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EQUILIBRIUM CONTRIBUTIONS AND "LOCALLY ENJOYED" PUBLIC GOODS

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Abstract

The main results of the traditional theory of private provision of public goods in the case of identical individuals are: 1) there exists a unique Nash equilibrium pattern of contributions in which everybody contributes the same amount (Bergstrom et al. [1986]); 2) this pattern is locally stable (Cornes [1980]). Under homothetic preferences, I show that these results generally no longer hold in the context of "locally enjoyed" public goods. In particular, when the symmetric Nash equilibrium is not the unique equilibrium pattern, it is locally unstable and there exists at least a locally stable asymmetric Nash equilibrium.

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Key words: Local Interaction, Public Goods, Nash Equilibria.

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I. INTRODUCTION: PUBLIC GOODS, NASH EQUILIBRIA AND LOCAL INTERACTION

It is widely accepted that Bergstrom et al. [1986] (henceforth BBV) represents one of the most important theoretical works on the private provision of public goods.[1] They refer to a situation in which N anonymous individuals have to decide how to share their personal income between consumption and contribution towards a public good which affects the well being of all the contributors in the society. Assuming that a) individuals are utility maximisers, b) individuals consider others' contributions as independent of their own, c) both public good and private consumption are normal goods, they prove that, for any size of the population, there exists a unique Nash equilibrium pattern of contributions. Moreover, by using a simple adjustment process, Cornes [1980] (henceforth CO) proves that the unique Nash equilibrium pattern is also locally stable.[2] If we assume that individuals are identical in terms of both preferences and incomes, the conclusion is that the unique, locally stable Nash equilibrium is represented by a pattern of identical contributions. Let us define this pattern *symmetric Nash equilibrium* (henceforth SNE).

However, in the real world, voluntary contribution regimes clearly do generate unequal contributions from people who are equal in objective circumstances such as preferences and incomes (i.e. some people free ride and some do not). How can this empirical puzzle be solved? One possible solution is to introduce an assumption on the altruistic attitude of individuals and allow it to vary across individuals.[3] An alternative solution which saves the hypothesis of identical preferences and incomes across individuals could consist of modifying the spatial structure assumed by BBV. In particular, it appears natural to ask whether the results of uniqueness and stability implied by the BBV model continue holding in a context in which agents who are identical in terms

[1] The literature on the private provision of public goods counts many contributions. Including Samuelson's seminal works (Samuelson [1954], [1955]), the reader can also refer to Warr [1983], Andreoni [1988], Cornes et al. [1984].

[2] Cornes [1980] proves local stability under a simple continuous adjustment process for a more general externalities model. The reader can also refer to Sandmo [1980].

[3] An example is the theory of warm-glow giving. See for example Andreoni [1989], [1990], Diamond [2003].

of both preferences and incomes present the following characteristics: a) they differ from each other for the neighbourhood they belong to, and b) they enjoy the level of public good collected by the members of his neighbourhood.[4]

Although there are many theories which move in this direction, none of them can be considered as an extension of the BBV model to the context of "locally enjoyed" public goods. One of the first attempts to take into account the spatial localisation of public goods is Tiebout's model of local expenditures (Tiebout [1956]). In this model individuals can freely move across communities which are endowed with a fixed quantity of some public good and which are characterised by an optimal community size. Communities below the optimal size try to attract new residents to lower average costs while communities above the optimal size do the opposite. Given this structure, the economy would converge autonomously towards an equilibrium since *'[…] except when the system is in equilibrium, there will be a subset of consumer-voters who are discontented with the patterns of their community. Another set will be satisfied.'* (page 420). It is clear that the model refers to a totally different problem with respect the one considered by BBV. Indeed, instead of introducing the local structure to explain how much individuals contribute, Tiebout uses it to analyse where individuals decide to move to.

More relevant for the topic of this paper is the model built by Eshel et al. [1998] (henceforth ESS). They adapt Ellison's model of local interaction (Ellison [1993]) to 2 by 2 public good games. There are N individuals spaced around a circle. Each individual has to decide how to contribute to a public good which affects the well being of those who belong to his neighbourhood. Individual i 's neighbourhood is composed by the first person on his right, the first person on his left and himself. Each individual can behave either altruistically or egoistically. An altruist contributes a quantity of public good which supplies one unit of utility to each member of his neighbourhood. The net cost to the altruist for providing that quantity of public good is expressed by a loss of utility equal to $c > 0$. An egoist does not contribute at all and does not bear any cost. He simply enjoys the total amount of

[4] For instance, the environment is a valid example of locally enjoyed public good.

public good supplied by the altruists. At the end of each period, each individual decides how to behave on the basis of a specific learning process. In particular, with probability μ (with $1 > \mu > 0$) he imitates the behaviour of those in his neighbourhood who earned the highest payoffs while with probability $(1-\mu)$ he retains his strategy. On the basis of these assumptions, the authors prove that altruists can survive if they are grouped together, so that the benefits of altruism are enjoyed primarily by other altruists, who then earn relatively high payoffs and are imitated. Moreover altruists manage to survive also in the presence of mutations that continually introduce egoists into the population.[5]

ESS model significantly differs from BBV model for twp reasons. Firstly, ESS obtain the result of coexistence between high contributors and low contributors assuming that people do not perfectly behave as rational agents. Secondly, individuals can only decide either to contribute fixed quantity of public good which increases the utility of the neighbours of one unit or to contribute nothing.

I am going to present a version of the BBV model based on the same spatial structure assumed by ESS. There are N individuals who are distributed around a circle. They are identical in terms of both preferences and incomes. Individual j's neighbourhood is defined as the first k individuals on his right, the first k individuals on his left and himself. The total contribution collected within his neighbourhood represents the level of public good enjoyed by individual j. Finally, individuals are assumed to be "traditional" utility maximisers. This simply means that, taking neighbours' behaviour has given, each individual decides how to share his income between contribution and consumption in order to maximise his utility function.

The main conclusion of my model is that the introduction of the assumption of "locally enjoyed" public goods dramatically affects both the results of uniqueness and stability of the SNE implied by BBV model in a context characterised by identical individuals. In particular, both these results continue being true in situations in which the preferences for the public good are unrealistically

[5] The existence of an evolutionary equilibrium characterised by coexistence of altruistic and egoistic contributors is also implied by Bergstrom et al. [1993].

strong. But not otherwise, when uniqueness does not hold, all the equilibria but the SNE are *asymmetric Nash equilibrium* patterns (henceforth ANE) in which high contributors coexist with low contributors. Coexistence is a *"natural"* result in the sense that, rather than being caused by *ad hoc* assumptions either on the behaviour of agents or on their personal characteristics, it appears as a direct consequence of the spatial structure assumed in my model.

The paper is organised as follows. In section II, I present the assumptions of the model. In section III, I prove an existence theorem for a generic size of the neighbourhood. In section IV, I study the conditions under which the SNE continues being unique and locally stable when individual j's neighbourhood is defined as the first individual on his left, the first individuals on his right and himself. In section V, I discuss a simple example which highlights the different implications of my model with respect those of the BBV model. Finally, in section VI, I generalise the results obtained in section IV to contexts characterised by larger neighbourhoods.

II. THE STRUCTURE OF THE MODEL: ASSUMPTIONS AND DEFINITIONS

Suppose that there are N individuals distributed around a circle. Individuals are identical in terms of both preferences and incomes. Each individual belongs to a neighbourhood. In particular, individual j's neighbourhood is defined as the first k individuals on his right, the first k individuals on his left and himself, so that $N_j = \{(j-k),(j-k-1),...,(j-1),j,(j+1),...,(j+k-1),(j+k)\}_{\mathrm{mod}(N)}$. The preferences of individual j are described by a homothetic utility function, $U_j = U(c_j, Q_{N_j})$, which is strictly increasing in both the level of private consumption, c_j, and the level of public good collected in his neighbourhood, Q_{N_j}. Note that the hypotheses on the utility function imply the one of normality of both public good and private consumption used by BBV in their model. Let us assume that Q_{N_j} is an additive function of the contributions of the members of j's neighbourhood, so that the level of public good enjoyed by individual j can be written as follows:

$$Q_{N_j} = \sum_{s \in N_j} q_s \qquad [1]$$

where q_s is the contribution of individual s.

Individual j has to decide how to share his income, I, between consumption and contribution by taking the contribution of his neighbours as given and by subjecting his decision to both a budget constraint and a non negativity constraint. Formally, if we assume that both the price for a unit of private consumption and the price for a unit of public good are equal to one, the decisional problem faced by individual j can be written as follows:

$$\begin{aligned} &\max_{c_j, q_j} \quad U_j = U(c_j, Q_{N_j}) \\ &s.t. \quad I = q_j + c_j \\ &\qquad\; q_j \geq 0 \end{aligned} \qquad [2]$$

Following BBV let us define $f(W_j)$ the demand function of contributor j when he is endowed with total wealth $W_j = I + Q_{N_j - j}$ where $Q_{N_j - j}$ is the total amount contributed by all the members of N_j but j. It simply represents the value of public good that individual j would choose for different values of W_j if he could ignore the non negativity constraint.[6] Given the assumption of homotheticity, $f(W_j)$ is linear. Then, it can be rewritten as follows:

$$f(W_j) = \alpha\left(I + Q_{N_j - j}\right), \ \forall j \in N \qquad [3]$$

The variable α, with $0 < \alpha < 1$, represents the proportion of income an individual would like to contribute to the public good if nobody contributes and it basically expresses the importance of the public good in the utility function.

For typical cases that model applies to, reasonable values of α are very low. For instance, in a "Cobb-Douglas" utility function of the type $U_j = c_j^{1-\alpha} Q_{N_j}^{\alpha}$, we could expect α to be lower than 0.5 meaning that the private consumption is more important than the public good.

[6] In other words, $f(W_j)$ represents the Engel curve of individual j.

Taking into account the non negativity constraint, we obtain the following expression which represents individual j's best response to $Q_{N_j - j}$:

$$q_j^{BR} = \max[\alpha(I + Q_{N_j - j}) - Q_{N_j - j};0] , \ \forall j \in N \qquad [4]$$

Let me introduce some simple definitions which will be widely used throughout the paper.

DEF.1. For given α, N, N_j, an equilibrium is a vector of contributions $(q_1^e,...,q_N^e)$ such that $q_j^e = q_j^{BR}$, $\forall j \in N$.

In particular, if $2k+1 \geq N$, then $N_j = N$, $\forall j$, and the spatial structure of the economy coincides with the one assumed by BBV. Therefore, the SNE is the unique equilibrium pattern of the model and it is globally stable.

DEF.2. A *plain* of size m is a set of m consecutive individuals $\{j,(j+1)_{\mathrm{mod}N},...,(j+m-1)_{\mathrm{mod}N}\}$ who contribute the same share of income.

DEF.3. For a given k, a *BBV community* of size z is a set of z consecutive individuals $\{j,(j+1)_{\mathrm{mod}\,N},...,(j+z-1)_{\mathrm{mod}\,N}\}$ such that a) $z \leq 2k$, b) individuals $\{(j-1)_{\mathrm{mod}\,N},...,(j-k)_{\mathrm{mod}\,N},(j+z)_{\mathrm{mod}\,N},...,(j+z+k)_{\mathrm{mod}\,N}\}$ are null contributors (i.e. individuals who contribute nothing).

The previous definition implies that if C is a *BBV community* of size z for a given k and $j \in C$, then a) everyone else in C is j's neighbour, b) if d is a neighbour of j but $d \notin C$, then he is a null contributor and finally c) the level of public good enjoyed by individual j is $Q_{N_j} = \sum_{s \in C} q_s$.

In other words, for a given k, a *BBV community* of size z is an "isolated" subgroup of individuals who enjoy the same level of public good. This means that the conclusions of the BBV model can be extended to each *BBV community*.

Now, let us turn to the dynamic structure of the model which will be used to study the stability properties of the equilibrium patterns. Let us suppose that individuals adjust their contribution over time according to the same continuous adjustment process adopted by CO:

$$\dot{q}_{j,t} = \frac{\partial q_{j,t}}{\partial t} = \mu[q_{j,t}^{BR} - q_{j,t}], \quad \mu > 0 \qquad [5]$$

Where μ represents the speed of adjustment which is assumed to be same across individuals. In other words, individual j adjusts his contribution to the equilibrium level gradually over time.[7]

III. EXISTENCE OF AN EQUILIBRIUM

Suppose that individual j's neighbourhood is composed by the first k individuals on his right, the first k individuals on his left and himself with $2k+1 \le N$. Given the definition of the neighbourhood, expression [4] becomes:

$$q_j^{BR} = \max[\alpha(I + \sum_{s=1}^{k} q_{(j-s)\text{mod}(N)} + q_{(j+s)\text{mod}(N)}) - (\sum_{s=1}^{k} q_{(j-s)\text{mod}(N)} + q_{(j+s)\text{mod}(N)});0], \ \forall j \in N \qquad [6]$$

Regarding the existence of an equilibrium, theorem 2 presented in BBV (page 33) can be innocuously extended to this context.

PROP.1. For any $k \ge 1$ and for any $N \ge 2k+1$, a Nash equilibrium exists.

[Proof in appendix]

The previous proposition does not say anything about the number and the stability properties of the equilibria. In the next sections, I shall show that both the properties of uniqueness and stability

7 Economists explain this adjustment mechanism in terms of learning process. For further information on learning processes in game theory see Fudenberg et al [1999].

significantly change when remove the assumption of globally enjoyed public goods in favour of local enjoyment. In particular, I shall begin with the case of $k=1$ and, after having presented a simple example, I will generalise the results to any $k>1$.

IV. NON UNIQUENESS AND STABILITY OF THE EQUILIBRIA WHEN k=1

When $k=1$, the best response function of individual j becomes:

$$q_j^{BR} = \max[\alpha(I + q_{(j-1)\mathrm{mod}(N)} + q_{(j+1)\mathrm{mod}(N)}) - (q_{(j-1)\mathrm{mod}(N)} + q_{(j+1)\mathrm{mod}(N)});0],\ \forall j \in N \qquad [7]$$

Given both the adjustment process followed by each individual (expression [5]) and expression [7], we can specify the following system which describes the dynamics of the model:

$$\begin{cases} \dot{q}_{1,t} = \mu\{\max[\alpha(I+q_{N,t}+q_{2,t})-(q_{N,t}+q_{2,t});0] - q_{1,t}\} \\ \vdots \\ \dot{q}_{j,t} = \mu\{\max[\alpha(I+q_{j-1,t}+q_{j+1,t})-(q_{j-1,t}+q_{j+1,t});0] - q_{j,t}\} \\ \vdots \\ \dot{q}_{N,t} = \mu\{\max[\alpha(I+q_{N-1,t}+q_{1,t})-(q_{N-1,t}+q_{1,t});0] - q_{N,t}\} \end{cases} \qquad [8]$$

System [8] is composed by non linear differential equations of the first order. Consider the system without the non negativity constraint. In this case, the dynamic system becomes linear and it can be rewritten in matrix notation as follows:

$$\underline{\dot{q}}_t = \mu\alpha\underline{I} + \mu A_N^1 \underline{q}_t \qquad [9]$$

Where $\underline{\dot{q}}_t, \underline{q}_t, \underline{I} \in \Re^N$, $\mu, \alpha \in \Re$ and finally:

$$A_N^1 = circulant\ \left[-1, -(1-\alpha), \overbrace{0,......,0}^{(N-3)\,terms}, -(1-\alpha)\right] \in \Re^N x \Re^N \qquad [10]$$

The circulant nature[8] of A_N^1 is important for two reasons. Firstly, since A_N^1 is symmetric, its eigenvalues are all real. Secondly, since A_N^1 is circulant, it can be diagonalised through the following normalized Fourier matrix $F_N^* \in \Re^N x \Re^N$:

$$F_N^* = \frac{1}{\sqrt{N}} F_N \qquad [11]$$

[8] For further references on circulant matrices, see Davis [1994].

The generic element of the matrix $F_N \in \Re^N x \Re^N$ situated on the h-th row k-th column is:

$$F_{h,k} = (w_{k-1})^{h-1} = e^{\frac{2\pi i(h-1)(k-1)}{N}} \qquad [12]$$

where i is the imaginary number $\sqrt{-1}$, and $w_{k-1} = e^{\frac{2\pi i(k-1)}{N}}$ is the k-th of the N roots of the cyclotomic equation $x^N = 1$. These roots (sometimes called the "De Moivre Numbers") represent the coordinates in the complex plane of the vertices of a regular polygon with N sides and unit radius. For example, when $N = 6$, we have:

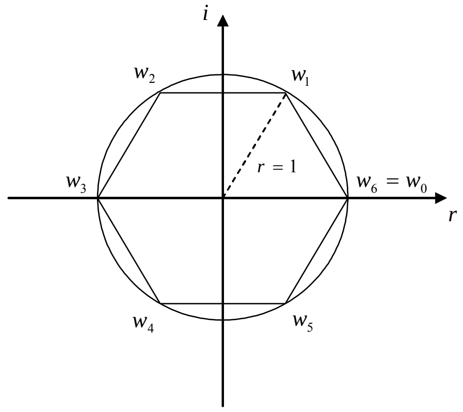


Fig.1. the coordinates in the complex plane of the vertices of a regular polygon with 6 sides and unit radius.

From the graph we have that $F_{h,k} = [w_{k-1}]^{h-1} = w_{(k-1)(h-1)\bmod(N)}$.

The h-th eigenvalue of a circulant matrix is obtained by multiplying the first row of A_N^1 by the h-th column of F_N. By applying this procedure, we obtain a general expression for the eigenvalues of A_N^1:

$$\lambda_h = -1-(1-\alpha)e^{\frac{2(h-1)\pi i}{N}} - (1-\alpha)e^{\frac{2(h-1)(N-1)\pi i}{N}} = -1-(1-\alpha)w_{h-1} - (1-\alpha)(w_{h-1})^{N-1},\ h = 1,...,N \quad [13]$$

Moreover, thanks to the properties of the De Moivre Numbers, we have that $(w_{h-1})^{N-1}$ is the complex conjugate of w_{h-1}.[9] Therefore, expression [13] can be rewritten as follows:

$$\lambda_h = -1-(1-\alpha)w_{h-1} - (1-\alpha)(w_{h-1})^{N-1} = -1-(1-\alpha)w_{h-1} - (1-\alpha)conj(w_{h-1}),\ h = 1,...,N \quad [14]$$

[9] If w_{h-1} is real, then its conjugate is itself.

By using the properties of the De Moivre Numbers, we can prove the following lemma:

LEMMA.1. Let O_N^1 be the set of values of $(1-\alpha)$ included between 0 and 1 such that at least one eigenvalue of A_N^1 is null.[10] Let N^{even} and N^{odd} be the dimension of the population when the number of individuals is even and when it is odd respectively, with $N^{even} \geq 4$ and $N^{odd} \geq 5$. We have that:

a) $\min \mathrm{O}_{N^{even}}^1 = 1/2$;

b) There exists a strictly decreasing succession of values $\{(1-a_5^1),(1-a_7^1),\dots,(1-a_{N^{odd}}^1)\}$ such that $\min \mathrm{O}_{N^{odd}}^1 = (1-a_{N^{odd}}^1)$ and $\lim_{N^{odd}\to\infty}(1-a_{N^{odd}}^1)=1/2$.

[Proof in appendix]

If $(1-\alpha) \notin \mathrm{O}_N^1$ and we do not consider the non negativity constraint, then the SNE in which everybody contributes

$$q_j^e = \frac{\alpha}{N_j-(N_j-1)\alpha} I = \frac{\alpha}{3-2\alpha} I, \ \forall j \in N \qquad [15]$$

is the unique equilibrium of system [9]. On the other hands, if we introduce the non negativity constraint and we allows $(1-\alpha)$ for varying in $(0,1)$, the SNE might lose the property of uniqueness. Indeed, although it is clear that, for any $(1-\alpha)$ and for any N, system [8] can have at most one and only one SNE pattern, there are particular conditions on $(1-\alpha)$ that imply (if satisfied) the existence of other ANE patterns. Before examining these conditions, let me introduce three further definitions which will be widely used in this section.

[10] The cardinality of this set depends on the size of the population.

DEF.4. An *"up and down" equilibrium pattern on N^{even} individuals* is an equilibrium pattern such that $q_j^e = q^e_{(j+2)\,\mathrm{mod}(N^{even})} = q^e_{(j+4)\,\mathrm{mod}(N^{even})} = \ldots = q^e_{(j-2)\,\mathrm{mod}(N^{even})} = a$ if and only if $q^e_{(j+1)\,\mathrm{mod}(N^{even})} = q^e_{(j+3)\,\mathrm{mod}(N^{even})} = q^e_{(j+5)\,\mathrm{mod}(N^{even})} = \ldots = q^e_{(j-1)\,\mathrm{mod}(N^{even})} = b$ with $a \neq b$.

DEF.5. A *"peak and hill" equilibrium pattern on N^{odd} individuals with an upwards spike* is an equilibrium pattern such that if the smallest local minima bare in positions $(j-1)\,\mathrm{mod}(N^{odd})$ and $(j+1)\,\mathrm{mod}(N^{odd})$ with $q^e_{(j-1)\,\mathrm{mod}(N^{odd})} = q^e_{(j+1)\,\mathrm{mod}(N^{odd})}$, then the second smallest local minima are in positions $(j-3)\,\mathrm{mod}(N^{odd})$ and $(j+3)\,\mathrm{mod}(N^{odd})$ with $q^e_{(j-3)\,\mathrm{mod}(N^{odd})} = q^e_{(j+3)\,\mathrm{mod}(N^{odd})}$, the third smallest local minima are in positions $(j-5)\,\mathrm{mod}(N^{odd})$ and $(j+5)\,\mathrm{mod}(N^{odd})$ with $q^e_{(j-5)\,\mathrm{mod}(N^{odd})} = q^e_{(j+5)\,\mathrm{mod}(N^{odd})}$ and so on until positions $\left(j + \left\lfloor \frac{N^{odd}}{2} \right\rfloor\right)\mathrm{mod}(N^{odd})$ and $\left(j - \left\lfloor \frac{N^{odd}}{2} \right\rfloor\right)\mathrm{mod}(N^{odd})$ which constitute a *plain* of size 2.

DEF.6. A *"peak and hill" equilibrium pattern on N^{odd} individuals with a downwards spike* is an equilibrium pattern such that if the smallest local minimum is in position i, then the second smallest local minima are in positions $(j-2)\,\mathrm{mod}(N^{odd})$ and $(j+2)\,\mathrm{mod}(N^{odd})$ with $q_{(j-2)\,\mathrm{mod}(N^{odd})} = q_{(j+2)\,\mathrm{mod}(N^{odd})}$, the third smallest local minima are in positions $(j-4)\,\mathrm{mod}(N^{odd})$ and $(j+4)\,\mathrm{mod}(N^{odd})$ with $q_{(j-4)\,\mathrm{mod}(N^{odd})} = q_{(j+4)\,\mathrm{mod}(N^{odd})}$, and so on until positions $\left(j + \left\lfloor \frac{N^{odd}}{2} \right\rfloor\right)\mathrm{mod}(N^{odd})$ and $\left(j - \left\lfloor \frac{N^{odd}}{2} \right\rfloor\right)\mathrm{mod}(N^{odd})$ which constitute a *plain* of size 2.

Now, I can state and prove under which conditions the SNE continues being the unique equilibrium pattern.

PROP.2. For $k=1$ and for any $N \geq 4$, the SNE is the unique equilibrium pattern of contributions if and only if either $(1-\alpha)<1/2$ in the even case or $(1-\alpha)<\left(1-a^{1}_{N^{odd}}\right)$ in the odd case.

[Proof in appendix]

The previous proposition states that if and only if either $(1-\alpha) \geq 1/2$ in the even case or $(1-\alpha) \geq \left(1-a^{1}_{N^{odd}}\right)$ in the odd case, we can find at least an ANE pattern in addition to the SNE one. This result is quite counterintuitive since it suggests the existence of asymmetric equilibrium patterns of contribution in a world populated by individuals who are identical in terms of preferences and incomes. Surprisingly, even the stability properties of the equilibrium patterns depend on the value of $(1-\alpha)$. In particular, the following proposition states that the conditions on $(1-\alpha)$ such that the SNE is the unique equilibrium pattern coincide with those which imply its local stability.

PROP.3. For $k=1$ and for any $N \geq 4$, the SNE pattern of contributions is stable if and only if either $(1-\alpha)<1/2$ in the even case or $(1-\alpha)<\left(1-a^{1}_{N^{odd}}\right)$ in the odd case. Moreover, there exists at least one ANE pattern of contributions which is locally stable if and only if $(1-\alpha) \notin \mathrm{O}^{1}_{N}$ and either $(1-\alpha)>1/2$ in the even case or $(1-\alpha)>\left(1-a^{1}_{N^{odd}}\right)$ in the odd case.

[Proof in appendix]

Combining the main implications of PROP.2. and PROP.3, we can state the following corollary.

COR.1. For $k=1$ and for any $N \geq 4$, if and only if $(1-\alpha) \geq 1/2$ in the even case or $(1-\alpha) \geq \left(1-a^{1}_{N^{odd}}\right)$ in the odd case, the SNE is not the unique equilibrium pattern of contributions and, moreover, it is locally unstable. Moreover, if and only if $(1-\alpha) > 1/2$ in the even case or $(1-\alpha) > \left(1-a^{1}_{N^{odd}}\right)$ in the odd case and $(1-\alpha) \notin \mathrm{O}^{1}_{N}$, then it is always possible to find at least an ANE pattern which is locally stable.

The previous corollary states that for "reasonable" values of the importance of the public good in the utility function, a) the main results on the SNE pattern of contributions obtained by BBV and CO no longer hold in a context characterised by "locally enjoyed" public goods with $k=1$, b) it is always possible to find at least one pattern of asymmetric contributions which represents a locally stable Nash equilibrium.

Finally, the following proposition states that, when the preferences for the private consumption are sufficiently strong, it is always possible to identify a particular class of locally stable ANE patterns of contributions which are obtained by combining of *BBV communities* of size 1 and size 2.

PROP.4. For $k=1$ and for any $N \geq 4$, if $(1-\alpha) \geq 0.61803$ and $(1-\alpha) \notin \mathrm{O}^{1}_{N}$, then the generic combination of *BBV communities* of size 1 with *BBV communities* of size 2 which exhausts N and such that it does not present two or more consecutive *BBV communities* of size 2 represents a locally stable ANE pattern.

[Proof in Appendix]

For instance, if $N=9$ and $(1-\alpha)=0.7$, then the following pattern of contributions obtained by combining three *BBV communities* of size 1 with one *BBV community* of size 2 is a locally stable ANE:

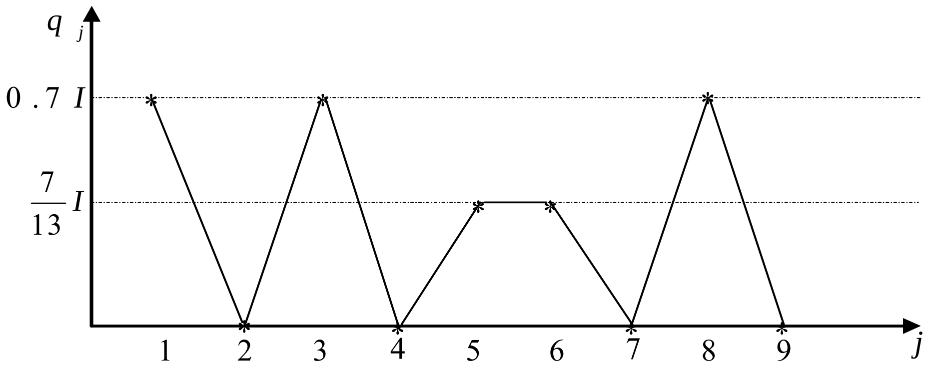


Fig.2. A locally stable ANE when $N = 9$ *and* $(1-\alpha) = 0.7$.

V. AN ILLUSTRATIVE EXAMPLE: THE BBV MODEL VS THE "LOCALLY ENJOYED" PUBLIC GOODS MODEL WHEN k=1

Consider a society of five individuals ($N = 5$). Let us start assuming that $k = 2$ so that $N_j = N$, $\forall j \in N$, and we have the BBV model. Given the expression of q_j^{BR} , we can specify the following dynamic system composed by five non linear differential equations of the first order:

$$\begin{cases} \dot{q}_{1,t} &= \mu \max[\alpha(I + q_2 + q_3 + q_4 + q_5) - (q_2 + q_3 + q_4 + q_5);0] - \mu q_{1,t} \\ \dot{q}_{2,t} &= \mu \max[\alpha(I + q_1 + q_3 + q_4 + q_5) - (q_1 + q_3 + q_4 + q_5);0] - \mu q_{2,t} \\ \dot{q}_{3,t} &= \mu \max[\alpha(I + q_1 + q_2 + q_4 + q_5) - (q_1 + q_2 + q_4 + q_5);0] - \mu q_{3,t} \\ \dot{q}_{4,t} &= \mu \max[\alpha(I + q_1 + q_2 + q_3 + q_5) - (q_1 + q_2 + q_3 + q_5);0] - \mu q_{4,t} \\ \dot{q}_{5,t} &= \mu \max[\alpha(I + q_1 + q_2 + q_3 + q_4) - (q_1 + q_2 + q_3 + q_4);0] - \mu q_{5,t} \end{cases} \quad [16]$$

Consider the system without the non negativity constraint. In this case, we have a dynamic system composed by linear differential equations which can be rewritten in matrix notation as follows:

$$\begin{bmatrix} \dot{q}_{1,t} \\ \dot{q}_{2,t} \\ \dot{q}_{3,t} \\ \dot{q}_{4,t} \\ \dot{q}_{5,t} \end{bmatrix} = \mu\alpha \begin{bmatrix} I \\ I \\ I \\ I \\ I \end{bmatrix} + \mu \begin{bmatrix} -1 & -(1-\alpha) & -(1-\alpha) & -(1-\alpha) & -(1-\alpha) \\ -(1-\alpha) & -1 & -(1-\alpha) & -(1-\alpha) & -(1-\alpha) \\ -(1-\alpha) & -(1-\alpha) & -1 & -(1-\alpha) & -(1-\alpha) \\ -(1-\alpha) & -(1-\alpha) & -(1-\alpha) & -1 & -(1-\alpha) \\ -(1-\alpha) & -(1-\alpha) & -(1-\alpha) & -(1-\alpha) & -1 \end{bmatrix} \begin{bmatrix} q_{1,t} \\ q_{2,t} \\ q_{3,t} \\ q_{4,t} \\ q_{5,t} \end{bmatrix} \quad [17]$$

Let us indicate with A_5^{BBV} the squared matrix of size 5 which appears in [17]. By imposing the equilibrium condition $\underline{\dot{q}}_t = 0$, we obtain:

$$\begin{bmatrix} 1 & (1-\alpha) & (1-\alpha) & (1-\alpha) & (1-\alpha) \\ (1-\alpha) & 1 & (1-\alpha) & (1-\alpha) & (1-\alpha) \\ (1-\alpha) & (1-\alpha) & 1 & (1-\alpha) & (1-\alpha) \\ (1-\alpha) & (1-\alpha) & (1-\alpha) & 1 & (1-\alpha) \\ (1-\alpha) & (1-\alpha) & (1-\alpha) & (1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} q_{1,t} \\ q_{2,t} \\ q_{3,t} \\ q_{4,t} \\ q_{5,t} \end{bmatrix} = \alpha \begin{bmatrix} I \\ I \\ I \\ I \\ I \end{bmatrix} \qquad [18]$$

The eigenvalues of $-A_5^{BBV}$ are $\lambda_1 = 5 - 4\alpha$, $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \alpha$. Therefore, given $0 < (1-\alpha) < 1$, the determinant of $-A_5^{BBV}$ is different from zero. The previous linear system has a unique solution in which everybody contributes the same amount (i.e. the SNE):

$$q_j^e = \frac{\alpha}{4-3\alpha} I, \qquad \forall j \in N \qquad [19]$$

The introduction of the non negativity constraint does not add any further equilibrium pattern.

Moreover, by evaluating the Jacobian matrix in the SNE pattern, we obtain that for $0 < (1-\alpha) < 1$ all the eigenvalues of A_5^{BBV} (which are the eigenvalues of $-A_5^{BBV}$ with opposite sign) are strictly negative. Therefore, the SNE is locally stable.

Now, let us assume that individual j 's neighbourhood is defined as the first individual on his right, the first individual on his left and himself. In this case, system [8] becomes:

$$\begin{cases} \dot{q}_{1,t} = \mu \max[\alpha(I + q_2 + q_5) - (q_2 + q_5); 0] - \mu q_{1,t} \\ \dot{q}_{2,t} = \mu \max[\alpha(I + q_1 + q_3) - (q_1 + q_3); 0] - \mu q_{2,t} \\ \dot{q}_{3,t} = \mu \max[\alpha(I + q_2 + q_4) - (q_2 + q_4); 0] - \mu q_{3,t} \\ \dot{q}_{4,t} = \mu \max[\alpha(I + q_3 + q_5) - (q_3 + q_5); 0] - \mu q_{4,t} \\ \dot{q}_{5,t} = \mu \max[\alpha(I + q_1 + q_4) - (q_1 + q_4); 0] - \mu q_{5,t} \end{cases} \qquad [20]$$

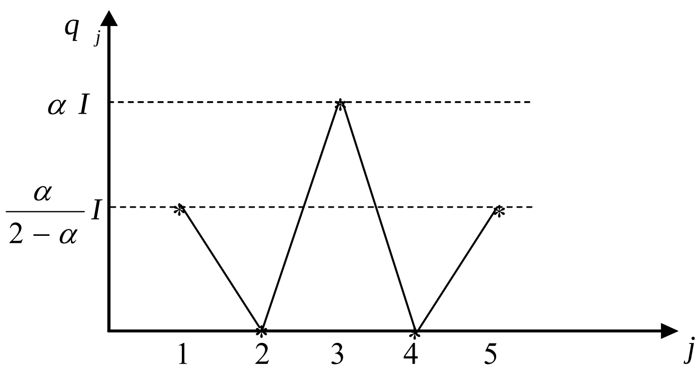
Again, by imposing the equilibrium condition without considering the non negativity constraint, we obtain the following linear system:

$$\begin{bmatrix} 1 & (1-\alpha) & 0 & 0 & (1-\alpha) \\ (1-\alpha) & 1 & (1-\alpha) & 0 & 0 \\ 0 & (1-\alpha) & 1 & (1-\alpha) & 0 \\ 0 & 0 & (1-\alpha) & 1 & (1-\alpha) \\ (1-\alpha) & 0 & 0 & (1-\alpha) & 1 \end{bmatrix} \begin{bmatrix} q_{1,t} \\ q_{2,t} \\ q_{3,t} \\ q_{4,t} \\ q_{5,t} \end{bmatrix} = \alpha \begin{bmatrix} I \\ I \\ I \\ I \\ I \end{bmatrix} \qquad [21]$$

If and only if the determinant of $-A_5^1$ is different from zero, the unconstrained system has a unique solution represented by the SNE in which everybody contributes the same amount

$$q_j^e = \frac{\alpha}{3-2\alpha} I \text{ , } \forall j \in N \qquad [22]$$

By using equation [14], it is possible to show that the eigenvalues of $-A_5^1$ are $\lambda_1 = 3 - 2\alpha$, $\lambda_2 = \lambda_5 = 1.618 - 0.61803\alpha$, $\lambda_3 = \lambda_4 = 1.618a - 0.61803$ and that the eigenvalues of A_5^1 are the eigenvalues of $-A_5^1$ with opposite sign. Clearly, $\mathrm{O}_N^1 = \{(1 - a_5^1) = 0.61803\}$. If and only if $(1-\alpha) = (1 - a_5^1)$, we have an infinite number of peak and hill equilibrium patterns in addiction to the SNE. Without considering the plain, the peak and hill equilibrium pattern with an upwards spike presents the highest number of local minima.[11] Among the infinite equilibrium patterns of this type let us consider the one in which, let us say, $q_2^e = q_4^e = 0$. By solving the linear system in which we impose these two constraints, we obtain that $q_3^e = \alpha I$ and $q_1^e = q_5^e = \frac{\alpha}{2-\alpha} I$. Given the previous expressions, the best response functions of individuals 2 and 4 imply that $q_2^e = q_4^e = 0$ if and only if $(1-\alpha) \geq (1 - a_5^1)$. Therefore, if and only if $(1-\alpha) < (1 - a_5^1)$, the SNE is the unique equilibrium pattern. On the contrary, if and only if $(1-\alpha) \geq (1 - a_5^1)$, the SNE is not unique and it is always possible to find at least an ANE pattern. For instance, the following pattern of contributions represents an equilibrium:



[11] Without considering the plain, a peak and hill equilibrium pattern with an upwards spike presents 2 local minima while a peak and hill equilibrium pattern with a downwards spike presents 1 local minimum.

Fig.3. Peak and hill equilibrium pattern with an upwards spike in $j = 3$ *and smallest local minima equal to zero.*

Now, let us turn to the stability properties of the equilibria. The Jacobian matrix in the SNE coincides with A_5^1. Therefore, the SNE is locally stable when all the eigenvalues of A_5^1 are strictly negative. This holds if and only if $0 < (1-\alpha) < (1-a_5^1)$. If and only if $(1-\alpha) = (1-a_5^1)$, there is an infinite number of locally unstable peak and hill equilibrium patterns. Finally, the Jacobian matrix in the peak and hill equilibrium pattern with an upwards spike in $j = 2$ and smallest local minima equal to zero is:

$$J_5^1(\underline{q}^e) = \mu \begin{bmatrix} -1 & -(1-\alpha) & 0 & 0 & -(1-\alpha) \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -(1-\alpha) & -1 & -(1-\alpha) & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -(1-\alpha) & 0 & 0 & -(1-\alpha) & -1 \end{bmatrix} \qquad [23]$$

If and only if $(1-a_5^1) < (1-\alpha) < 1$, all the eigenvalues of $J_5^1(\underline{q}^e)$ are strictly negative and, therefore, the peak and hill equilibrium pattern with an upwards spike and smallest local minima equal to zero is locally stable.

VI. NEIGHBOURHOODS COMPOSED BY MORE THAN THREE INDIVIDUALS (k>1)

What happens when the size of the neighbourhood increases? Suppose that the neighbourhood of individual j is composed by the first k individuals on his right, the first k individuals on his left and himself, with $k > 1$ and $2k+1 < N$. In this case, the best response function of individual j is expressed by expression [6]. Given both the adjustment process stated in the assumptions and expression [6], we can specify the following dynamic system:

$$\begin{cases} \dot{q}_{1,t} = \mu\{\max[\alpha(I + \sum_{s=1}^{k} q_{(1-s)\text{mod}(N),t} + q_{(1+s)\text{mod}(N),t}) - (\sum_{s=1}^{k} q_{(1-s)\text{mod}(N),t} + q_{(1+s)\text{mod}(N),t});0] - q_{1,t}\} \\ \vdots \\ \dot{q}_{j,t} = \mu\{\max[\alpha(I + \sum_{s=1}^{k} q_{(j-s)\text{mod}(N),t} + q_{(j+s)\text{mod}(N),t}) - (\sum_{s=1}^{k} q_{(j-s)\text{mod}(N),t} + q_{(j+s)\text{mod}(N),t});0] - q_{j,t}\} \\ \vdots \\ \dot{q}_{N,t} = \mu\{\max[\alpha(I + \sum_{s=1}^{k} q_{(N-s)\text{mod}(N),t} + q_{(N+s)\text{mod}(N),t}) - (\sum_{s=1}^{k} q_{(N-s)\text{mod}(N),t} + q_{(N+s)\text{mod}(N),t});0] - q_{N,t}\} \end{cases} \quad [24]$$

If we do not consider the non negativity constraint, system [24] becomes a linear dynamic system of the type:

$$\underline{\dot{q}}_t = \mu\alpha\underline{I} + \mu A_N^k \underline{q}_t \quad [25]$$

Where $\underline{\dot{q}}_t, \underline{q}_t, \underline{I} \in \Re^N$, $\mu, \alpha \in \Re$ and

$$A_N^k = circulant\ [-1, \underbrace{-(1-\alpha),\dots,-(1-\alpha)}_{k\ terms}, \overbrace{0,\dots\dots,0}^{(N-2k-1)\ terms}, \underbrace{-(1-\alpha),\dots,-(1-\alpha)}_{k\ terms}] \in \Re^N x \Re^N \quad [26]$$

By using the properties of the circulant matrix, the expression of the eigenvalues of A_N^k can be written as follows:

$$\lambda_{j+1} = -1 - 2(1-\alpha)\,\text{Re}\left[\sum_{s=1}^{k} e^{\frac{2js\pi i}{N}}\right], \qquad j = 0,\dots,N-1 \quad [27]$$

where $\text{Re}[d]$ is the real part of the complex number d.

Specularly to what said in section II, if $(1-\alpha) \notin \text{O}_N^k$ and we do not consider the non negativity constraint, then the SNE in which everybody contributes

$$q_j^e = \frac{\alpha}{N_j - (N_j - 1)\alpha} I = \frac{\alpha}{(2k+1) - 2k\alpha} I, \ \forall j \in N \quad [28]$$

is the unique equilibrium of system [24]. In the other cases, the SNE remains the unique equilibrium pattern if and only if specific conditions on $(1-\alpha)$ are satisfied. Moreover, as in the case of $k = 1$, these conditions coincide with those which allow the SNE for being locally stable. In particular, the following proposition generalises the results of uniqueness and stability of the SNE stated in COR.1. to larger neighbourhoods:

PROP.5. Let O_N^k be the set of values of $(1-\alpha)$ included between 0 and 1 such that at least one eigenvalue of A_N^k is null. Let $\left(1-a_N^k\right)$ be the lowest value in O_N^k. For any $k>1$ and for any $N \geq 2k+2$, the SNE is the unique equilibrium pattern and it is locally stable if and only if $(1-\alpha) < \left(1-a_N^k\right)$. Moreover, if and only if $(1-\alpha) \notin O_N^k$ and $(1-\alpha) > \left(1-a_N^k\right)$, it is always possible to find a ANE which is locally stable.

[Proof in appendix]

Can we say anything else on the value of $\left(1-a_N^k\right)$? Unfortunately, when $k>1$ the monotonic relation between $\left(1-a_N^k\right)$ and the number of individuals highlighted in LEMMA.1. no longer holds. However, the following property states, for any $k \geq 1$ and for any $N \geq 2k+2$, the existence of a lower bound such that if $(1-\alpha)$ is greater than or equal to such a bound, then the SNE in not unique and unstable.

PROP.6. For any $k>1$ and for any $N \geq 2k+2$, if $(1-\alpha) \geq 1/2$, then the SNE is not the unique equilibrium pattern and, moreover, it is unstable.

[Proof in appendix]

Finally, in order to identify a particular class of ANE patterns which are obtained by combining *BBV communities* of size 1 with *BBV communities* of size 2, we can extend PROP.4. to any $k>1$.

PROP.7. For any $k>1$ and for any $N \geq 2k+2$, if $(1-\alpha) \geq 0.61803$ and $(1-\alpha) \notin O_N^k$, then the generic combination of *BBV communities* of size 1 with *BBV communities* of size 2 which exhausts N represents a locally stable ANE pattern.

[Proof in appendix]

VII. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCHES

The statement "*identical individuals contribute the same amount of resources*" is the most natural and reasonable result of the traditional theory of private provision of public goods. Unfortunately, the real world induces public economists to be very sceptical with this implication. Voluntary contribution regimes clearly do generate unequal contributions from people who are equal in objective circumstances such as preferences and incomes. Could we use the framework of the traditional models to get rid of this embarrassing puzzle? Rather than introducing heterogeneity in preferences and incomes, I have shown that symmetry, uniqueness and stability which characterise the equilibrium pattern of contributions in the traditional models dramatically depend on the implicit hypothesis of "globally enjoyed" public goods. In particular, by introducing the Ellison's "local interaction structure" (Ellison [1993]) in the BBV model (Bergstrom et al. [1986]) with agents who have identical homothetic preferences and are endowed with the same income, I have proved that: a) the SNE is the unique equilibrium pattern and it is locally stable if and only if the preferences are unrealistically biased towards the public good; b) all the other equilibrium patterns but the SNE are characterised by asymmetric contributions; c) if and only if the SNE is locally unstable, it is always possible to find a locally stable ANE pattern; d) the previous results hold for any size of the neighbourhood such that it does not include the entire population.

It is possible to identify at least two aspects of my model which require further developments. Firstly, it could be interesting to analyse the results of uniqueness and stability of the SNE under different specifications of the utility function which satisfy the minimal requirements highlighted by BBV. Secondly, one could try to remove the assumption of identical individuals assuming either heterogeneous incomes or heterogeneous preferences.

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APPENDIX

Proof of PROP.1.

Let $\underline{q}^{BR} \in \Re^N$ be the vector which contains the best responses functions of the N contributors in the economy. $\underline{q}^{BR}$ defines a continuous function from the compact and convex set $\mathrm{I} = \{x \in \Re^N : 0 \le x_j \le I, j = 1,..., N\}$ to itself. Therefore, by the Brouwer's Fixed Point Theorem there must exist a fixed point $(q_1^e,..., q_N^e)$, which is a Nash equilibrium vector of contributions. ■

Proof of LEMMA.1.

When N is even, the coordinates of the vertex $w_{\frac{N^{even}}{2}}$ are $(0,-1)$. Then, expression [14] implies both that $\lambda_{\frac{N^{even}}{2}+1}$ is the greatest eigenvalue and that $(1-\alpha) = 1/2$ (the value of $(1-\alpha)$ such that $\lambda_{\frac{N^{even}}{2}+1} = 0$) is the smallest element in $\mathrm{O}^1_{N^{even}}$. When N is odd, the vertexes $w_{\left\lfloor \frac{N^{odd}}{2} \right\rfloor}$ and $w_{\left\lceil \frac{N^{odd}}{2} \right\rceil}$ (where $\left\lfloor \frac{N^{odd}}{2} \right\rfloor$ and $\left\lceil \frac{N^{odd}}{2} \right\rceil$ are the minor and the major integer of $\frac{N^{odd}}{2}$ respectively) are one the complex conjugate of the other and, moreover, they have the greatest real part in absolute value. Therefore, $\lambda_{\left\lfloor \frac{N^{odd}}{2} \right\rfloor+1}$ and $\lambda_{\left\lceil \frac{N^{odd}}{2} \right\rceil+1}$ are the greatest eigenvalues and the value of $(1-\alpha)$ such that $\lambda_{\left\lceil \frac{N^{odd}}{2} \right\rceil+1} = \lambda_{\left\lfloor \frac{N^{odd}}{2} \right\rfloor+1} = 0$ is the smallest element in $\mathrm{O}^1_{N^{odd}}$. Let us call this value $\left(1-a^1_{N^{odd}}\right)$. Moreover, as N^{odd} tends to infinitive, the vertexes $w_{\left\lfloor \frac{N^{odd}}{2} \right\rfloor}$ and $w_{\left\lceil \frac{N^{odd}}{2} \right\rceil}$ get closer and closer to the vertex $(0,-1)$. Therefore, the succession of $\left(1-a^1_{N^{odd}}\right)$ is monotone decreasing in N^{odd} with $\lim_{N^{odd}\to\infty}\left(1-a^1_{N^{odd}}\right)=1/2$. ■

Proof of PROP.2.

The equilibria of the "unconstrained" system [9] are obtained by imposing its LHS equal to zero, so that:

$$\mu\alpha\underline{I} + \mu A^1_N \underline{q}_t = 0 \qquad \text{[A1]}$$

$\forall(1-\alpha) \in \mathrm{O}^1_N$, matrix A^1_N is not full rank. Therefore, [A1] has an infinite number of solutions which includes the SNE. $\forall(1-\alpha) \notin \mathrm{O}^1_N$, matrix A^1_N is full rank. Therefore [A.1] has a unique solution which is equal to the SNE:

$$\underline{q}^e = -\alpha (A_N^1)^{-1} \underline{I} \qquad \text{[A2]}$$

with $\underline{q}^e \in \Re^N$ the generic element of which is $q_j^e = \frac{\alpha}{3-2\alpha} I$.

Now, let us identify ANE patterns implied by the non negativity constraint. Suppose that the dimension of the population is N^{even}. Consider the pattern in which $q_j = q_{(j+2)\bmod(N^{even})} = \ldots = q_{(j-2)\bmod(N^{even})} = 0$, and $q_{(j+1)\bmod(N^{even})} = q_{(j+3)\bmod(N^{even})} = \ldots = q_{(j-1)\bmod(N^{even})} = \alpha I$. Given $q_{(j-1)\bmod(N^{even})}$ and $q_{(j+1)\bmod(N^{even})}$, the best reply function of individual j implies that $q_j^e = 0$ if and only if $(1-\alpha) \geq 1/2$. Therefore, if and only if $(1-\alpha) < 1/2$, the non negativity constraint does not bind and the SNE is the unique equilibrium pattern. Suppose that the dimension of the population is N^{odd} and $(1-\alpha) = \left(1 - a^1_{N^{odd}}\right)$. Starting from the SNE, let q_j to change from $q_j^e = \frac{\alpha}{3-2\alpha} I$ to $q_j = \vartheta^u$ with $\vartheta^u \in \left(\frac{\alpha}{3-2\alpha} I; \alpha I\right]$. By solving the system composed by the $N^{odd} - 1$ best response functions of the "unconstrained" contributors and the condition $q_j = \vartheta^u$, we find the expression of the equilibrium contributions $\forall s \in N^{odd}$ with $s \neq j$. Then, given $q^e_{(j-1)\bmod(N)}$ and $q^e_{(j+1)\bmod(N)}$, the best response function of individual j implies that $q_j^e = \vartheta$ if and only if $(1-\alpha) = \left(1 - a^1_{N^{odd}}\right)$. The set of equilibrium contributions identifies a peak and hill equilibrium pattern with an upwards spike. Since $\vartheta^u \in \left(\frac{\alpha}{3-2\alpha} I; \alpha I\right]$, there exists an infinite number of equilibrium patterns of this type. Among them, consider the one such that the smallest local minima are represented by two null contributors. Because of the non negativity constraint, the condition on $(1-\alpha)$ such that this pattern is an ANE becomes $(1-\alpha) \geq \left(1 - a^1_{N^{odd}}\right)$. By repeating the previous procedure with $\vartheta^d \in \left[0; \frac{\alpha}{3-2\alpha} I\right)$, we end up with a peak and hill equilibrium pattern with a downwards spike represented by a null contributor. The condition on $(1-\alpha)$ such that it is an equilibrium pattern is the same of the one obtained in the previous case. Among the two ANE we have identified, let us consider the one which presents the highest number of local minima without considering the plateau.[12] By induction we have that, if and only if $\left(1 - a_5\right) > (1-\alpha) \geq \left(1 - a^1_{N^{odd}}\right)$, then $\exists (1-\alpha)'$ with $\left(1 - a_5^1\right) \geq (1-\alpha)' > \left(1 - \alpha\right)$ such that it is possible to find an equilibrium pattern associated with $(1-\alpha)'$ which presents at least one further null contributor. Indeed, the second smallest local minima are equal to zero if and only if $(1-\alpha) \geq \left(1 - a^1_{N^{odd}-2}\right)$, the third smallest local minima are equal to zero if and only if $(1-\alpha) \geq \left(1 - a^1_{N^{odd}-2}\right)$, and so on until $(1-\alpha) = \left(1 - a_5^1\right)$. When $(1-\alpha) \geq \left(1 - a_5^1\right)$, then for any N^{odd}, there always exists an equilibrium

[12] For example, if $N = 5$, a peak and hill equilibrium pattern with an upwards spike presents 2 local minima (without considering the plateau) while a peak and hill equilibrium pattern with a downwards spike presents 1 local minimum (without considering the plateau).

pattern which is composed by a combination of a unique *BBV community* of size 2 and $\frac{N^{odd}-3}{2}$ *BBV community* of size 1. ■

Proof of PROP.3.

If and only if $(1-\alpha)\in \mathrm{O}_N^1$, at least one eigenvalue of matrix A_N^1 is null. Therefore we have an infinite number of unstable equilibrium patterns. If and only if $(1-\alpha)\notin \mathrm{O}_N^1$ and either $(1-\alpha)<1/2$ in the even case or $(1-\alpha)<\left(1-a_{N^{odd}}^1\right)$ in the odd case, the SNE is the unique equilibrium pattern. In order to study the stability of the SNE, we directly study the sign of the eigenvalues of A_N^1 . LEMMA.1. implies that they are strictly negative. Therefore the SNE is locally stable. In order to study the stability properties of the equilibria when $(1-\alpha)\notin \mathrm{O}_N^1$ and either $(1-\alpha)>1/2$ in the even case or $(1-\alpha)>\left(1-a_{N^{odd}}^1\right)$ in the odd case, we use the linearization procedure. Indeed, let us rewrite the generic differential equation of the dynamic system [11] as follows:

$$\dot{q}_j=\mu\left\{m_j(q_j^e)[\alpha I-(1-\alpha)(q_{(j-1)\,\mathrm{mod}(N)}+q_{(j+1)\,\mathrm{mod}(N)})]-q_j\right\} \qquad \text{[A3]}$$

where $m_j(q_j^e)$ is equal to 1 if and only if $\alpha I\geq(1-\alpha)(q_{(j-1)\mathrm{mod}(N)}^e+q_{(j+1)\mathrm{mod}(N)}^e)$ and it is null if and only if $\alpha I<(1-\alpha)(q_{(j-1)\mathrm{mod}(N)}^e+q_{(j+1)\mathrm{mod}(N)}^e)$.

The generic equilibrium pattern $(q_1^e,...,q_N^e)$ is locally stable if and only if all the eigenvalues of the Jacobian matrix evaluated in it are strictly negative. In particular, the Jacobian matrix $J_N^1(\underline{q}^e)$ is a squared matrix of size N such that its j -th row is obtained by multiplying all the terms equal to $-(1-\alpha)$ which appear in the j -th row of A_N^1 by $m_j(q_j^e)$. Let us consider the SNE. In this case the Jacobian matrix coincides with A_N^1 . Therefore, for $(1-\alpha)\notin \mathrm{O}_N^1$ with either $(1-\alpha)>1/2$ in the even case or $(1-\alpha)>\left(1-a_{N^{odd}}^1\right)$ in the odd case, the SNE equilibrium is locally unstable. Now, suppose that the dimension of the population is N^{odd} . Let us assume that N^{odd} is such that the configuration which presents the highest number of local minima without considering the plateau is a peak and hill equilibrium pattern with an upwards spike. As already said, if and only if $\left(1-a_{N^{odd}}^1\right)\leq(1-\alpha)<\left(1-a_{N^{odd}-2}^1\right)$, we can identify other two equilibria: a "peak and hill" equilibrium pattern with a downwards spike represented by a null contributor and a "peak and hill" equilibrium pattern with an upwards spike surrounded by two null contributors. The condition such that all the eigenvalues of the Jacobian matrix evaluated in the first equilibrium pattern are strictly negative is $(1-\alpha)<\left(1-a_{N^{odd}}^1\right)$ which clearly represents a contradiction. The condition such that all the eigenvalues of the Jacobian matrix evaluated in the second equilibrium pattern are strictly negative is $(1-\alpha)<\left(1-a_{N^{odd}-2}^1\right)$. Therefore, if and only if $(1-\alpha)\notin \mathrm{O}_{N^{odd}}^1$ and $\left(1-a_{N^{odd}}^1\right)<(1-\alpha)<\left(1-a_{N^{odd}-2}^1\right)$, a "peak and hill" equilibrium pattern with an upwards spike such that the first smallest local minima are represented by null contributors is locally stable. Now, let $(1-\alpha)$ to increase. If and only if $\left(1-a_{N^{odd}-2}^1\right)\leq(1-\alpha)<\left(1-a_{N^{odd}-4}^1\right)$, we can identify a peak and hill equilibrium pattern with an upwards spike such that the first and the second smallest local minima are null. By using the Jacobian matrix evaluated in this equilibrium pattern we have that the condition such that all the eigenvalues are strictly negative is $(1-\alpha) < \left(1 - a^1_{N^{odd}-4}\right)$. Therefore, if and only if $(1-\alpha) \notin \mathrm{O}^1_{N^{odd}}$ and $\left(1 - a^1_{N^{odd}-2}\right) \le (1-\alpha) < \left(1 - a^1_{N^{odd}-4}\right)$, a "peak and hill" equilibrium pattern with an upwards spike such that the first and the second smallest local minima are represented by null contributors is locally stable. The rest of the proof consists in repeating the same procedure letting $(1-\alpha)$ to increase until $\left(1 - a^1_5\right)$. If and only if $(1-\alpha) \notin \mathrm{O}^1_{N^{odd}}$ and $\left(1 - a^1_5\right) \le (1-\alpha)$, the "peak and hill" pattern with an upwards spike such that all the local minima are equal to zero is locally stable. Suppose that the dimension of the population is N^{even}. If and only if $(1-\alpha) \ge 1/2$ we can identify an up and down equilibrium pattern composed by the combination of $N^{even}/2$ *BBV community* of size 1. By applying the linearisation procedure we can easily prove that this equilibrium pattern is locally stable if and only if $(1-\alpha) \notin \mathrm{O}^1_{N^{even}}$ and $(1-\alpha) > 1/2$. ■

Proof of PROP.4.

Firstly, let us prove that an equilibrium pattern cannot contain two consecutive *BBV communities* of size 2. Let C be a *BBV community* of size 2. By definition, it is straightforward to prove that, if C is a *BBV community* of size z then, in equilibrium:

$$q^e_j = \frac{\alpha}{z-(z-1)\alpha} I \text{ , } \forall j \in C \qquad \text{[A4]}$$

Moreover, it has to be a null contributor between two *BBV communities* of size 2. Let us call this individual x. Given $q^e_{(x-1)\bmod(N)} = q^e_{(x+1)\bmod(N)} = \frac{\alpha}{2-\alpha} I$, we have that the best response function of individual x implies $q^e_x = 0$ if and only if $2\frac{\alpha}{2-\alpha} I \ge \alpha I$. The previous condition is satisfied when $\alpha \le 0$ which is impossible.

Secondly, let us prove that if $k = 1$, then, $\forall (1-\alpha) \ge 0.61803$ such that $(1-\alpha) \notin \mathrm{O}^1_N$ and $\forall N \ge 4$, the generic combination of *BBV communities* of size 1 with *BBV communities* of size 2 which exhausts N and such that it does not present two or more consecutive *BBV communities* of size 2 represents a locally stable ANE pattern. When the size of the population is such that it can be decomposed into *BBV communities* of size 1 exclusively, then the neighbourhood of the generic null contributor b is composed by two individuals who belong to a *BBV community* of size 1. By using the best response function of individual b, we obtain that $q_b = 0$ if and only if $(1-\alpha) \ge 0.5$. When the size of the population is such that it cannot be decomposed into *BBV communities* of size 1 exclusively, we can always find a combination of *BBV communities* of size 1 with *BBV communities* of size 2 which exhausts N and such that it does not present two or more consecutive *BBV communities* of size 2 . Any of these combinations is characterised by the presence of at least one individual x such that: a) he is a null contributor; b) his neighbourhood is composed by himself, an individual who belongs to a *BBV community* of size 1 and, finally, an individual who belongs to *BBV community* of size 2. Note that, among the null contributors who separate a *BBV community* of size 1 form a *BBV* *community* of size 2, we have that $Q_{N_x - x} \le Q_{N_j - j}$, $\forall j \in N$ with $j \ne x$. By using the best response function of individual x, we obtain that $q_x^e = 0$ if and only if $(1-\alpha) \ge 0.61803$. Finally, by using the linearisation procedure presented in the proof of PROP.3., we have that, $\forall (1-\alpha) \ge 0.61803$ such that $(1-\alpha) \notin \mathrm{O}_N^1$, any combination of *BBV communities* of size 1 with *BBV communities* of size 2 which exhausts N and which does not present two or more consecutive *BBV communities* of size 2 is locally stable. ■

Proof of PROP.5.

$\forall (1-\alpha) \in \mathrm{O}_N^k$, we have an infinite number of ANE patterns in addition to the SNE. In this case, there is not any locally stable equilibrium pattern.

$\forall (1-\alpha) \notin \mathrm{O}_N^k$, the SNE is the unique solution of system [24]. In order to find other equilibria patterns implied by the non negativity constraint, let us replicate the procedure presented in PROP.2. Starting from the SNE associated with $(1-\alpha) = (1 - a_N^k)$, let q_j to change from $q_j^e = \dfrac{\alpha}{(2k+1) - 2k\alpha} I$ to $q_j = \vartheta^u$ with $\vartheta^u \in \left(\dfrac{\alpha}{(2k+1) - 2k\alpha} I; \alpha I \right]$. By solving the system composed by the $N-1$ best response functions of the "unconstrained" individuals and the condition $q_j = \vartheta^u$, we find the expression of the equilibrium contributions $\forall s \in N$ with $s \ne j$. Then, by using the equilibrium contributions of the individuals who belong to j's neighbourhood, expression [6] implies that $q_j^e = \vartheta^u$ if and only if $(1-\alpha) = (1 - a_N^k)$. Therefore, we have identified a particular pattern of contributions the shape of which depends the size of the population and the value of $(1-\alpha)$. Since $\vartheta^u \in \left(\dfrac{\alpha}{(2k+1) - 2k\alpha} I; \alpha I \right]$, we have an infinite number of equilibrium patterns of this type. Let ϑ^u increase until the smallest local minimum of the pattern of contributions gets value zero. Let us call this pattern ℓ^u. Because of the non negativity constraint, the condition on $(1-\alpha)$ such that ℓ^u is an ANE becomes $(1-\alpha) \ge (1 - a_N^k)$. By repeating the same procedure with $\vartheta^d \in \left[0; \dfrac{\alpha}{(2k+1) - 2k\alpha} I \right)$, we find another ANE with null contributors as smallest local minima, ℓ^d, which is implied by the condition $(1-\alpha) \ge (1 - a_N^k)$. By applying the linearisation procedure, we have that if and only if $(1-\alpha) \notin \mathrm{O}_N^k$ and $(1-\alpha) > (1 - a_N^k)$, one and only one between ℓ^u and ℓ^d is locally stable since all the eigenvalues of the Jacobian matrix evaluated in it are strictly negative. In particular, the Jacobian matrix $J_N^k(\underline{q}^e)$ is a squared matrix of size N such that its j-th row is obtained by multiplying all the terms equal to $-(1-\alpha)$ which appear in the j-th row of A_N^k by $m_j(q_j^e)$. Starting from the locally stable equilibrium pattern, by induction we have that if and only if $(1 - \overline{a}_N^k) > (1-\alpha) \ge (1 - a_N^k)$, then

$\exists(1-\alpha)'$ with $\left(1-\overline{a}_N^k\right)\geq(1-\alpha)'>(1-\alpha)$ such that the equilibrium pattern associated with $(1-\alpha)'$ presents at least one further null contributor, where $\left(1-\overline{a}_N^k\right)$ is the highest value of $(1-\alpha)$ such that $\forall(1-\alpha)>\left(1-\overline{a}_N^k\right)$ the number of null contributors associated with $(1-\alpha)$ is the same of the one associated with $\left(1-\overline{a}_N^k\right)$. By applying the linearisation procedure, we have that the equilibrium patterns identified by induction are locally stable if and only if $(1-\alpha)\notin \mathrm{O}_N^k$ and the conditions on $(1-\alpha)$ such that they are equilibria are satisfied. Finally, the SNE equilibrium is locally stable if and only if the eigenvalues of matrix A_N^k are strictly negative which holds if and only if $(1-\alpha)<\left(1-a_N^k\right)$. ■

Proof of PROP.6.

By defining $\beta=\frac{2\pi j}{N}$, we can rewrite the term $\mathrm{Re}\left[\sum_{s=1}^{k} e^{\frac{2js\pi i}{N}}\right]$ of expression [27] as follows:

$$\mathrm{Re}\left[\sum_{s=1}^{k} e^{i\beta s}\right]=\mathrm{Re}\left[\sum_{s=0}^{k} e^{i\beta s}\right]-1=\mathrm{Re}\left[\frac{1-e^{i\beta(k+1)}}{1-e^{i\beta}}\right]-1 \quad \text{[A5]}$$

By using both the Euler's formula $\cos\ \beta s\ =\ \mathrm{Re}\left[\sum_{s=1}^{k} e^{i\beta s}\right]$ and the trigonometric identities, we obtain:

$$\mathrm{Re}\left[\frac{1-e^{i\beta(k+1)}}{1-e^{i\beta}}\right]-1=\mathrm{Re}\left(\frac{1-\cos\beta(k+1)-i\sin\beta(k+1)}{1-\cos\beta-i\sin\beta}\right)-1=\frac{\sin\beta\left(k+\frac{1}{2}\right)}{2\sin\frac{\beta}{2}}-\frac{1}{2} \quad \text{[A6]}$$

Therefore, by combining equation [A6] with the expression of the generic eigenvalue λ_{j+1}, we obtain:

$$\lambda_{j+1}=-1+(1-\alpha)-(1-\alpha)\left[\frac{\sin(2k+1)\frac{\beta}{2}}{\sin\frac{\beta}{2}}\right]=-1+(1-\alpha)-(1-\alpha)g(\beta) \quad \text{[A7]}$$

Suppose for the sake of simplicity that the parameter j is a real variable. It is clear that, in order to maximise λ_{j+1} with respect j, we have to minimise the expression $g(\beta)$. Apart from a pure numerical factor, the function $g(\beta)$ is a Dirichlet kernel of order k and its minimum value is attained at its first local minimum. Using standard calculus techniques, we have to solve the following equation:

$$\begin{aligned}\frac{\partial g(\beta)}{\partial\beta}=0 &\Leftrightarrow (2k+1)\cos(2k+1)\frac{\beta}{2}\sin\frac{\beta}{2}-\cos\frac{\beta}{2}\sin(2k+1)\frac{\beta}{2}=0\\ &\Leftrightarrow \tan(2k+1)\frac{\beta}{2}=(2k+1)\tan\frac{\beta}{2}\end{aligned} \quad \text{[A8]}$$

Let us define the minimum point β_m. Then:

$$\tan\frac{\beta_m}{2} = \frac{\tan(2k+1)\frac{\beta_m}{2}}{(2k+1)} \quad \text{[A9]}$$

Using the trigonometric relations, we have:

$$\sin\frac{\beta_m}{2} = \frac{\tan\frac{\beta_m}{2}}{\sqrt{1+\tan^2\frac{\beta_m}{2}}} = \frac{\frac{\tan(2k+1)\frac{\beta_m}{2}}{(2k+1)}}{\sqrt{1+\frac{\tan^2(2k+1)\frac{\beta_m}{2}}{(2k+1)^2}}} = \frac{\left|\cos(2k+1)\frac{\beta_m}{2}\right|\tan(2k+1)\frac{\beta_m}{2}}{\sqrt{(2k+1)^2\cos^2(2k+1)\frac{\beta_m}{2}+\sin^2(2k+1)\frac{\beta_m}{2}}} \quad \text{[A10]}$$

and coming back to $g(\beta)$, we obtain:

$$\min g(\beta) = g(\beta_m) = -\frac{\tan(2k+1)\frac{\beta_m}{2}}{\sqrt{1+\tan^2(2k+1)\frac{\beta_m}{2}}}\frac{\sqrt{(2k+1)^2\cos^2(2k+1)\frac{\beta_m}{2}+\sin^2(2k+1)\frac{\beta_m}{2}}}{\left|\cos(2k+1)\frac{\beta_m}{2}\right|\tan(2k+1)\frac{\beta_m}{2}} = \quad \text{[A11]}$$
$$= -\sqrt{(4k^2+4k)\cos^2(2k+1)\frac{\beta_m}{2}+1}$$

We conclude that:

$$\max_j \lambda_{j+1} = -1+(1-\alpha)+(1-\alpha)\sqrt{(4k^2+4k)\cos^2(2k+1)\frac{\beta_m}{2}+1} \quad \text{[A12]}$$

Thanks to the properties of both $g(\beta)$ and the tan function, we have that the first non trivial root (different from zero) of [A8] is included in the set $[\pi, 3\pi/2]$. Therefore, given the periodicity of the tangent function we obtain:

$$\frac{2\pi}{(2k+1)} \leq \frac{\beta_m}{2} \leq \frac{3\pi}{2(2k+1)} \quad \text{[A13]}$$

By combining the previous condition with [A12], we obtain both an upper bond and a lower bond for the greatest eigenvalue:

$$-1+2(1-\alpha) \leq \max_j \lambda_{j+1} \leq -1+(1-\alpha)+(1-\alpha)(2k+1) \quad \text{[A14]}$$

From the LHS of [A14] we have that $\max_j \lambda_{j+1} \geq 0$ if and only if $(1-\alpha) \geq 1/2$. Therefore, if and only if $(1-\alpha) \geq 1/2$, the SNE is both not unique and unstable. ■

Proof of PROP.7.

The proof of PROP.7. is very similar to that of PROP.4. The only difference consists of the fact that, for $k > 1$, an ANE can present two consecutive BBV communities of size 2. Indeed, consider a combination which presents two consecutive BBV communities of size 2 and at least one BBV community of size 1. Then, it has to be characterised by the presence of at least one individual x and at least k individuals s such that: a) they are both null contributors; b) x's neighbourhood is composed by himself, an individual who belongs to a *BBV community* of size 1, an individual who belongs to a *BBV community* of size 2 and $2(k-1)$ null contributors while s 's neighbourhood is composed by himself, 3 individuals who belong to *BBV communities* of size 2 and $2k-3$ null contributors. From the previous consideration, it follows that $Q_{N_x - x} \leq Q_{N_s - s}$. Therefore, among the null contributors who separate a *BBV community* of size 1 form a *BBV community* of size 2, we have that $Q_{N_x - x} \leq Q_{N_j - j}$, $\forall j \in N$ with $j \neq x$. The rest of the proof is identical to the previous case.

■

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