provides the Euler equation of the system:

$$\begin{aligned} \beta^t \left\{ \rho_t q + (1-q)(\omega_t - z) - u - \alpha \left[(1-q)^2 D_t - (1-q)F_t + (1-q)NW \right] + \right. \\ \left. - \frac{1 - \kappa(1-q)}{\kappa} \left\{ \alpha \left[(1-q)D_t - F_t + NW_t \right] - (\omega_t - z) + v_t \right\} + \right. \\ \left. + \frac{\gamma_Y \beta}{\kappa} E_t \left\{ \alpha \left[(1-q)D_{t+1} - F_{t+1} + NW \right] - (\omega_{t+1} - z) + v_{t+1} \right\} \right\} = 0. \quad (62) \end{aligned}$$

$$\begin{aligned} & \beta^t \left\{ \rho_t q + (1-q)(\omega_t - z) - u - \alpha \left[(1-q)^2 D_t - (1-q)F_t + (1-q)NW \right] + \right. \\ & \quad \left. - \frac{1 - \kappa(1-q)}{\kappa} \left\{ -\omega_t + \alpha \left[(1-q)D_t - F_t + NW \right] + v_t + z \right\} + \right. \\ & \quad \left. + \frac{\gamma_Y \beta}{\kappa} E_t \left\{ -\omega_{t+1} + \alpha \left[(1-q)D_{t+1} - F_{t+1} + NW \right] + v_{t+1} + z \right\} \right\} = 0. \end{aligned} \quad (63)$$

Rearranging and dividing everything by β^t we obtain a difference equation for F_t and D_t . Since $X_t = E_t[X_t]$ we can include everything under the expectation, from which we will from now on omit the time index, since all expectations are at time t .

$$\begin{aligned} & E \left\{ \gamma_Y \beta \alpha F_{t+1} = \alpha F_t + \gamma_Y \beta (1-q) \alpha D_{t+1} - \alpha (1-q) D_t + [\beta \gamma_Y \alpha - \alpha] NW + \right. \\ & \quad \left. + \kappa [\rho_t q + (1-q)(\omega_t - z) - u] + [1 - \kappa(1-q)] [\omega_t - v_t - z] - \gamma_Y \beta [\omega_{t+1} - v_{t+1} - z] \right\}. \end{aligned} \quad (64)$$

The next step is to simplify the resulting expression and to substitute for the value of D_t in Equation (64), the expression of the dynamic constraint that is obtained from Equation (58).

$$\begin{aligned} & E \left\{ \gamma_Y \beta \alpha F_{t+1} = \alpha F_t + \gamma_Y \beta (1-q) \alpha \left[\frac{\gamma_Y}{1 - (1-q)\kappa} D_t - \frac{\kappa}{1 - (1-q)\kappa} F_{t+1} + \right. \right. \\ & \quad \left. \left. + \frac{\kappa}{1 - (1-q)\kappa} NW + \frac{g_3 r_{t+1}^D - g_4 r_{t+1}^B}{1 - \kappa(1-q)} \right] - \alpha (1-q) D_t + (\beta \gamma_Y \alpha - \alpha) NW + \right. \\ & \quad \left. + \kappa [\rho_t q + (1-q)(\omega_t - z) - u] + [1 - \kappa(1-q)] [\omega_t - v_t - z] - \gamma_Y \beta [\omega_{t+1} - v_{t+1} - z] \right\}, \end{aligned} \quad (65)$$

and

$$\begin{aligned} & E \left\{ \gamma_Y \beta \frac{\alpha}{1 - (1-q)\kappa} F_{t+1} = \alpha F_t + \gamma_Y \beta (1-q) \alpha \left\{ \frac{\gamma_Y}{1 - (1-q)\kappa} D_t + \right. \right. \\ & \quad \left. \left. + \frac{\kappa}{1 - (1-q)\kappa} NW + \frac{g_3 r_{t+1}^D - g_4 r_{t+1}^B}{1 - \kappa(1-q)} \right\} - \alpha (1-q) D_t + (\beta \gamma_Y \alpha - \alpha) NW + \right. \\ & \quad \left. + \kappa [\rho_t q + (1-q)v_t - u] + (\gamma_Y \beta - L) [v_{t+1} - \omega_{t+1}] + (\beta \gamma_Y - 1)z \right\}. \end{aligned} \quad (66)$$

Dividing both sides of the former equation for α , and introducing the lag operator L , we obtain:

$$\begin{aligned} E[F_{t+1}] &= \frac{1 - (1-q)\kappa}{\gamma_Y \beta} F_t + \frac{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]}{\gamma_Y \beta} D_t + \\ & \quad + \frac{\beta \gamma_Y + (1-q)\kappa - 1}{\gamma_Y \beta} NW + (1-q)[g_3 r_t^D - g_4 r_t^B] + \\ & \quad + E \left\{ \frac{1 - (1-q)\kappa}{\gamma_Y \beta \alpha} \left\{ \kappa [\rho_t q + (1-q)v_t - u] + (\gamma_Y \beta - L) [v_{t+1} - \omega_{t+1}] + (\beta \gamma_Y - 1)z \right\} \right\}. \end{aligned} \quad (67)$$

As before we end up with a system of two equations, the other is obtained from the dynamic constraint:

$$D_t = \frac{\gamma_Y}{1 - \kappa(1-q)} D_{t-1} - \frac{\kappa}{1 - \kappa(1-q)} F_t + \frac{\kappa}{1 - \kappa(1-q)} NW + \frac{g_3 r_t^D - g_4 r_t^B}{1 - \kappa(1-q)} = 0. \quad (68)$$

We can simplify the value of the second intercept term as:

$$\begin{aligned} Z_{t+1} &= \kappa [\rho_t q + (1-q)v_t - u] + (\gamma_Y \beta - L) [v_{t+1} - \omega_{t+1}] + (\beta \gamma_Y - 1)z = \\ & (\gamma_Y \beta - L) \left[r_{t+1}^B - r_{t+1}^D - \frac{a}{b} - \frac{d}{b} r_{t+1}^B + r_{t+1}^D - \epsilon_{t+1}^L \right] + \kappa [(r_t^R - r_t^D)q + (1-q)(r_t^B - r_t^D) - u] + \\ & \quad + (\beta \gamma_Y - 1)z = \left\{ \gamma_Y \beta \left(1 - \frac{d}{b} \right) + \left[(1-q)\kappa - \left(1 - \frac{d}{b} \right) \right] L \right\} r_{t+1}^B + \\ & \quad - (\gamma_Y \beta - L) \epsilon_{t+1}^L + \kappa [q \rho_t - r_t^D - u] + (\beta \gamma_Y - 1) \left(z - \frac{a}{b} \right) \end{aligned} \quad (69)$$

Solution of the system

Defining $X_t = g_3 r_t^D - g_4 r_t^B$, the first equation can be rewritten as:

$$D_t = \frac{\gamma_Y \beta}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E[F_{t+1}] - \frac{[1 - (1-q)\kappa]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} F_t + \\ - \frac{\beta \gamma_Y + [(1-q)\kappa - 1]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} NW - \frac{\gamma_Y \beta}{[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} X_t + \\ - \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \quad (70)$$

Substituting the former in the other equation, we obtain a second order difference equation in $E[F_t]$.

$$\frac{\gamma_Y \beta}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E[F_{t+1}] - \frac{[1 - (1-q)\kappa]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} F_t + \\ - \frac{\beta \gamma_Y + [(1-q)\kappa - 1]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} NW - \frac{\gamma_Y \beta}{[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} X_t + \\ - \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_{t+1}\right] = \\ = \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{\gamma_Y \beta}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} F_t - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{[1 - (1-q)\kappa]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} F_{t-1} + \\ - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{\beta \gamma_Y + [(1-q)\kappa - 1]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} NW + \\ - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{\gamma_Y \beta}{[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} X_{t-1} - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{1 - (1-q)\kappa}{(1-q)\alpha[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_t\right] + \\ + \\ - \frac{\kappa}{1 - \kappa(1-q)} F_t + \frac{\kappa}{1 - \kappa(1-q)} NW + \frac{X_t}{1 - \kappa(1-q)}. \quad (71)$$

It can be expressed as:

$$E[F_{t+1}] - \frac{[1 - (1-q)\kappa]}{\gamma_Y \beta} F_t + \frac{\kappa}{1 - \kappa(1-q)} \frac{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]}{\gamma_Y \beta} F_t + \\ - \frac{\gamma_Y}{1 - \kappa(1-q)} F_t + \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{[1 - (1-q)\kappa]}{\gamma_Y \beta} F_{t-1} = \\ = \frac{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]}{\gamma_Y \beta} \\ \left[\left[-\frac{\gamma_Y}{1 - \kappa(1-q)} + 1 \right] \frac{\beta \gamma_Y + [(1-q)\kappa - 1]}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} NW + \frac{\kappa}{1 - \kappa(1-q)} NW + \right. \\ \left. - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{\gamma_Y \beta}{[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} X_{t-1} + \frac{\gamma_Y \beta}{[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} X_t + \frac{1}{1 - \kappa(1-q)} X_t + \right. \\ \left. - \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_t\right] + \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_{t+1}\right] \right]. \quad (72)$$

We now study separately the left-hand side.

$$E[F_{t+1}] - \frac{[1 - (1-q)\kappa]}{\gamma_Y \beta} F_t + \frac{\kappa}{1 - \kappa(1-q)} \frac{(1-q)[\gamma_Y^2 \beta - [1 - (1-q)\kappa]]}{\gamma_Y \beta} F_t + \\ - \frac{\gamma_Y}{1 - \kappa(1-q)} F_t + \frac{\gamma_Y}{1 - \kappa(1-q)} \frac{[1 - (1-q)\kappa]}{\gamma_Y \beta} F_{t-1} = E[F_{t+1}] - \left[\frac{1}{\gamma_Y \beta} + \gamma_Y \right] F_t + \frac{1}{\beta} F_{t-1}. \quad (73)$$

We split the right-end side in different pieces, to begin with net worth.

$$\begin{aligned} & \frac{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]}{\gamma_Y\beta} \\ \left[\left[-\frac{\gamma_Y}{1 - \kappa(1-q)} + 1 \right] \frac{\beta\gamma_Y + [(1-q)\kappa - 1]}{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]} NW + \frac{\kappa}{1 - \kappa(1-q)} NW \right] = & (74) \\ & = \frac{[1 - \gamma_Y][\beta\gamma_Y - 1]}{\gamma_Y\beta} NW. \end{aligned}$$

The value of Z_t is:

$$\begin{aligned} & \frac{(1-q)\{\gamma_Y^2\beta - [1 - (1-q)\kappa]\}}{\gamma_Y\beta} \times \\ \left[-\frac{\gamma_Y}{1 - \kappa(1-q)} \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha}\right] Z_t + \right. \\ & \left. + \frac{1 - (1-q)\kappa}{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]} E\left[\frac{1}{\alpha} Z_{t+1}\right] \right] = \\ & = -\frac{1}{\beta} E\left[\frac{1}{\alpha}\right] Z_t + \frac{1 - (1-q)\kappa}{\gamma_Y\beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \end{aligned}$$

The value of X_t is:

$$\begin{aligned} & \left[\frac{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]}{\gamma_Y\beta} \right] \\ \left[-\frac{\gamma_Y}{1 - \kappa(1-q)} \frac{\gamma_Y\beta}{[\gamma_Y^2\beta - [1 - (1-q)\kappa]]} X_{t-1} + \right. \\ & \left. + \frac{\gamma_Y\beta}{[\gamma_Y^2\beta - [1 - (1-q)\kappa]]} X_t + \frac{1}{1 - \kappa(1-q)} X_t \right] = \\ & = \frac{(1-q)(\gamma_Y\beta - 1)}{\gamma_Y\beta} X_t + \frac{(1-q)\gamma_Y}{1 - \kappa(1-q)} (\epsilon_t^D - \epsilon_t^B). \end{aligned}$$

The final result is the following:

$$\begin{aligned} E[F_{t+1}] - \left[\frac{1}{\gamma_Y\beta} + \gamma_Y \right] F_t + \frac{1}{\beta} F_{t-1} = & \frac{[1 - \gamma_Y][\beta\gamma_Y - 1]}{\gamma_Y\beta} NW + \\ -\frac{1}{\beta} E\left[\frac{1}{\alpha}\right] Z_t + \frac{1 - (1-q)\kappa}{\gamma_Y\beta} E\left[\frac{1}{\alpha} Z_{t+1}\right] + & \frac{(1-q)(\gamma_Y\beta - 1)}{\gamma_Y\beta} X_t + \frac{(1-q)\gamma_Y}{[1 - \kappa(1-q)]} (\epsilon_t^D - \epsilon_t^B). \end{aligned} \quad (75)$$

Alternative solution of the system

$$\begin{aligned} E[F_{t+1}] = & \frac{1 - (1-q)\kappa}{\gamma_Y\beta} F_t + \frac{(1-q)[\gamma_Y^2\beta - [1 - (1-q)\kappa]]}{\gamma_Y\beta} D_t + \\ + \frac{\beta\gamma_Y + [(1-q)\kappa - 1]}{\gamma_Y\beta} NW + (1-q)X_t + & \frac{1 - (1-q)\kappa}{\gamma_Y\beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \end{aligned} \quad (76)$$

$$D_t = \frac{\gamma_Y}{1 - \kappa(1 - q)} D_{t-1} - \frac{\kappa}{1 - \kappa(1 - q)} F_t + \frac{\kappa}{1 - \kappa(1 - q)} NW + \frac{X_t}{1 - \kappa(1 - q)}. \quad (77)$$

The second equation can be exposed as:

$$F_t = \frac{\gamma_Y}{\kappa} D_{t-1} - \frac{1 - \kappa(1 - q)}{\kappa} D_t + NW + \frac{X_t}{\kappa}. \quad (78)$$

$$\begin{aligned} & \frac{\gamma_Y}{\kappa} D_t - \frac{1 - \kappa(1 - q)}{\kappa} E[D_{t+1}] + NW + \frac{E[X_{t+1}]}{\kappa} = \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} \times \\ & \left[\frac{\gamma_Y}{\kappa} D_{t-1} - \frac{1 - \kappa(1 - q)}{\kappa} D_t + NW + \frac{X_t}{\kappa} \right] + \frac{(1 - q)[\gamma_Y^2 \beta - [1 - (1 - q)\kappa]]}{\gamma_Y \beta} D_t + \\ & + \frac{\beta \gamma_Y + (1 - q)\kappa - 1}{\gamma_Y \beta} NW + (1 - q)X_t + \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \end{aligned} \quad (79)$$

Putting all the terms in D on the left-hand side, we obtain the following expression:

$$\begin{aligned} & -\frac{1 - \kappa(1 - q)}{\kappa} E[D_{t+1}] + \frac{\gamma_Y}{\kappa} D_t + \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} \frac{1 - \kappa(1 - q)}{\kappa} D_t + \\ & - E\frac{(1 - q)[\gamma_Y^2 \beta - [1 - (1 - q)\kappa]]}{\gamma_Y \beta} D_t - \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} \frac{\gamma_Y}{\kappa} D_{t-1} = \\ & = \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} NW + \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} \frac{X_t}{\kappa} + \frac{\beta \gamma_Y + (1 - q)\kappa - 1}{\gamma_Y \beta} NW + \\ & - NW + (1 - q)X_t - \frac{E[X_{t+1}]}{\kappa} + \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \end{aligned} \quad (80)$$

It can be simplified as:

$$\begin{aligned} & E[D_{t+1}] - \frac{\gamma_Y}{1 - \kappa(1 - q)} D_t - \frac{1 - (1 - q)\kappa}{\gamma_Y \beta} D_t + \frac{(1 - q)[\gamma_Y^2 \beta - [1 - (1 - q)\kappa]]}{\gamma_Y \beta} \frac{\kappa}{1 - \kappa(1 - q)} D_t + \\ & + \frac{1}{\beta} D_{t-1} = -\frac{\kappa}{\gamma_Y \beta} NW - \frac{1}{\gamma_Y \beta} X_t - \frac{\beta \gamma_Y - [(1 - q)\kappa - 1]}{\gamma_Y \beta} \frac{\kappa}{1 - \kappa(1 - q)} NW + \frac{\kappa}{1 - \kappa(1 - q)} NW + \\ & - \frac{\kappa}{1 - \kappa(1 - q)} (1 - q)X_t + \frac{1}{1 - \kappa(1 - q)} E[X_{t+1}] - \frac{\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \end{aligned} \quad (81)$$

The left-hand side of the equation has exactly the same structure as the one in F_t and clearly the simplification is identical. We will now study the different parts of the right-end side, starting with net worth.

$$-\frac{\kappa}{\gamma_Y \beta} NW - \frac{\beta \gamma_Y - [(1 - q)\kappa - 1]}{\gamma_Y \beta} \frac{\kappa}{1 - \kappa(1 - q)} NW + \frac{\kappa}{1 - \kappa(1 - q)} NW. \quad (82)$$

Under a common denominator we obtain the following result:

$$\frac{-\kappa[1 - \kappa(1 - q)] - \beta \gamma_Y \kappa + [(1 - q)\kappa - 1]\kappa + \gamma_Y \beta \kappa}{\gamma_Y \beta [1 - \kappa(1 - q)]} NW = 0.$$

As a consequence the impact of net worth on deposits is null, as we would intuitively expect. The direct effect of interest rates is due to:

$$-\frac{1}{\gamma_Y \beta} X_t - \frac{\kappa}{1 - \kappa(1 - q)} (1 - q)X_t + \frac{1}{1 - \kappa(1 - q)} E[X_{t+1}].$$

Or:

$$-\frac{\gamma_Y \beta \kappa (1 - q) + [1 - \kappa(1 - q)]}{\gamma_Y \beta [1 - \kappa(1 - q)]} X_t + \frac{1}{1 - \kappa(1 - q)} X_{t+1}.$$

Since we know that $X_{t+1} = X_t + \epsilon_{t+1}^D - \epsilon_{t+1}^B$, we can write:

$$-\frac{\gamma_Y \beta \kappa (1-q) + [1 - \kappa(1-q)]}{\gamma_Y \beta [1 - \kappa(1-q)]} X_t + \frac{X_t + \epsilon_{t+1}^D - \epsilon_{t+1}^B}{1 - \kappa(1-q)}.$$

And, since $E_t[\epsilon_{t+1}] = 0$:

$$\frac{-\gamma_Y \beta \kappa (1-q) - [1 - \kappa(1-q)] + \gamma_Y \beta}{\gamma_Y \beta [1 - \kappa(1-q)]} X_t.$$

It can be simplified as:

$$\frac{\gamma_Y \beta - 1}{\gamma_Y \beta} X_t.$$

The other intercept term is $-\frac{\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]$. We can finally write the value of D_t as:

$$E[D_{t+1}] - \left[\frac{1}{\gamma_Y \beta} + \gamma_Y\right] D_t + \frac{1}{\beta} D_{t-1} = \frac{\gamma_Y \beta - 1}{\gamma_Y \beta} X_t - \frac{\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \quad (83)$$

Deposits

We can obtain the value of D_t from:

$$E[D_{t+1}] - \left[\frac{1}{\gamma_Y \beta} + \gamma_Y\right] D_t + \frac{1}{\beta} D_{t-1} = \frac{\gamma_Y \beta - 1}{\gamma_Y \beta} X_t - \frac{\kappa}{\gamma_Y \beta} E\left[\frac{1}{\alpha} Z_{t+1}\right]. \quad (84)$$

The solution is given by:

$$D_{t+j+1} = \frac{1}{\gamma_Y \beta} D_{t+j} + \frac{\kappa}{\beta \gamma_Y} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y}\right)^i E_{t+i} \left[\frac{Z'_{t+j+i+2}}{\alpha}\right] - \frac{\kappa}{\beta \gamma_Y \alpha (1 - \gamma_Y)} C + \\ - \frac{\gamma_Y \beta - 1}{\gamma_Y \beta} \sum_{i=1}^{\infty} \left(\frac{1}{\gamma_Y}\right)^i E_{t+i} [X_{t+j+i+1}], \quad (85)$$

Under the assumption that interest rates follow a random walk process, we can for simplicity assume that deterministic component of the rates remain constant and treat the values of the rates at time t and $t + 1$ that enter in the solution as constants. In this case the result is the following:

$$D_{t+j+1} = \frac{1}{\gamma_Y \beta} D_{t+j} - \frac{\kappa}{\beta \gamma_Y \alpha (1 - \gamma_Y)} \left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] \left[r_{t+j+1}^B + COV\left(r^B, \frac{1}{\alpha}\right) \right] + \\ - \frac{\kappa}{\beta \gamma_Y \alpha (1 - \gamma_Y)} \kappa (r_{t+j}^R q - r_{t+j}^D) + \frac{\gamma_Y \beta - 1}{\gamma_Y \beta (1 - \gamma_Y)} (g_3 r_{t+j}^D - g_4 r_{t+j}^B) - \frac{\kappa}{\beta \gamma_Y \alpha (1 - \gamma_Y)} C + \\ - \frac{(\gamma_Y \beta - 1) \kappa}{\beta \gamma_Y (1 - \gamma_Y)} COV\left(L^D, \frac{1}{\alpha}\right); \quad (86)$$

it can finally be simplified as:

$$D_{t+j+1} = \frac{1}{\gamma_Y \beta} D_{t+j} + \frac{\kappa \left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] + (\gamma_Y \beta - 1) \alpha g_4 H}{\beta \gamma_Y \alpha (\gamma_Y - 1)} r_{t+j+1}^B + \\ + \frac{\kappa}{\beta \gamma_Y \alpha (\gamma_Y - 1)} \kappa r_{t+j}^R q - \frac{\kappa^2 + \alpha (\gamma_Y \beta - 1) g_3}{\beta \gamma_Y \alpha (\gamma_Y - 1)} r_{t+j}^D + \frac{\kappa \left\{ (\beta \gamma_Y - 1) \left[z - \frac{q}{b} \right] - \kappa u \right\}}{\beta \gamma_Y \alpha (\gamma_Y - 1)} + \\ + \frac{\kappa}{\beta \gamma_Y \alpha (\gamma_Y - 1)} \left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - 1) + (1 - q) \kappa \right] COV\left(r^B, \frac{1}{\alpha}\right) - \frac{(\gamma_Y \beta - 1) \kappa}{\beta \gamma_Y (\gamma_Y - 1)} COV\left(L^D, \frac{1}{\alpha}\right). \quad (87)$$

The interest rate on loans

The interest rate on loans can easily be obtained substituting the solution (44) for the quantity of loans in the demand condition:

$$L_{t+j} = a - br_t^L + dr_{t+j}^B + \eta_{t+j}, \quad (88)$$

For simplicity we use the solution of (26), but the result is general.

$$L_{t+j+1} = \frac{1}{\gamma_Y \beta} L_{t+j} + \frac{1}{\gamma_Y \beta} NW - \frac{\left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] r_{t+j+1}^B + \kappa (r_{t+j}^R q - r_{t+j}^D)}{\gamma_Y \beta \alpha} +$$

$$- \frac{(\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u}{\gamma_Y \beta \alpha} - \frac{\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - 1) + (1 - q) \kappa}{\gamma_Y \beta \alpha} COV\left(r^B, \frac{1}{\alpha}\right) + \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{\alpha}\right), \quad (89)$$

$$r_{t+j}^L = \frac{1}{b} \left\{ a + dr_{t+j}^B + \eta_{t+j} - \frac{1}{\gamma_Y \beta} L_{t+j} - \frac{1}{\gamma_Y \beta} NW + \right.$$

$$+ \frac{\left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] r_{t+j+1}^B + \kappa (r_{t+j}^R q - r_{t+j}^D)}{\gamma_Y \beta \alpha} + \frac{(\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u}{\gamma_Y \beta \alpha} +$$

$$\left. + \frac{\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - 1) + (1 - q) \kappa}{\gamma_Y \beta \alpha} COV\left(r^B, \frac{1}{\alpha}\right) - \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{\alpha}\right) \right\}; \quad (90)$$

focusing just on the relationship with the rate on bonds, we obtain:

$$r_{t+j}^L = \frac{1}{b} \left\{ dr_{t+j}^B + \frac{\left[\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - H) + (1 - q) \kappa H \right] r_{t+j+1}^B}{\gamma_Y \beta \alpha} + G \right\}, \quad (91)$$

where

$$G = a + \eta_{t+j} - \frac{1}{\gamma_Y \beta} L_{t+j} - \frac{1}{\gamma_Y \beta} NW + \frac{\kappa (r_{t+j}^R q - r_{t+j}^D)}{\gamma_Y \beta \alpha} + \frac{(\beta \gamma_Y - 1) \left[z - \frac{a}{b} \right] - \kappa u}{\gamma_Y \beta \alpha} +$$

$$+ \frac{\left(1 - \frac{d}{b}\right) (\gamma_Y \beta - 1) + (1 - q) \kappa}{\gamma_Y \beta \alpha} COV\left(r^B, \frac{1}{\alpha}\right) - \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{\alpha}\right); \quad (92)$$

It is useful to study in first instance the result when the bank has no market power and the rate on loans is taken as exogenous. Considering the bank as representative of the sector and aggregating we would obtain:

$$r_{t+j}^L = \frac{\left[-(\gamma_Y \beta - H) r_{t+j+1}^L + (1 - q) \kappa H \right] r_{t+j+1}^B}{\gamma_Y \beta v} + G', \quad (93)$$

and

$$r_{t+j}^L + \frac{\gamma_Y \beta - H}{\gamma_Y \beta v} r_{t+j+1}^L = \frac{(1 - q) \kappa r_{t+j}^B}{\gamma_Y \beta v} + G'. \quad (94)$$

Finally,

$$r_{t+j+1}^L = \frac{1 - \gamma_Y \beta v}{\gamma_Y \beta} r_{t+j}^L + \frac{(1 - q) \kappa}{\gamma_Y \beta} r_{t+j}^B + vG', \quad (95)$$

where

$$G' = -\frac{1}{\gamma_Y \beta} L_{t+j} - \frac{1}{\gamma_Y \beta} NW + \frac{\kappa (r_{t+j}^R q - r_{t+j}^D - u)}{\gamma_Y \beta v} +$$

$$+ \frac{(1 - q) \kappa}{\gamma_Y \beta v} COV\left(r^B, \frac{1}{v}\right) - \frac{\gamma_Y \beta - 1}{\beta \gamma_Y} COV\left(L^D, \frac{1}{v}\right); \quad (96)$$